A NOTE ON NON-COMMUTATIVE MARKOV STATES

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For a long time, several mathematicians have studied the properties of product states on UHF C^*-algebras. To the best of our knowledge, few results have been obtained on non-product states on UHF C^* algebras. In this note, which is an attempt in this direction, we prove as an example some properties of states defined by L. Accardi and called Markov states.

These states are a generalization to the non-commutative case of Markov measures of the classical ergodic theory. Moreover, they allow us to construct non-commutative dynamical systems generalizing Bernoulli shifts.

Recall that a matrix $P = (p_{ij}) \in M_m(\underline{C})$ is a stochastic matrix if $p_{ij} \geq 0$ and $\Sigma_j p_{ij} = 1$ for all i,j. For all positive integers n let $P^n = (p_{ij}^{(n)})$; P is called <u>irreducible</u> if for each pair i,j there is $n > 0$ such that $p_{ij}^{(n)} > 0$. If P is irreducible it is well known that there exists a unique vector $\Lambda = (\lambda_1, .., \lambda_m)$ with $\Lambda P = \Lambda$ and $\Sigma_i \lambda_i = 1$, $\lambda_i > 0$ (see for example [D.G.S.]). Moreover, one says that P is <u>aperiodic</u> if there exists $n_0 > 0$ such that $p_{ij}^{(n)} > 0$ for all $n \geq n_0$ and all i,j.

Given an irreducible stochastic matrix $P \in M_0 = M_m(\underline{C})$, we construct a shift-invariant state ϕ on the C^*-algebra $C = \otimes_{\underline{Z}} M_0$ which we call a Markov state on C.

We prove that the von Neumann algebra obtained by the GNS construction of C for ϕ is a factor if and only if P is aperiodic. Assuming that ϕ is faithful, we then prove that the centralizer of ϕ in M is the hyperfinite II_1 factor R and that the Connes-Størmer entropy of the restriction θ of the shift to R is

$$H(\theta) = -\sum_{i,j} \lambda_i p_{ij} \log p_{ij}.$$

This result has been obtained in [Be]. Finally we show that the dynamical system (R,θ) can be obtained using the Krieger's crossed product.

Similar results have been announced in [St2], but they have not been published.

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Let M_0 be the I_m-factor $(m>1)$ and $\{e_{ij}\}_{i,j=1,\dots,m}$ be a complete system of matrix units in M_0. Let $P = \Sigma_{i,j} p_{ij} e_{ij}$ be an irreducible stochastic matrix and $\Lambda = (\lambda_1,\dots,\lambda_m)$ be the left eigenvector for the eigenvalue 1. Denote by ϕ_0 the state on M_0 defined by $h = \Sigma_i \lambda_i e_{ii}$, $\phi_0 = \mathrm{Tr}(h\cdot)$ where Tr is the usual trace on M_0.

Let $W_i \in M_0$ be defined by $W_i = \Sigma_j p_{ij} e_{jj}$ and $W \in M_0 \otimes M_0$ be $W = \Sigma_i e_{ii} \otimes W_i^{\frac{1}{2}}$ and let γ be the completely positive linear map from $M_0 \otimes M_0$ to M_0, defined by

$$\gamma(x \otimes y) = E_1(W(x\otimes y)W)$$

where $E_1 \colon M_0 \otimes M_0$ is given by $E_1(x\otimes y) = x\,\mathrm{Tr}(y)$.

Let C be the C^*-algebra $C = \otimes_{\underline{Z}} M_0$; we will denote by π_j the canonical injection of M_0 in the j-th factor of C. For $k \leq \ell$ let M_k^ℓ be the C^*-algebra generated by $\{\pi_j(M_0), j=k,\dots,\ell\}$. If $x_k \in M_0$, $k = 0,\dots,n$ we define the state ϕ_0^n on M_0^n by

$$\phi_0^n(\pi_0(x_0)\dots\pi_n(x_n)) = \phi_0(\gamma(x_0 \otimes \gamma(x_1 \otimes \dots \otimes \gamma(x_{n-1} \otimes x_n)\dots)))$$

and if α is the shift on C, we define the state ϕ_k^ℓ on M_k^ℓ by

$$\phi_k^\ell(x) = \phi_0^{\ell-k}(\alpha^{-k}(x)) \;\; \forall x \in M_k^\ell.$$

<u>Definition 1</u> [Ac] The state ϕ on C defined by the family $\{\phi^n_{-n}\}$ is called a Markov state on C.

Notice that we can obtain the same definition for ϕ, using [Pi].

<u>Lemma 2</u>. For $x_0,\ldots,x_n \in M_0$, $x_k = \Sigma_{i,j} x^{(k)}_{ij} e_{ij}$, we have

$$\phi^n_0(\pi_0(x_0)\ldots\pi_n(x_n)) = \sum_{i_0,\ldots,i_n} \lambda_{i_0} p_{i_0,i_1} \ldots p_{i_{n-1},i_n} x^{(0)}_{i_0,i_0} \ldots x^{(n)}_{i_n,i_n}.$$

The proof is easy and is left to the reader.

<u>Proposition 3</u>. If $W_i = h_0$ for all $i = 1,\ldots,m$, then ϕ is a product state.

<u>Proof</u>. We have $W = \Sigma_i e_{ii} \otimes h_0^{\frac{1}{2}} = 1 \otimes h_0^{\frac{1}{2}}$, so

$$\begin{aligned}\gamma(x\otimes y) &= E_1((1\otimes h_0^{\frac{1}{2}})(x\otimes y)(1\otimes h_0^{\frac{1}{2}})) \\ &= x\ \mathrm{Tr}(h_0^{\frac{1}{2}}\ y\ h_0^{\frac{1}{2}}) = x\ \phi_0(y)\end{aligned}$$

Hence $\phi^n_0(\pi_0(x_0)\ldots\pi_n(x_n)) = \phi_0(x_0)\ldots\phi_0(y)$.

q.e.d.

Let M be the Neumann algebra obtained by the GNS construction for the Markov state ϕ of the C^*-algebra C.

<u>Porposition 4</u>. M is a factor if and only if the matrix P is aperiodic.

<u>Proof</u>. a) Assume that ϕ is factorial. It is clear that the system (C,α) is asymptotically abelian, i.e.

$$\| x\alpha^n(y)-\alpha^n(y)x \| \underset{n\to\infty}{\to} 0 \quad \forall x,y \in C.$$

Hence by [Pe, 7.13.4] we deduce that

$$\phi(x\alpha^n(y)) \to \phi(x)\phi(y) \quad \forall x,y \in C.$$

In particular, if $x = \pi_0(e_{ii})$ and $y = \pi_0(e_{jj})$ then

$$\phi(\pi_0(e_{ii})\pi_n(e_{jj})) = \lambda_i\, p_{ij}^{(n)} \to \lambda_i\lambda_j.$$

Hence $p_{ij}^{(n)} \to \lambda_j$ so P is aperiodic [D.G.S.,8.16].

b) Now assume that P is aperiodic. Then $\lim_{n\to\infty} p_{ij}^{(n)} = \lambda_j\ \forall i,j$.

By [Po,2.5], ϕ is factorial if and only if for all $x \in C$ there is $n \geq 0$ such that

$$|\phi(xy)-\phi(x)\phi(y)| \leq \|y\|$$

for all $y \in (M_{-n}^{n})^{c} = (M_{-n}^{n})' \cap C$.

Let $x_0 \in C$ and $\varepsilon > 0$ be given and let $x \in M_{-k}^{k}$ be such that $\|x-x_0\| < \varepsilon$ and $\|x\| \leq \|x_0\|$ (Kaplansky's density theorem). Let $n_0 > k$ be such that $|p_{ij}^{(n-k-1)}p_{k\ell}^{(n-k-1)}-\lambda_j p_{i\ell}^{(2n)}| < \varepsilon\, \lambda_j\, p_{i\ell}^{(2n)}$ for all i,j,k,ℓ and all $n \geq n_0$.

Let $n > n_0$ be fixed and let $y_0 \in (M_{-n+1}^{n-1})^{c}$; there exist $q > n$ and $y \in (M_{-n+1}^{n-1})^{c} \cap M_{-q}^{q}$ such that $\|y-y_0\| < \varepsilon\|y_0\|$ and $\|y\| \leq \|y_0\|$. It is easy to see that

$$|\phi(x_0y_0)-\phi(x_0)\phi(y_0)| \leq 2\varepsilon\|y_0\|(1+\|x_0\|)+|\phi(xy)-\phi(x)\phi(y)|.$$

We will see that $|\phi(xy)-\phi(x)\phi(y)| \leq 16\varepsilon\|x\|\|y\|$. We will then have

$$|\phi(x_0y_0)-\phi(x_0)\phi(y_0)| \leq 2\varepsilon\|y_0\|(1+9\|x_0\|).$$

So by choosing $\varepsilon \leq (2(1+9\|x_0\|))^{-1}$, we will obtain that ϕ is factorial.

By polarization and linearity, it is sufficient to prove that $|\phi(xy)-\phi(x)\phi(y)| \leq \varepsilon\|x\|\|y\|$ for $x \in M_{-k}^{k}$, $x \geq 0$ of the form

$$x = \pi_{-k}(x_{-k})\ldots\pi_k(x_k) \quad \text{with} \quad x_\ell = \sum_{i,j} x_{ij}^{(\ell)} e_{ij}$$

and for $y \in (M_{-n+1}^{n-1})' \cap M_{-q}^{q}$, $y \geq 0$ of the form

$$y = \pi_{-q}(y_{-q})\ldots\pi_{-n}(y_{-n})\pi_{n}(y_{n})\ldots\pi_{q}(y_{q})$$

with $y_\ell = \Sigma_{i,j}\, y_{i,j}^{(\ell)}\, e_{i,j}$.

By Lemma 2 we have

$$\phi(x) = \sum_{i_{-k},\ldots,i_k} \lambda_{i_{-k}} p_{i_{-k},i_{-k+1}}\ldots p_{i_{k-1},i_k} x_{i_{-k},i_{-k}}^{(-k)}\ldots x_{i_k,i_k}^{(k)}$$

$$\phi(y) = \sum_{\substack{\ell_{-q}\ldots\ell_{-n}\\ \ell_n\ldots\ell_q}} \lambda_{\ell_{-q}} p_{\ell_{-q},\ell_{-q+1}}\ldots p_{\ell_{-n-1},\ell_{-n}} p_{\ell_{-n},\ell_n}^{(2n)} p_{\ell_n,\ell_{n+1}}\ldots p_{\ell_{q-1},\ell_q}$$
$$\cdot\; y_{\ell_{-q},\ell_{-q}}^{(-q)}\ldots y_{\ell_{-n},\ell_{-n}}^{(-n)}\, y_{\ell_n,\ell_n}^{(n)}\ldots y_{\ell_q,\ell_q}^{(q)}$$

and

$$\phi(xy) = \sum_{i_k,\ldots,i_k} \sum_{\substack{\ell_{-q}\ldots\ell_{-n}\\ \ell_n\ldots\ell_q}} \lambda_{\ell_{-q}} p_{\ell_{-q},\ell_{-q+1}}\ldots p_{\ell_{-n-1},\ell_{-n}} p_{\ell_{-n},i_{-k}}^{(n-k-1)} \cdot$$
$$\cdot\; p_{i_{-k},i_{-k+1}}\ldots p_{i_{k-1},i_k} p_{i_k,\ell_n}^{(n-k-1)} p_{\ell_n,\ell_{n+1}}\ldots p_{\ell_{q-1},\ell_q} \cdot$$
$$\cdot\; y_{\ell_{-q},\ell_{-q}}^{(-q)}\ldots y_{\ell_{-n},\ell_{-n}}^{(-n)} x_{i_{-k},i_{-k}}^{(-k)}\ldots x_{i_k,i_k}^{(k)} y_{\ell_n,\ell_n}^{(n)}\ldots y_{\ell_q,\ell_q}^{(q)} .$$

So we have

$$|\phi(xy)-\phi(x)\phi(y)| =$$

$$\Big| \sum_{i_k\ldots i_k} \sum_{\substack{\ell_{-q}\ldots\ell_{-n}\\ \ell_n\ldots\ell_q}} \lambda_{\ell_{-q}} p_{\ell_{-q},\ell_{-q+1}}\ldots p_{\ell_{-n-1},\ell_{-n}} p_{i_{-k},i_{-k+1}}\ldots$$
$$\ldots p_{i_{k-1},i_k} p_{\ell_n,\ell_{n+1}}\ldots p_{\ell_{q-1},\ell_q} \cdot$$

$$\cdot\; x_{i_{-k},i_{-k}}^{(-k)}\ldots x_{i_k,i_k}^{(k)} y_{\ell_{-q},\ell_{-q}}^{(-q)}\ldots y_{\ell_{-n},\ell_{-n}}^{(-n)} y_{\ell_n,\ell_n}^{(n)}\ldots y_{\ell_q,\ell_q}^{(q)} \cdot$$

$$\cdot\; (p_{\ell_{-n},i_{-k}}^{(n-k-1)} p_{i_k,\ell_n}^{(n-k-1)} - \lambda_{i_{-k}} p_{\ell_{-n},\ell_n}^{(2n)}) \Big| .$$

By the choice of n we have

$$\left| p_{\ell_{-n},i_{-k}}^{(n-k-1)} p_{i_k,\ell_n}^{(n-k-1)} - \lambda_{i_{-k}} p_{\ell_{-n},\ell_n}^{(2n)} \right| \leq \varepsilon\, \lambda_{i_{-k}} p_{\ell_{-n},\ell_n}^{(2n)} .$$

Hence

$$|\phi(xy)-\phi(x)\phi(y)| \leq \varepsilon\; \phi(x)\phi(y) \leq \varepsilon \|x\| \|y\| .$$

q.e.d.

From now we will assume that ϕ is faithful and therefore $p_{ij} > 0$ for all i,j. Let $h_{p,q}$ be the Randon-Nikodym derivative of ϕ_p^q with respect to the usual trace Tr_p^q on M_p^q. By definition of ϕ_p^q we have $h_{p,q} = \alpha^p(h_{0,q-p})$.

Lemma 5. With the above notations we have

a) $h_{0,n} = \sum_{i_0,\dots,i_n} \lambda_{i_0} p_{i_0,i_1} \cdots p_{i_{n-1},i_n} \pi_0(e_{i_0,i_0}) \cdots \pi_n(e_{i_n,i_n})$.

b) The unitary operator $u_t^{(p,q)} = h_{p-1,q+1}^{-it} h_{p,q}^{it}$ belongs to the C^*-algebra generated by M_{p-1}^p and M_q^{q+1}.

Proof. The proof of a) is easy and is omitted.

If $n = q-p$, we have

$$h_{p-1,q+1}^{-it} h_{p,q}^{it} = \alpha^{p-1}(h_{0,n+2}^{-it} \alpha(h_{0,n}^{it}))$$

and

$$h_{0,n+2}^{-it} = \sum_{j_0,\dots,j_{n+2}} (\lambda_{j_0} p_{j_0,j_1} \cdots p_{j_{n+1} j_{n+2}})^{-it} \pi_0(e_{j_0,j_0}) \cdots \pi_{n+2}(e_{j_{n+2},j_{n+2}})$$

$$\alpha(h_{0,n}^{it}) = \sum_{i_1,\dots,i_{n+1}} (\lambda_{i_1} p_{i_1,i_2} \cdots p_{i_n,i_{n+1}})^{it} \pi_1(e_{i_1,i_1}) \cdots \pi_{n+1}(e_{i_{n+1},i_{n+1}}).$$

So $h_{0,n+2}^{-it} \alpha(h_{0,n}^{it}) =$

$$= \sum_{j_0,j_1} \lambda_{j_0}^{-it} \lambda_{j_1}^{it} p_{j_0,j_1}^{-it} \pi_0(e_{j_0,j_0}) \pi_1(e_{j_1,j_1}) \sum_{j_{n+1},j_{n+2}} p_{j_{n+1},j_{n+2}}^{-it}$$
$$\pi_{n+1}(e_{j_{n+1},j_{n+1}}) \pi_{n+2}(e_{j_{n+2},j_{n+2}}).$$

So $h_{0,n+2}^{-it} \alpha(h_{0,n}^{it})$ belongs to the C^*-algebra generated by M_0^1 and M_{n+1}^{n+2} and therefore $h_{p-1,q+1}^{it} h_{p,q}^{-it}$ is in the C^*-algebra generated by M_{p-1}^p and M_q^{q+1}. q.e.d.

Remark 6. As P is aperiodic, by a similar proof of Proposition 4, one can see that ϕ is strongly mixing with respect to α.

Then using Lemma 5a) and Corollary 4.3 of [St1], it is easy to see that M is of type III_1 if the quotients λ_i/λ_j and $p_{ij}/p_{k\ell}$ are not all contained in the same cyclic subgroup of the group of positive real numbers.

Let σ^{ϕ} be the modular group for ϕ in M and $\sigma^{(p,q)}$ be the modular group for ϕ_p^q in M_p^q. As $\phi\circ\alpha = \phi$ and $\phi_p^q = \phi_{p+k}^{q+k}\circ\alpha^k$ for all $k \in \underline{Z}$, we have

$$\sigma_t^{\phi}\circ\alpha = \alpha\circ\sigma_t^{\phi} \quad \text{and} \quad \alpha^k\circ\sigma_t^{(p,q)} = \sigma_t^{(p+k,q+k)}\circ\alpha^k$$

for all $t \in \underline{R}$.

<u>Proposition 7</u>. For all $x \in M_{-n}^n$ and all $t \in \underline{R}$ we have

$$\sigma_t^{\phi}(x) = \sigma_t^{(-n-1,n+1)}(x).$$

<u>Proof</u>. We have $\sigma_t^{\phi}(x) = \sigma_t^{\phi}\circ\sigma_{-t}^{(-k,k)}\circ\sigma_t^{(-k,k)}(x)$ and for $k > n+1$

$$\begin{aligned}\sigma_t^{(-k,k)}(x) &= h_{-k,k}^{it}\; x h_{-k,k}^{-it}\\ &= h_{-k+1,k-1}^{it}\; h_{-k+1,k-1}^{-it}\; h_{-k,k}^{it} x h_{-k,k}^{-it}\; h_{-k+1,k-1}^{it}\; h_{-k+1,k-1}^{-it}\\ &= h_{-k+1,k-1}^{it}\,(u_t^{(-k+1,k-1)})^{*}\, x u_t^{(-k+1,k-1)} h_{-k+1,k-1}^{-it}\\ &= h_{-k+1,k-1}^{it}\; x h_{-k+1,k-1}^{-it}\\ &= \sigma_t^{(-k+1,k-1)}(x).\end{aligned}$$

So for all $k > n+1$ we obtain

$$\sigma_t^{\phi}(x) = \sigma_t^{\phi}\circ\sigma_{-t}^{(-k,k)}\circ\sigma_t^{(-n-1,n+1)}(x).$$

But by [Lo, Lemma 4], $\sigma_t^{(-k,k)}(x)$ converges strongly to $\sigma_t^{\phi}(x)$ when $k \to \infty$. So

$$\sigma_t^{\phi}(x) = \sigma_t^{(-n-1,n+1)}(x)$$

for all $t \in \underline{R}$.

q.e.d.

Let N be the centralizer of ϕ in M.

Definition 8. The restriction θ of α to N is called a Markov shift on N.

As an immediate consequence of Proposition 3 we have

Corollary 9. If $W_i = h_0$ for all $i = 1,\ldots,m$, then the automorphism θ is a Bernoulli shift.

For all $n \in \underline{N}$ we define

$$N_{-n}^{n} = \{x \in M_{-n}^{n} \mid \sigma_t^{(-n-1,n+1)}(x) = x \text{ for all } t \in \underline{R}\}$$

The following proposition is an easy consequence of Proposition 7.

Proposition 10. Let E_ϕ be the normal and faithful conditional expectation from M to N which preserves ϕ. Then $E_\phi(M_{-n}^{n}) = N_{-n}^{n}$ so N is generated by the sequence $\{N_{-n}^{n}\}$.

Now our aim is to show that N is a factor, so it will be the hyperfinite II_1 factor. To prove this, we will see that N can be obtained as the Krieger's crossed product of a standard Borel space by an countable locally finite ergodic group.

Let $X_0 = \{1,\ldots,m\}$, $X = \Pi_{\underline{Z}} X_0$ and μ be the shift-invariant Markov measure on X with initial distribution Λ and transition matrix P. We will still assume that the p_{ij}'s are strictly positive.

Let $X_k^\ell = \Pi_k^\ell X_0$, μ_k^ℓ be the restriction of μ to X_k^ℓ and let G_k^ℓ be the the group of automorphisms g of X_k^ℓ such that

$$(g\omega)_k = \omega_k \ , \quad (g\omega)_\ell = \omega_\ell \quad \text{for all } \omega \in X_k^\ell$$

and

$$g\,\mu_k^\ell = \mu_k^\ell \ .$$

In [Kr2] W. Krieger has proved the following theorem.

Theorem 11. The group $G = \bigcup_{n \in \underline{N}} G^{n}_{-n}$ acts ergodically on (X,μ).

We recall now briefly the construction of the Krieger's crossed product [Kr1] as it is done in [Gui].

Let Y be a standard Borel space with non atomic probability measure ν. Let H be a countable ergodic group of automorphisms of Y preserving the measure ν.

For all $\omega \in Y$, let $H\omega$ be the orbit of ω under the action of H and let $K_\omega = \ell^2(H\omega)$ with canonical Hilbert basis $(\varepsilon_{\omega,\phi})$, $\phi \in H\omega$. If $\varepsilon^{(g)}_\omega = \varepsilon_{\omega, g\omega}$, then the set of $\varepsilon^{(g)}$ is a fundamental family of mesurable vector fields [Di, II.1]. One can therefore define the Hilbert space $K = \int_X^{\oplus} K_\omega \, d\mu(\omega)$. For $a \in L^\infty(Y,\nu)$ and $g \in H$, let M_a and U_g be the operators on K defined by

$$(M_a\xi)_\omega = a(\omega)\xi_\omega \quad \text{and} \quad (U_g\xi)_\omega = \Psi_{g,g^{-1}\omega}(\xi_{g^{-1}\omega})$$

where $\Psi_{g,\omega}$ is the isomorphism from K_ω onto $K_{g\omega}$ defined by

$$\Psi_{g,\omega}(\varepsilon_{\omega,\phi}) = \varepsilon_{g\omega,\phi} .$$

Then U is a unitary representation of H in K and we have the relations

$$U_g\xi^{(h)} = \xi^{(hg^{-1})} \quad \text{and} \quad U_g M_a U_g^{*} = M_{ga}$$

where $ga(\omega) = a(g^{-1}\omega)$.

The von Neumann algebra $B = \{M_a,\ a \in L^\infty(Y,\nu)\}$ is isomorphic to $L^\infty(Y,\nu)$, so we will identify them.

By hypothesis on the group H, the von Neumann algebra $R = R(Y,H)$ generated by B and $\{U_g, g \in H\}$ is a factor of type II_1, hyperfinite if H is amenable, which will be called the Krieger's crossed product of Y by H.
In our case, as G is locally finite, $R = R(X,G)$ is the hyperfinite II_1-factor.

Let A_0 be the maximal abelian subalgebra of M_0 generated by the $\{e_{ii}\}$ and let A_p^q be the canonical image of $\otimes_p^q A_0$ in M. The von Neumann algebra A generated by $\{A_{-n}^n\}$ is maximal abelian in M and clearly $A \subset N$. As A can be identified with $L^\infty(x,\mu)$, the group G acts on A. Since any element of G_p^q gives rise to a permutation of the minimal projections of A_p^q, there exists a unitary representation $g \to v_g$ of G_p^q in M_p^q. Moreover the canonical conditional expectation E_k from M_{-k}^k onto A_{-k}^k preserves ϕ_{-k}^k. For all $g \in G_{-k}^k$ and all $x \in M_{-k-1}^{k+1}$ we have

$$\begin{aligned}\phi_{-k-1}^{k+1}(v_g x v_g^*) &= \phi_{-k-1}^{k+1}(E_{k+1}(v_g x v_g^*)) \\ &= \phi_{-k-1}^{k+1}(v_g E_{k+1}(x) v_g^*) \\ &= \mu_{-k-1}^{k+1}(g(E_{k+1}(x))) \\ &= \mu_{-k-1}^{k+1}(E_{k+1}(x)) = \phi_{-k-1}^{k+1}(x).\end{aligned}$$

Therefore $v_g \in N$ for all $g \in G$; thus the Krieger's crossed product $R = R(X,G)$ is a subfactor of N.

Let now R_k be the finite dimensional subalgebra of R generated by A_{-k}^k and $\{v_g, g \in G_{-k}^k\}$. To see that N is the hyperfinite II_1 factor, it is sufficient to show that $N_{-k}^k \subset R_{k+1}$. As $A_{-k}^k \subset R_{k+1}$ and A_{-k}^k is regular in N_{-k}^k, it is sufficient to see that the normalizer of A_{-k}^k in N_{-k}^k, $N(A_{-k}^k)$, is in R_{k+1}. Let $u \in N(A_{-k}^k)$ then $\sigma_t^{(-k-1,k+1)}(u) = u$ for all $t \in \underline{R}$, thus $\phi_{-k-1}^{k+1}(uxu^*) = \phi_{-k-1}^{k+1}(x)$ for all $x \in M_{-k-1}^{k+1}$. In particular, for all $a \in A_{-k-1}^{k+1}$, $\phi_{-k-1}^{k+1}(uau^*) = \phi_{-k-1}^{k+1}(a)$, so u defines an element of G_{-k-1}^{k+1} and therefore $u \in R_{k+1}$. Thus we have proved the following theorem.

<u>Theorem 12.</u> N is the hyperfinite II_1-factor.

<u>Theorem 13.</u> Let θ be the Markov shift on N. Then the entropy of θ is

$$H(\theta) = -\sum_{i,j} \lambda_i \, p_{ij} \log p_{ij}.$$

Proof. Henceforth we will use the notations of [C.S.] for the entropy. By Kolmogolov-Sinai's theorem of Connes and Størmer [C.S.] and Proposition 10 we have

$$H(\theta) = \lim_{n\to\infty} H(N^n_{-n},\theta).$$

For all $k \in \underline{Z}$ let $N^{n+k}_{-n+k} = \theta^k(N^n_{-n})$. For a fixed n we have

$$H(N^n_{-n},\theta) = \lim_{q\to\infty} (2q)^{-1} H(N^n_{-n},\theta(N^n_{-n}),\ldots,\theta^{2q}(N^n_{-n}))$$

$$= \lim_{q\to\infty} (2q)^{-1} H(N^n_{-n},N^{n+1}_{-n+1},\ldots,N^{n+2q}_{-n+2q}).$$

For all $k = 1,\ldots,2q$ we have $N^{n+k}_{-n+k} \subset N^{n+2q}_{-n}$. Indeed for all $x \in N^{n+k}_{-n+k}$ and all $t \in \underline{R}$

$$\sigma_t^{(-n-1,n+2q+1)}(x) = \sigma_t^{(-n-1,n+2q+1)}\circ\sigma_{-t}^{(-n+k-1,n+k+1)}(x)$$

$$= h^{it}_{-n-1,n+2q+1}\, h^{-it}_{-n+k-1,n+k+1}\; x\; h^{it}_{-n+k-1,n+k+1}\, h^{-it}_{-n-1,n+2q+1}$$

and by a same argument as in Lemma 5 b) we see that $h^{it}_{-n-1,n+2q+1}\cdot h^{-it}_{-n+k-1,n+k+1}$ belongs to the C^*-algebra generated by M^{k-n-1}_{-n-1} and M^{n+2q+1}_{k+n+1}; thus this operator commutes with x and therefore $x \in N^{n+2q}_{-n}$.

By the properties (C) and (D) of [C.S.] we obtain

$$H(N^n_{-n},N^{n+1}_{-n+1},\ldots,N^{n+2q}_{-n+2q}) \leq H(N^{n+2q}_{-n}) = H(A^{n+2q}_{-n}) = H(A^{2n+2q}_0).$$

Furthermore for all $r > 0$

$$H(A^r_0) = \sum_{i_0,\ldots,i_r} \eta\phi(\pi_0(e_{i_0,i_0})\ldots\pi_r(e_{i_r,i_r}))$$

$$= \sum_{i_0,\ldots,i_r} \eta(\lambda_{i_0} p_{i_0,i_1}\cdots p_{i_{r-1},i_r})$$

$$= \sum_i \lambda_i \log\lambda_i - r\sum_{i,j}\lambda_i p_{ij}\log p_{ij}.$$

So $\qquad H(N^n_{-n},\theta) \leq -\sum_{i,j} \lambda_i p_{ij} \log p_{ij}$

and then $\qquad H(\theta) \leq -\sum_{i,j} \lambda_i p_{ij} \log p_{ij}.$

On the other hand, for all n we have $H(\theta) \geq H(A^n_{-n},\theta)$ and

$$\begin{aligned} H(A^n_{-n},\theta) &= \lim_{q\to\infty} q^{-1} H(A^n_{-n},\ldots,\theta^q(A^n_{-n})) \\ &= \lim_{q\to\infty} q^{-1} H(A^n_{-n},\ldots,A^{n+q}_{-n+q}) \\ &= \lim_{q\to\infty} q^{-1} H(A^{n+q}_{-n}) \\ &= -\sum_{i,j} \lambda_i\, p_{ij} \log p_{ij}. \end{aligned}$$

q.e.d.

<u>Proposition 14</u>. Let (X,G,μ) be as before, and let S be the shift on (X,μ). Then S extends to an automorphism σ of $R = R(X,G)$ and the dynamical systems (N,θ) and (R,σ) are conjugate.

<u>Proof</u>. It is clear that $SGS^{-1} = G$ because $SG^n_{-n}S^{-1} \subset G^{n+1}_{-n-1}$. Thus $GS^{-1}\omega = S^{-1}G\omega$ for all $\omega \in X$. Using the same notations as before Theorem 12, the linear mapping $\Phi_\omega : K_\omega \to K_{S^{-1}\omega}$ defined by $\Phi_\omega(\varepsilon_{\omega,\psi}) = \varepsilon_{S^{-1}\omega,S^{-1}\psi}$ is an isomorphism, and by [Di,II.2] the field $\omega \to \Phi_\omega$ is mesurable. Furthermore it is easy to see that the operator V on K defined by

$$(V\xi)_\omega = \Phi_\omega^{-1}\, \xi_{S^{-1}\omega}$$

is unitary and has the properties

$$VaV^* = S(a) \quad \text{for all} \quad a \in L^\infty(X,\mu)$$

$$VU_gV^* = U_{SgS^{-1}} \quad \text{for all} \quad g \in G.$$

Therefore the automorphism σ of R defined by $\sigma(x) = VxV^*$, $x \in R$ extends S.

Moreover if J is the isomorphism from N to R identifying A with $L^{\infty}(X,\mu) \subset R$ in the canonical way, and sending v_g onto u_g, then $J\theta J^{-1} = \sigma$.

q.e.d.

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