THE RECURSION THEORY OF PTYKES

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1. Introduction

The ptyx was introduced in Girard [2] and it is a higher type version of the dilator introduced in Girard [1]. Girard and Ressayre [5] gave an alternative approach to the ptykes and they gave several applications.

In this paper we will investigate a category of generalized binary relations and we will see how the ptykes can be represented as objects in this category.

We will review the decomposition of a ptyx and prove a hierarchy-theorem for the corresponding decomposition trees. Employing the functorial bounding theorem from Girard and Normann [4] we will see how the recursion theorem provides us with a general notion of functorial recursion over the decomposition trees. Parts of the paper will be a review of known results or simple generalizations of such. In these cases we omit or give minor hints to the proofs. Familiarity with an introduction to dilators or denotation systems (Girard [1], [2] or Girard-Normann [3]) will be an advantage.

2. Types and Classes

We will base our study of Π^1_2-logic, the ptykes and related objects on one category, the universal type U. U is a generalization of the category of binary relations:

2.1 Definition

a) A pair $\langle f,X\rangle$ is in the universal type U if X is a set and $f:X^2 \to \underline{\underline{N}}$.

b) If $\langle f,X\rangle$ and $\langle g,Y\rangle$ are two elements of U then

$$\phi : X \to Y$$

is a called an imbedding if ϕ is 1-1 and for all $x_1, x_2 \in X$ we have

$$f(x_1,x_2) = g(\phi(x_1), \phi(x_2))$$

When no information is lost we will write f for $\langle f,X\rangle$, and if nothing else is made explicit, f,g etc. will denote elements of U.

2.2 Definition

If $f,g \in U$, then $I(f,g)$ is the set of imbeddings from f to g.

2.3 Definition

a) A *class* C is a subcollection of U such that if $g \in C$ and $\phi \in I(f,g)$ then $f \in C$.

b) A *pretype* T is a class of finite objects

c) If T is a pretype, then the type TP(T) of T is the class of all objects such that each finite subfunction is in T.

d) If C is a class then PT(C) is the class of all finite elements of C and TP(C), the type of C, is TP(PT(C)).

As a trivial observation we get

2.4 Lemma

Let C be a class. Every $f \in TP(C)$ is the limit of a directed system from PT(C).

We let U_0 denote the class of all finite elements of U. If $f \in U_0$ then f is isomorphic to some $g:n^2 \to \underline{\underline{N}}$ $(n=\{0,\ldots,n-1\})$ Using some standard enumeration of finite sequences, we may code g as a natural number.

2.5 Definition

a) For each $f \in U_0$, let $D(f)$ be the $g:n^2 \to \underline{\underline{N}}$ isomorphic to f with the lowest number code. We call $D(f)$ the distinguished version of f.

b) Let $U_D = \{D(f) \mid f \in U_0\}$

If C is a class, let

$$C_D = \{D(f) \mid f \in C \cap U_0\} = C \cap U_D$$

c) A class C is recursively based if C_D is recursive.

2.6 Definition

a) Let A_0 be the subclass of U_0 containing all f with no nontrivial automorphisms.

b) Let $A_D = A_0 \cap U_D$ and let $A = TP(A_0)$.

c) A class C is an A-class if $C \subseteq A$.

Let C be an A-class, $f \in A$. Let J_f be the set of finite subfunctions of f. For each $g \in J_f$ there will be a unique isomorphism $\gamma_g:D(g) \to g$. If $g \subseteq h \in J_f$, let $\gamma_{gh} = \gamma_h^{-1} \cdot \gamma_g$. Then

$$f = \varinjlim \langle D(g), \gamma_{gh} \rangle.$$

We will write $f = \varinjlim \langle f_i, \gamma_{ij} \rangle$ and call it the canonical limit-construction of f.

Now let C_1 and C_2 be two A-classes. Let $F:C_1 \to C_2$ be a functor commuting to pullbacks and direct limits.

Let $f:X^2 \to \underline{\underline{N}}$ be in C_1 and let $F(f) = g:Y^2 \to \underline{\underline{N}}$.

Let $\langle f_i, \phi_{ij} \rangle$ be the canonical limit-construction of f with imbeddings $\phi_i:f_i \to f$. Let $g_i = F(f_i)$ and $F(\phi_i) = \gamma_i:g_i \to g$.

Let $y \in Y$. Since F commmutes to pullbacks there is a unique minimal i such that $y \in Im(\gamma_i)$. Let Y_i be such that

$g_i : Y_i^2 \to \underline{\underline{N}}$. Then there is a unique $c \in Y_i$ such that

$$y = \gamma_i(c)$$

We call $(c;\phi\ ;f)$ a <u>denotation</u> for y. Given F and f, this denotation will be unique.

2.7 <u>Definition</u>

Let C_1, C_2 and F be as above.

a) A <u>bone</u> for F is a pair (c,f) where $f \in U_D$, c is in the domain of $F(f)$ and for all $g \in U_D$, $\phi \in I(g,f)$, if $g \neq f$ then $c \notin Im(F(\phi))$.

b) The <u>skelleton</u> of F is the set of bones for F.

c) The <u>dimention</u> of F is the cardinality of the skelleton.

Now let C_1 and C_2 be A-classes and let $F:C_1 \to C_2$ be a functor commuting with pullbacks and direct limits. We will see how the skelleton supports a natural binary function.

Let (c,f) and (d,g) be two bones for F. The interesting relation between (c,f) and (d,g) is the set of values

$$F(h)((c;\phi;h),\ (d;\gamma;h))$$

seen as a function of how the imbeddings ϕ and γ are related.

Recursively in f and g we have a list $h_1,\ldots,h_k$ from U_D and imbeddings $\phi_i : f \to h_i$, $\gamma_i : g \to h_i$ such that for all $(c;\phi;h)$, $(d;\gamma;h)$ there is a number i and an imbedding $\psi : h_i \to h$ such that

$$\phi = \psi \circ \phi_i,\ \gamma = \psi \circ \gamma_i.$$

Then $F(h)((c;\phi;h),\ (d:\gamma;h))$

$= \quad F(h_i)((c;\phi_i;h_i),\ (d;\gamma_i;h_i)).$

Let $N_F((c,f),(d,g)) = \langle n_1, n_2, m_1, \dots, m_k \rangle$ where n_1 and n_2 are numbercodes for f,g resp. and

$$m_i = F(h_i)((F(\phi_i)(c),\ F(\gamma_i)(d)).$$

From $N_F((c,f),(d,g))$ we can recover f and g but not c and d.

2.8 Definition

Let C_1 and C_2 be A-classes. We let $C_1 \to C_2$ be the collection of all functions

$$N: Z^2 \to \underline{\underline{N}}$$

isomorphic to some N_F as described above.

2.9 Theorem

If C_1 and C_2 are A-classes then $C_1 \to C_2$ is an A-class.

Proof

Let $N_F: Z^2 \to \underline{\underline{N}}$ be given. We have to prove that any substructure of N_F is isomorphic to some N_G. Let $Y \subseteq Z$. For each $h \in C_1$, let $G(h)$ be the substructure of $F(h)$ with domain the set of elements with denotations $(c;\phi;h)$ where the image of ϕ is in Y. G naturally extends to a functor commuting to pullbacks and direct limits and N_G is isomorphic to $N_F \upharpoonright Y^2$ This shows that $C_1 \to C_2$ is a class.

It remains to show that it is an A-class. Let $N: Z^2 \to \underline{\underline{N}}$ be a finite element of $C_1 \to C_2$ with corresponding functor G_N. Let ϕ be an automorphism on N, $h: X^2 \to \underline{\underline{N}}$ be a non-empty, finite element of C_1.

Let ψ be the automorphism on $G_N(h)$ induced by ϕ. Since C_2 is an A-class, ψ is the identity. But then ϕ is the identity, since denotations are unique.

2.10 Remarks

a) We used that C_1 is an A-class to get a unique denotation system and that C_2 is an A-class to prove that $C_1 \rightarrow C_2$ is an A-class.

b) It is not in general correct that $C_1 \rightarrow C_2$ is recursively based when C_1 and C_2 are recursively based. The classes we will study later will be recursively based, the methods used to prove this can be found in Girard [2].

c) The construction of $C_1 \rightarrow C_2$ is uniform. Thus there is one class coding all partial functors from U_A to U_A with a class as the domain.

3. WO-classes

3.1 Definition

Let $f:\underline{\underline{N}} \rightarrow \underline{\underline{N}}$. By f^- we mean

$$f^-(x,y) = \begin{cases} f(\langle x,y\rangle)-1 & \text{if } f(x,y) > 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

b) Let C be a class. By C_ω we mean

$$C_\omega = \{f:\underline{\underline{N}}\rightarrow\underline{\underline{N}} \mid f^- \in C\}$$

c) A subset A on $\underline{\underline{N}}^{\underline{\underline{N}}}$ can be reduced to C if there is a recursive $F:\underline{\underline{N}}^{\underline{\underline{N}}} \rightarrow \underline{\underline{N}}^{\underline{\underline{N}}}$ such that

$$g \in A \iff F(g) \in C_\omega$$

3.2 Lemma

Let C,D be classes, let $f \in (C\rightarrow D)_\omega$ and let $g \in (C_\omega)$. Uniformly recursive in f and g we may find $h \in D_\omega$ such that

$$h^- = f^-(g^-).$$

The proof is easy and is left for the reader.

The litterature contains several theorems about effective operators from ptykes to ordinals that can be bounded by ptykes, the most recent and general is due to Kechris [6]. A standard method of proof is to functorially find well-founded trees that dominates the ordinal in question and then linearizing it by e.g. a Kleene-Brouwer ordering. The WO-classes will be a general family of classes for which these kinds of arguments works.

3.3. Definition

a) Let WO be the class of (characteristic functions of) well-orderings

b) A class C can be well-ordered if there is a function $\tau : \underline{\underline{N}} \to \underline{\underline{N}}$ such that for all $f \in C$, $\tau \circ f \in WO$.

c) Assume that C can be well-ordered. We call C a WO-class if for each well-ordered family $\{f_i\}_{i<\beta}$ there is an $f \in C$ and imbeddings $\phi_i : f_i \to f$ into pairwise disjoint subsets of f such that the well ordering of f puts $\phi_i(x) < \phi_j(y)$ if $i < j$.

3.4 Lemma

If C is a class then $C \to WO$ is a WO-class.

Proof

Let $\{f_i\}_{i \in \underline{\underline{N}}}$ be some enumeration of

$$(C \to WO) \cap U_D$$

and let $f_\omega = \sum_{i \in \underline{\underline{N}}} f_i$. Each f_i has a canonical imbedding

$$\gamma_i : f_i \to f_\omega$$

If $F \in C \to WO$, let (c,f) and (d,g) be two bones for F. We order them by the values of the denotations

$$(c;\ \gamma_f;\ f_\omega) \qquad (d;\ \gamma_g;\ f_\omega)$$

We leave the details for the reader.

Our classes are not only closed under function-spaces but also under Cartesian products:

3.5 Definition

a) Let $f_1,\dots,f_n$ be elements of U.
$f_i{:}X_i^2 \to \underline{\underline{N}}$. Let $\langle f_1,\dots,f_n\rangle$ be the function

$$f{:}(\{1\}\times X_1 \cup \dots \cup \{n\}\times X_n)^2$$

defined by

$$f((i,x_1),\ (i,x_2)) = \langle i, f_i(x_1,x_2)\rangle + 1$$

$$f((i,x),\ (j,y)) = 0 \quad \text{if} \quad i \neq j.$$

b) If $C_1,\dots,C_n$ are classes, let $C_1\times\dots\times C_n$ be the class of those f that are isomorphic to some $\langle f_1,\dots,f_n\rangle$ where $f_i \in C_i$ for $i=1,\dots,n$

It is easily seen that this really defines a class. If $C_1,\dots,C_n$ are WO-classes then $C_1\times\dots\times C_n$ is a WO-class, using lexicographical orderings.

We have the following:

3.6 Theorem

Let C be a WO-class and let A be reducible to C via the recursive function F. Then the complement $\sim A$ can be reduced to $C \to WO$ via some recursive G such that if $g \in A$ there is an infinite descending sequence in $G(g)^{-}$ $(F(g)^{-})$ uniformly recursive in g.

Proof

Let g be given and let $f \in C$. Let $h:\underline{\underline{N}} \to \text{dom}(f)$. Define a tree $T_g(f)$ such that h is a branch in $T_g(f)$ if and only if $g' = F(g)^{-}$ is an element of U and $h \restriction \text{dom } g'$ is an imbedding of g' into f. This can be done functorially. $T_g(f)$ is a tree on f and using the well-ordering of f it can be linearized to $O_g(f)$ which will be an element of $C \to WO$ if $g \in A$. If $g \in A$ then take f to be $F(g)^{-}$. The identity -imbedding of $F(g)^{-}$ extended to a map from $\underline{\underline{N}}$ to f will be a branch in $T_g(f)$, so $O_g(f)$ is effectively not well-founded.

3.7 Definition

Let C be a WO-class.

We call $A \subseteq \underline{\underline{N}}^{\underline{\underline{N}}}$ a Π^1_C-set if A can be reduced to $C \to WO$ via a recursive F. A is $\underset{\sim}{\Pi}^1_C$ if it can be reduced to $C \to WO$ via a continuous F.

The Π^1_C-sets have many properties in common with the Π^1_k-sets. The following result is stated without proofs:

3.8 Theorem

a) The union of a recursively enumerated family of Π^1_C-sets is a Π^1_C-set

b) The intersection of recursively enumerated family of Π^1_C-sets is a Π^1_C-set

c) If A is a Π^1_C-set, then

$B = \{x \mid \forall y \langle x,y\rangle \in A\}$

is a Π^1_C-set

d) There is a Π^1_C-set that is universal for $\underset{\sim}{\Pi}^1_C$-sets.

In a) we construct a tree $T(f)$ such that a branch in $T(f)$ will contain a branch in all the orderings $F_i(f)$. If no $F_i \in C \to WO$ we can take witnesses f_i, and since C is a WO-class we can imbedd all the f_i's into one f which will give a branch in $T(f)$. In b) we observe

$$\sum_{i \in \underline{\underline{N}}} F_i \in C \to WO \iff F_i \in C \to WO \text{ for all } i.$$

c) is proved by a tedious but simple coding of $\forall x$ into $C \to WO$ and d) is based on the fact that the set of continuous functionals is Π^1_1 and thus reducible to WO. We are now ready to define the ptykes. The main structural properties are given inductively using theorems 3.6 and 3.8.

3.9 <u>Definition</u>

a) Let $Pt(0) = WO$

$Pt(k+1) = Pt(k) \to WO.$

b) The elements of $Pt(k)$ are called ptykes of pure type.

3.10 <u>Definition</u>

Let $\Xi_k = \sum\{P_i \mid P_i$ is a recursive element of $Pt(k)\}$

Ξ_0 is essentially ω_1^{CK}. Girard and Ressayre [5] has shown that $\Xi_k(\Xi_{k-1})$ is the ordinal π_k, the supremum of all Π^1_k-well-orderings of subsets of $\underline{\underline{N}}$.

We can effectively decide if an ordinal $\alpha = \omega_1^{CK}$, but a similar fact does not hold for Ξ_k in general. We will find a remedy for that and indicate how it can be used.

3.11 Definition

Let $\{f_i\}_{i\in\underline{\underline{N}}}$ be a recursive enumeration of all the partial recursive functions. Let $A_k = \{i \mid f_i$ is total and $\bar{f_i} \in Pt(k)\}$.

a) (Precise definition)

$$\Xi_k = \sum_{i\in A_k} \bar{f_i}$$

b) Let F be recursive such that

$i \in A_k \iff F(i) \in A_{k+1}$

Let $\Theta_k = \sum_{i\in A_k} \bar{f}_{F(i)}$.

3.12 Lemma

Let $P \in Pt(k)$, $Q \in Pt(k+1)$. We may set-recursively decide if

$$(P,Q) = (\Xi_k, \Theta_k)$$

Proof

By the well-ordering we may recursively decide if two ptykes are isomorphic. Now let P and Q be given. Let $\{f_i\}_{i\in\underline{\underline{N}}}$ be as in 3.11. Inductively let

$$i \in A_P \text{ if } \sum_{\substack{j<i \\ j\in A_P}} \bar{f_j} + \bar{f_i}$$

is isomorphic to an initial seqment of P.

If $P \neq \sum_{i\in A_P} \bar{f_i}$ then $P \neq \Xi_k$.

If $P = \sum_{i\in A_P} \bar{f_i}$, let $Q_P = \sum_{i\notin A_P} \bar{f}_{F(i)}$.

If $Q_P = Q$ then we have $P = \Xi_k$, $Q = \Theta_k$

otherwise not.

Set-recursion is defined in Normann [7]

3.13 Corollary

If $\prec$ is a well-ordering of $\underline{\underline{N}}$ recursive in a complete Π^1_{k+1}-set, then there is a recursive functor $F \in Pt(k)\times Pt(k+1) \to WO$ such that $\|F(E_k,\Theta_k)\| > \|\prec\|$.

Proof

This follows from 3.12 and well-known bounding theorems.

4. Decomposition of a ptyx

The ptykes of type 1 are also called dilators. One of the main structural properties of dilators is the well-founded decomposition of a dilator into "smaller" dilators. A principal tool is recursion over this decomposition and a principal obstacle is the need of getting functorial operators out of these recursions.

Girard constructed a similar decomposition of ptykes. We will review this decomposition and prove a hierarchy-theorem for it. In the next section we will combine it with the functorial bounding theorem from Girard-Normann [4] to give a general method for recursion over the decomposition.

1. Definition

a) If $\{P_i\}_{i<\alpha}$ is a family from $Pt(k)$ we define $\sum_{i<\alpha} P_i$ in the usual way.

b) $P \in Pt(k)$ is called connected if P is not a nontrivial sum $P = P_1 + P_2$.

We have

4.2 Lemma

All ptykes is the unique sum of a well-ordered family of connected ptykes.

The proof is essentially as in the dilator case.

From now on assume that $k>1$.

4.3 Definition

Let $P \in Pt(k)$ and let (c,f) be a bone for P. Let $f = \sum_{i<n} f_i$ where f_i is connected.

Let $h = \sum_{i<n} (f_i+f_i)$ and let $\phi_i : f \to h$ be the imbedding that for $j \neq i$ sends f_j on the first corresponding occurence in h while it sends f_i on the second.

We say that f_i is more important than f_j if

$$(c;\phi_i;h) > (c;\phi_j;h).$$

Note that it is the indexed occurence of f_i that is more important than the ditto of f_j.

As in the dilator-case we will slow down a connected Ptyx by laying restrictions on its most important part.

4.4 Definition

a) Let $P \in Pt(k)$ be connected, $P \neq \underline{1}$. Let $h_0 \in Pt(k-1)$. Let

$P^{h_0}(h_1)$

be the order-type of the subset of $P(h_0+h_1)$ given by the denotations

$(c;\phi+\psi;h_0+h_1)$

where (c,f) is a bone for P, i is the most important index,

$$\phi : \sum_{j<i} f_j \to h_0$$

$$\psi : \sum_{i<j<n} f_j \to h_1$$

b) If $P \in Pt(k)$ and $P = \sum_{i<\beta} P_i$ where $\beta > 1$ and each P_i is connected, then each P_i is a <u>component</u> of P.

c) If $P \in Pt(k)$ is connected and $h \in Pt(k-1)$ then P^h is a <u>component</u> of P if $P^h \neq \underline{0}$.

d) A component of a component of P is itself a component of P.

4.5 <u>Theorem</u> (Girard, unpublished)

The decomposition tree of P, i.e. the tree of sequences $(P_1,\ldots,P_n)$ where $P_1 = P$ and each P_{i+1} is a component of P_i, is well-founded.

The proof is an elaboration of the proof in the dilator case, and is based on a sequence of lemmas leading up to that result.

We would not gain much if the decomposition trees of a Ptyx could be dominated by that of a dilator. Our next task will be to show that this is not the case.

We first define a family of projections $\pi_k : Pt(k) \to Pt(k-1)$ and inverses $\nu_k : Pt(k-1) \to Pt(k)$:

4.6 <u>Definition</u>

a) Let $\pi_1(D)$ be the collapse of a dilator D to a well-ordering. Let $\nu_1(\alpha)$ be the constant α dilator.

b) Assume that π_k and ν_k are defined for some $k \geq 1$. Let

$$\pi_{k+1}(P)(h) = P(\nu_k(h))$$

where $P \in Pt(k+1)$, $h \in Pt(k-1)$. Let $\nu_k(D)(E) = D(\pi_k(E))$. These maps are extended to functors in the canonical way.

4.7 Lemma

If $D \in Pt(k)$ and $k \geq 0$ then

$$\pi_{k+1}(\nu_{k+1}(D)) = D$$

Proof

Use induction on k. The induction start is obvious and the induction step is standard.

4.8 Remark

Observe that π_k and ν_k will commute with sums, with pull-backs and with direct limits.

4.9 Definition

let $D \in Pt(k)$. Let

$$P_D(E) = D(\pi_k(E))$$

4.10 Lemma

$P_D \in Pt(k+1)$ and the decomposition tree of D can be imbedded into the decomposition -tree of P_D.

Proof

We use induction on the decomposition of D.

1. If $D = \underline{1}_k$ then $P_D = \underline{1}_{k+1}$ and the decomposition trees are isomorphic.
2. If $D = \sum_{i<\alpha} D_i$ then $P_D = \sum_{i<\alpha} P_{D_i}$ and the induction is trivial.
3. Let $D \neq \underline{1}_k$ be connected, let $h \in Pt(k-1)$. Let $h_1 = \nu_k(h)$.

   We will show that $P_D h$ can be imbedded into $P_D^{h_1}$, and by the induction-hypothesis the lemma will follow.

Let us consider

$P_D h(E) = D^h(\pi_k(E))$.

The value is a subset of

$D(h+\pi_k(E))$

where the most important part of each denotation is taken from h, the rest from $\pi_k(E)$.

Now $P_D^{h_1}(E) \subseteq P_D(h_1+E)$ where the most important part of the denotation is taken from h_1 and the rest from E. Since

$$P_D(h_1+E) = D(\pi_k(h_1+E)) = D(h+\pi_k(E)),$$

to each denotation for P_D we find a denotation for D using π_k, so ν_k gives us the desired imbbedding.

4.11 <u>Theorem</u>

There is an element $P \in Pt(k+1)$ that is connected and such that P_D is isomorphic to P^D for all $D \in Pt(k)$.

<u>Proof</u>

We will describe the denotation-system for P_D. Each bone (c,f) will be on the form (c,f_1+f_2) where f_1 is any singleton (d,g) that may serve as a possible bone for some element in $Pt(k-1)$, and f_2 is minimal such that $g \subseteq \pi_k(f_2)$.

To be more precise, for each $g \in U_D \cap Pt(k-1)$ take any $f_2 \in Pt(k)$ such that $g \subseteq \pi_k(f_2)$ while for no proper subfunction $f_3 \subset f_2$ we have $g \subseteq \pi_k(f_3)$. Let f_1 be any element of $Pt(k)$ with exactly one bone of the form (d,g). Let $f \in U_D$ be isomorphic to $f_1 + f_2$.

For each such choice of f, let (c,f) be a bone for P. So far c can be anything, its canonical value will be determined when we have described the ordering between denotations.

Let $(c_1; \phi_1+\phi_2; h)$ and $(c_2; \phi_3+\phi_4; h)$ be two denotations based on

$$(c_1, f_1+f_2), g_2 \quad \text{and} \quad (c_2, f_3+f_4), g_4$$

as above.

If ϕ_1 and ϕ_3 sends f_1 and f_3 into different addends of h, then we order the denotations by the order of the addends (This makes f_1 resp. f_3 to the most important parts). Now assume that h_1 is an addend of h and essentially $\phi_1: f_1 \to h_1$, $\phi_3: f_3 \to h_1$. Let h_2 be the part of h that is above h_1. The maps ϕ_1 and ϕ_3 give us two h_1-bones (d_1, g_1) and (d_3, g_3). Let

$$\psi_2 = \pi_k(\phi_2)\restriction g_1 \quad , \quad \psi_4 = \pi_k(\phi_4)\restriction g_3$$

Then

$$(d_1; \psi_2; \pi_k(h_2)) \quad \text{and} \quad (d_2: \psi_4; \pi_k(h_2))$$

are two $h_1(\pi_k(h_2))$-denotations. We order $(c_1: \phi_1+\phi_2; h)$ and $(c_2: \phi_3+\phi_4; h)$ by the value of these denotations.

It is now easy to see that P_D and P^D will be isomorphic for all D.

4.12 Corollary

For each k there is a recursive ptyx P of type $k+1$ such that the decomposition tree of D can be imbedded into the decomposition tree of P for all $D \in Pt(k)$.

5. The functional recursion scheme

One important aspect of the decomposition of a dilator is the functorial recursion one may define over it. One problem is to arrange the definitions in such away that the result is a functor. The Λ-operator of Girard [1] is an example of a successful inductive definition.

Let us make a crude atempt to generalize Λ. We will define $\Lambda(P)$ to be an operator from $Pt(k-1)$ to $Pt(k-1)$, where $P \in Pt(k)$, and we will use induction on P:

$\Lambda(\underline{1}_k)(h) = h+h \qquad$ for $h \in Pt(k-1)$

$\Lambda(P+Q)(h) = \Lambda(P)(\Lambda(Q)(h))$

if Q is connected

$\Lambda(\sum_{i<\beta} P_i)(h) = \sum_{i<\beta} \Lambda(\sum_{j<i} (P_j))(h)$

if β is a limit ordinal and each P_i are connected.

$\Lambda(P)(h) = \Lambda(P^{\Lambda(P^h)(h)})(\Lambda(P^h)(h))$

If $P \neq \underline{1}_k$ is connected.

This is of course nonsense, this Λ is not going to be functorial, and the recursive definition will break down because if wont't make any sense.

The problem is the equalities, but for most applications it will be satisfactory just to find a Λ such that the righthandside can be <u>imbedded</u> in the lefthandside, and we will show that there is indeed a recursive Λ satisfying this. The method is quite general and will be called the <u>functorial recursion scheme</u>

The construction is based on the following result from Girard-Normann[4]:

5.1 <u>Proposition</u>

Let k,n be given. Let F be a partial set-recursive function. Then uniformly recursive in an index for F there is a functor P commuting to pullbacks and direct limits such that for all $D \in Pt(k)$, if $F(E) \in Pt(n)$ for all E that can be imbedded into D then $P(D) \in Pt(n)$ and $F(D)$ can be imbedded into $P(D)$. This will also hold if we replace $Pt(k)$ with a mixed type of ptykes.

Now assume that we have a partial recursive functor Λ. Uniformly in the index for Λ we define the set-recursive function $\Phi_\Lambda: Pt(k)\times Pt(k-1) \to Pt(k-1)$ by

$$\Phi_\Lambda(P,h) = h+h \quad \text{if } P = \underline{1}_k$$

$$\Phi_\Lambda(P,h) = \Lambda(P_1)(\Lambda(P_2)(h))$$

if $P_2 \neq \underline{0}_k$ is connected.

If $P = \sum_{i<\beta} P_i$, β is a limit ordinal

and each P_i are connected, then

$$\Phi_\Lambda(P,h) = \sum_{i<\beta} \Lambda\ (\sum_{<i} (P_j))(h)$$

If $P \neq \underline{1}_k$ is connected, let

$$\Phi_\Lambda(P,h) = \Lambda(P^{\Lambda(P^h)(h)})(\Lambda(P^h)(h)).$$

We then use proposition 5.1 to find a functor $\Psi_\Lambda(P)$ such that

$$\lambda h \Phi_\Lambda(P,h) \text{ can be imbedded into } \Psi_\Lambda(P).$$

By the recursion theorem there will be an index e for a partial operator Λ such that $\Psi_\Lambda = \Lambda$. It is then not difficult to see by induction on P that Λ will be defined everywhere and Λ will be functorial since Ψ_Λ is functorial.

5.2 Remark

We will not state the functorial recursion scheme as a precise result since we have not found a good optimal formulation. Any combination of the recursion theorem and a functorial bounding prinsiple in order to bound an operator recursively defined over the decomposition of a ptyx will be an instance of the scheme.

We will end this paper by showing that the decomposition of ptykes is optimal in a certain sense.

As a trivial observation we see

5.3 Lemma

Let P be recursive Ptyx of type $k \geq 1$. Then the decomposition tree of P restricted to countable elements of $Pt(k-1)$ can be realized as a well-founded Π^1_k-relation

Proof

The decomposition is Δ^1_1 and the restriction to elements of $Pt(k-1)$ is Π^1_k.

We will show that any Π^1_k-well-founded relation can be imbedded into the decomposition-tree of a ptyx. We need the following.

5.4 Lemma

Let $K: Pt(k-1) \to Pt(k)$ be functorial. Then there is a connected functor $P \in Pt(k)$ such that for all $h \in Pt(k-1)$

$$K(h) \text{ can be imbedded into } P^{1+h}$$

The proof is simple and is left for the reader.

5.5 Theorem

Let $\prec$ be a well-founded Π^1_k-relation. Then there is a recursive $P \in Pt(k)$ and an order preserving map $\phi : dom(\prec) \to$ countable part of the decomposition tree of P.

Proof

We replace $\prec$ by its tree T of descending sequences.

Uniformly recursive in each node σ in T we will define P_σ such that the tree T_σ of nodes below σ can be mapped into the decomposition-tree of P_σ.

The definition will be by induction on σ and we will use the recursion theorem to tie the whole definition together.

Without loss of generality we may assume

1) The domain of $\prec$ is Π^1_k

2) There is a fixed recursive least element a_0 in $\prec$.

We are then ready to give the definition: If τ ends with a_0 we let $P_\tau = \underline{1}_k$.

Now let $\sigma \in T$. $T_\sigma = \{\tau \mid \tau \in T \wedge \tau \text{ extends } \sigma\}$ is Π^1_k. Thus there is a continuous function F_σ uniformly recursive in σ such that

$$F_\sigma(\tau) \in Pt(k-1) \iff \tau \in T_\sigma$$

By a version of the functorical bounding theorem from Girard-Normann [4], there is a functor $Q_\sigma : Pt(k-1) \to Pt(k)$ such that for all τ and $h \in P(k-1)$:

If $F_\sigma(\tau)$ can be imbedded into h

then P_τ can be imbedded into $Q_\sigma(h)$.

Let P_σ be such that $Q_\sigma(h)$ can be imbedded into P_σ^{1+h}, by lemma 5.4. Then T_σ can be imbedded into the decomposition tree of P_σ.

Finally let $P = P_{\langle\rangle}$. Then $\prec$ can be mapped into the decomposition-tree of P.

5.6 Remark

When we reduce a Π^1_k-set to Pt(k-1) we can make the reduction 1-1 by coding the real into the ptyx (k> 2) and then this proof relly gives an imbedding.

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