

A note on the solution of the two-dimensional
slip-boundary layer problem

by

Arnold F. Bertelsen

Abstract

The general homogeneous solution of the differential equation associated with an arbitrary term of the Blasius series expansion of the stream function, is given. This result is used to establish the solution of higher order terms of the Blasius series, than earlier published. A numerical example is also given.

I. Introduction and formulation of the problem

The two-dimensional flow induced in a viscous fluid (kinematic viscosity ν) around a circular cylinder (radius a) by the following slip velocity on the cylinder surface,

$$(1) \quad v(a, \theta) = V_0 c_0^2 \left[\theta + \sum_{N=1}^{\infty} a_N \theta^{2N+1} \right]$$

is considered (r, θ denote two-dimensional polar coordinates V_0 characteristic velocity).

The velocity field (u, v) is related to the stream function Ψ by,

$$(2) \quad (u, v) = \left(-\frac{1}{r} \frac{\partial \Psi}{\partial \theta}, \frac{\partial \Psi}{\partial r} \right)$$

Introducing the following dimensionless quantities,

$$(3) \quad \begin{cases} \zeta = c_0 \sqrt{R} \frac{r-a}{a} = c_0 \sqrt{R} \frac{y}{a} \\ \Psi = \frac{V_0 a}{\sqrt{R}} c_0 \psi(\zeta, \theta; R), \quad R = \frac{V_0 a}{\nu} \end{cases}$$

into the vorticity equation, it is well known that the following asymptotic expansion of $\psi(\zeta, \theta; R)$ can be carried out,

$$(4) \quad \psi(\zeta, \theta; R) \underset{\substack{\zeta \text{ fixed} \\ R \rightarrow \infty}}{\sim} \psi_0(\zeta, \theta) + \frac{1}{\sqrt{R}} \psi_1(\zeta, \theta) + \dots$$

This expansion leads to after partial integration and some manipulations are carried out,

$$(5) \quad \frac{\partial^3 \psi_0}{\partial \zeta^3} = -\frac{\partial \psi_0}{\partial \theta} \frac{\partial^2 \psi_0}{\partial \zeta^2} + \frac{\partial \psi_0}{\partial \zeta} \frac{\partial^2 \psi_0}{\partial \zeta \partial \theta}$$

$$(6) \quad \frac{\partial^3 \psi_1}{\partial \zeta^3} = -\frac{\partial \psi_0}{\partial \theta} \frac{\partial^2 \psi_1}{\partial \zeta^2} + \frac{\partial \psi_0}{\partial \zeta} \frac{\partial^2 \psi_1}{\partial \zeta \partial \theta} - \frac{\partial \psi_1}{\partial \theta} \frac{\partial^2 \psi_0}{\partial \zeta^2} + \frac{\partial \psi_1}{\partial \zeta} \frac{\partial^2 \psi_0}{\partial \zeta \partial \theta} - \frac{\partial^3 \psi_0}{\partial \zeta^3}$$

(higher order terms of ψ are not considered) with the boundary conditions,

$$(7) \quad \begin{cases} \psi_0(0, \theta) = 0 \\ \left[\frac{\partial \psi_0}{\partial \zeta} \right]_{\zeta=0} = \theta + \sum_{N=1}^{\infty} a_N \theta^{2N+1} \\ \left[\frac{\partial \psi_0}{\partial \zeta} \right]_{\zeta=\infty} = 0 \end{cases}$$

$$(8) \quad \begin{cases} \psi_1(0, \theta) = 0 \\ \left[\frac{\partial \psi_1}{\partial \zeta} \right]_{\zeta=0} = 0 \\ \left[\frac{\partial \psi_1}{\partial \zeta} \right]_{\zeta=\infty} = \sum_{N=0}^{\infty} b_N \theta^{2N+1} \end{cases},$$

where of course the flow outside the boundary layer must be calculated before $\{b_N\}$ is known ($N = 0, 1, 2, \dots$).

The problem stated by (5, 6, 7 & 8) appears for example in connection with oscillatory boundary layers where the time averaged Reynolds stresses induce the slip velocity (1) on the cylinder (see Stuart 1966). Riley (1965) studied the problem and solved the three first terms in the Blasius series expansions,

$$(9) \quad \psi_0(\zeta, \theta) = \psi_{0,0}(\zeta)\theta + \sum_{N=1}^{\infty} a_N \psi_{0,N}(\zeta) \theta^{2N+1}$$

We attempt a similar expansion of $\psi_1(\zeta, \theta)$,

$$(10) \quad \psi_1(\zeta, \theta) = \sum_{N=0}^{\infty} b_N \psi_{1,N}(\zeta) \theta^{2N+1}$$

These expansions give the following equations

$$(11) \quad \psi_{0,0}'''' + \psi_{0,0} \psi_{0,0}'' - \psi_{0,0}'^2 = 0$$

$$(12) \quad L_N \psi_{0,N} = \begin{cases} 0, & N=1 \\ \frac{1}{a_N} \sum_{k=1}^{N-1} a_k a_{N-k} (2k+1) \{-\psi_{0,k} \psi_{0,N-k}'' + \psi_{0,k}' \psi_{0,N-k}'\}, & N=2,3,4,\dots \end{cases}$$

$$(13) \quad L_N \psi_{1,N} = \begin{cases} -\frac{\zeta}{b_0} \psi_{0,0}''', & N=0 \\ \frac{1}{b_N} \sum_{k=1}^N a_k b_{N-k} \{- (2k+1) \psi_{0,k} \psi_{1,N-k}'' + 2(N+1) \psi_{0,k}' \psi_{0,N-k}' \\ - [2(N-k)+1] \psi_{0,k}'' \psi_{1,N-k}\} - \frac{a_N}{b_N} \zeta \psi_{0,N}''', & N=1,2,3,\dots \end{cases}$$

$$(14) \quad L_N \equiv \frac{d^3}{d\zeta^3} + (1-e^{-\zeta}) \frac{d^2}{d\zeta^2} - 2(N+1)e^{-\zeta} \frac{d}{d\zeta} - (2N+1)e^{-\zeta}$$

where the solution $\psi_{0,0} = 1 - e^{-\zeta}$ (Riley 1965, eq. 27, but deviant notations) has been used to obtain (12), (13) and (14). The boundary conditions are,

$$(15) \quad \begin{cases} \psi_{0,N}(0) = 0 \\ [\psi_{0,N}]_{\zeta=0} = 1 \\ [\psi_{0,N}]_{\zeta=\infty} = 0 \end{cases}$$

$$(16) \quad \begin{cases} \psi_{1,N}(0) = 0 \\ [\psi_{1,N}]_{\zeta=0} = 0 \\ [\psi_{1,N}]_{\zeta=\infty} = 1 \end{cases}$$

Inspection of equations (12) and (13) reveals that $\psi_{0,N}$ and $\psi_{1,N}$ have identical general homogeneous solutions for $N \geq 1$.

II. Solution

An important step in achieving the general solution of an arbitrary term of the Blasius series is to establish the general homogeneous solution of the equation concerned. The general homogeneous solution $\psi_{0,N}^{(H)}(\zeta)$ of $\psi_{0,N}(\zeta)$ can be constructed by superposition of terms $a_{m,n} \zeta^m e^{-n\zeta}$. Some details of the calculations determining $\{a_{m,n}\}$ are given in appendix A. The results of these calculations are,

$$\begin{aligned}
 \psi_{0,N}^{(H)}(\zeta) &= A_N e^{-\zeta} + B_N [1 + (2N+1)\zeta e^{-\zeta} \\
 &\quad - \sum_{n=2}^{2N+1} \frac{(2N+1)2N(2N-1)\cdots(2N-n+2)}{(n!)^2(n-1)} e^{-n\zeta}] \\
 (17) \quad &+ C_N [-(6N+4) + \zeta + \frac{2N+1}{2} \zeta^2 e^{-\zeta} \\
 &\quad + \sum_{n=2}^{2N+1} (K_n - \frac{(2N+1)2N(2N-1)\cdots(2N-n+2)}{(n!)^2(n-1)} \zeta) e^{-n\zeta} \\
 &\quad + (2N+1)((2N)!) \sum_{k=2}^{\infty} \frac{(-1)^{k-1} (k-2)!}{((2N+k)!)^2(2N+k-1)} e^{-(2N+k)\zeta}]
 \end{aligned}$$

where

$$K_2 = -\frac{(2N+1)(4N+1)}{4}$$

and for $n \geq 3$

$$\begin{aligned}
 K_n &= \frac{(2N-n+2)(n-2)}{n^3 - n^2} K_{n-1} \\
 &\quad + \frac{(2N+1)(2N)(2N-1)\cdots(2N-n+3)}{((n-1)!)^2(n^3-n^2)} \left[\frac{2(N-n+2)}{n-2} \right. \\
 &\quad \left. - \frac{n(3n-2)(2N-n+2)}{n^2(n-1)} \right]
 \end{aligned}$$

According to equation (12) and (13)

$$(18) \quad \psi_{0,N}^{(H)} = \psi_{1,N}^{(H)}$$

when $N \geq 1$. For $N = 0$ we find,

$$(19) \quad \psi_{1,0}^{(H)} = A_0 e^{-\zeta} + B_0 [1 + \zeta e^{-\zeta}] \\ + C_0 [-4 + \zeta + \frac{1}{2} \zeta^2 e^{-\zeta} + \sum_{k=2}^{\infty} (-1)^{k-1} \frac{(k-2)!}{(k!)^2 (k-1)} e^{-k\zeta}]$$

The particular solutions can now be found by variation of the parameters, but also by inspection matching the residual terms

$$(20) \quad R_{m,n}(\zeta; N) = L_N(a_{m,n} \zeta^m e^{-n\zeta})$$

to the inhomogeneity terms of (12) and (13) by choosing special values of $\{a_{m,n}\}$.

Applications

The results obtained above are now used to establish the fourth term $\psi_{0,3}$ of the Blasius series (9) subject to conditions (15). In this context we need to quote three first terms given by Riley (1965) equations (27), (28), (29) and (31), respectively, which in our notations can be written

$$(21) \quad \psi_{0,0} = 1 - e^{-\zeta}$$

$$(22) \quad \psi_{0,1} = \frac{1}{68} [12 + (7 + 36\zeta)e^{-\zeta} - 18e^{-2\zeta} - e^{-3\zeta}]$$

$$(23) \quad \psi_{0,2} = f_2(\zeta) + \frac{3a_1^2}{a_2} g_2(\zeta)$$

where

$$(24) \quad f_2(\zeta) = \frac{1}{18552} [1440 + (7063 + 7200\zeta)e^{-\zeta} \\ - 7200e^{-2\zeta} - 1200e^{-3\zeta} - 1200e^{-3\zeta} - 100e^{-4\zeta} - 3e^{-5\zeta}]$$

$$(25) \quad g_2(\zeta) = \frac{1}{53615280} [99216 - (5392720 + 477900\zeta + 2504520\zeta^2)e^{-\zeta} \\ + (5486940 + 5009040\zeta)e^{-2\zeta} - (140655 - 417420\zeta)e^{-3\zeta} \\ - 53270e^{-4\zeta} + 489e^{-5\zeta}]$$

As usual, $\psi_{0,3}$ is written as a sum of functions f_3 , g_3 and h_3 which are independent of $\{a_N\}$, i.e.,

$$(26) \quad \psi_{0,3} = f_3(\zeta) + \frac{a_1 a_2}{a_3} g_3(\zeta) + \frac{3a_1^3}{a_3} h_3(\zeta)$$

which give,

$$(27) \quad \left\{ \begin{array}{l} L_3 f_3(\zeta) = 0 \\ L_3 g_3(\zeta) = -5\psi''_{0,1} f_2 + 8\psi'_{0,1} f_2' - 3\psi_{0,1} f_2'' \\ L_3 h_3(\zeta) = -5\psi''_{0,1} g_2 + 8\psi'_{0,1} g_2' - 3\psi_{0,1} g_2'' \\ [f_3'(\zeta)]_{\zeta=0} = 1, \quad f_3(0) = [f_3'(\zeta)]_{\zeta=\infty} = 0 \\ [g_3'(\zeta)]_{\zeta=0} = g_3(0) = [g_3'(\zeta)]_{\zeta=\infty} = 0 \\ [h_3'(\zeta)]_{\zeta=0} = h_3(0) = [h_3'(\zeta)]_{\zeta=\infty} = 0 \end{array} \right.$$

The solutions are,

$$(28) \quad f_3(\zeta) = \frac{1}{3927660} [151200 + (1957814 + 1058400\zeta)e^{-\zeta} \\ - 1587600e^{-2\zeta} - 441000e^{-3\zeta} - 73500e^{-4\zeta} - 6615e^{-5\zeta} \\ - 294e^{-6\zeta} - 5e^{-7\zeta}] ,$$

$$(29) \quad g_3(\zeta) = \psi_{0,3}^{(g)}(\zeta) + \frac{1}{210256} [7254 - 43200\zeta^2 e^{-\zeta} + (112611 + 129600\zeta)e^{-2\zeta} \\ + (-7426 + 25200\zeta)e^{-3\zeta} + (-\frac{46315}{12} + 2400\zeta)e^{-4\zeta} \\ + (-\frac{30299}{80} + 90\zeta)e^{-5\zeta} - \frac{2581}{200} e^{-6\zeta} + \frac{39}{560} e^{-7\zeta}]$$

$$\begin{aligned}
 (30) \quad h_3(\zeta) = & \psi_{0,3}^{(h)}(\zeta) + \frac{1}{91145976} \left\{ -\frac{4766634}{7} + (-8181\zeta^2 + 751356\zeta^3) e^{-\zeta} \right. \\
 & - \left(\frac{19164177}{4} + 9430020\zeta + 4508136\zeta^2 \right) e^{-2\zeta} \\
 & + \left(\frac{23985321}{8} + \frac{833949}{2}\zeta - 563517\zeta^2 \right) e^{-3\zeta} \\
 & + \left(\frac{1690601}{4} + 191772\zeta \right) e^{-4\zeta} \\
 & - \left(\frac{14325}{400} + \frac{4401}{2}\zeta \right) e^{-5\zeta} \\
 & \left. + \frac{10074}{40} e^{-6\zeta} - \frac{2301}{112} e^{-7\zeta} \right\}
 \end{aligned}$$

where,

$$\psi_{0,3}^{(g)}(\zeta) = \psi_{0,3}^{(H)}(\zeta; A_3; B_3; C_3)$$

with

$$A_3 \approx -0.704077008, \quad B_3 \approx -0.96801791 \times 10^{-7}, \quad C_3 = 0;$$

and,

$$\psi_{0,3}^{(h)}(\zeta) = \psi_{0,3}^{(H)}(\zeta; A_3; B_3; C_3)$$

with

$$A_3 \approx 0.0880122017, \quad B_3 \approx 0.334608072 \times 10^{-7}, \quad C_3 = 0.$$

($\psi_{0,3}^{(H)}$ is given by equation 17.)

The first term of the second order approximation ψ_1 is treated in the same way, i.e.,

$$(31) \quad \psi_{1,0}(\zeta) = p_0(\zeta) + \frac{1}{b_0} q_0(\zeta)$$

giving,

$$(32) \quad \left\{ \begin{array}{l} L_0 p_0(\zeta) = 0 \\ L_0 q_0(\zeta) = \zeta e^{-\zeta} \\ p_0(0) = [p_0'(\zeta)]_{\zeta=0} = 0 \\ [p_0'(\zeta)]_{\zeta=0} = 1 \\ q_0(0) = [q_0'(\zeta)]_{\zeta=0} = 0 \\ [q_0'(\zeta)]_{\zeta=\infty} = 0 \end{array} \right.$$

The solutions are,

$$(33) \quad p_0(\zeta) = \psi_{1,0}^{(h)}(\zeta; A_0; B_0; C_0)$$

with

$$A_0 = \frac{5}{2} - \frac{1}{2} \sum_{k=2}^{\infty} (-1)^{k-1} \frac{1}{k!(k-1)^2}$$

$$B_0 = \frac{3}{2} + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^{k-1} \frac{(k-2)!}{(k!)^2}$$

$$C_0 = 1,$$

and,

$$(34) \quad q_0(\zeta) = \psi_{1,0}^{(H)}(\zeta; A_0; B_0; C_0) - 2 - \zeta$$

with

$$A_0 = 3 + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^k \frac{(k-2)!(k+1)}{(k!)^2 (k-1)}$$

$$B_0 = 3 + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^{k-1} \frac{(k-2)!}{(k!)^2}$$

$$C_0 = 1$$

where $\psi_{1,0}^{(H)}(\zeta; A_0; B_0; C_0)$ is given by equation (19).

Numerical examples

The slip velocity induced by the time averaged Reynolds stresses in the Stokes layer at a long circular cylinder placed orthogonal to a oscillatory flow field, generates a steady slip boundary layer outside the Stokes layer where (see Riley 1975 equation (9), note deviant notations),

$$\left[\frac{\partial \psi_0}{\partial \zeta} \right]_{\zeta=0} = \frac{3}{2} \sin 2\theta$$

which give,

$$c_0 = \sqrt{3} \quad , \quad a_0 = 1 \quad , \quad a_1 = -\frac{2}{3} \quad , \quad a_2 = \frac{2}{15} \quad , \quad a_3 = -\frac{8}{945} \quad .$$

An approximate expression of the dimensionless momentumflux in the slip boundary layer is,

$$M(\theta) = c_0^2 \int_0^\infty \left\{ \sum_{N=0}^3 [a_N \psi'_{0,N}(\zeta) \theta^{2N+1}] \right\}^2 d\zeta$$

which give,

$$M\left(\frac{\pi}{2}\right) \approx 0.975$$

This is unexpected close to a result given by Riley (1975, pp 807) based on numerical integration which gave

$$M\left(\frac{\pi}{2}\right) \approx 0.991$$

In figure 2 the dimensionless tangential velocity,

$$V = c_0^2 \sum_{N=0}^{N_1} a_N \frac{\partial \psi_{0,N}}{\partial \zeta} \theta^{2N+1}$$

for $N_1 = 2$ (Riley 1965) and $N_1 = 3$.

This figure indicate a three term Blasius series to give the tangential velocity with resonable accuracy for

$$|\theta| \leq \frac{\pi}{3}$$

while a four term series seems to be applicable for

$$|\theta| \leq \frac{5}{12}\pi$$

Appendix A

The general residual term of a test solution $a_{m,n}\zeta^m e^{-n\zeta}$ is,

$$\begin{aligned}
 (A1) \quad R_{m,n}(\zeta;N) &= L_N\{a_{m,n}\zeta^m e^{-n\zeta}\} \\
 &= a_{m,n} \{ [m(m-1)(m-2)\zeta^{m-3} + m(m-1)(1-3n)\zeta^{m-2} \\
 &\quad + mn(3n-2)\zeta^{m-1} + (n^2-n^3)\zeta^m] e^{-n\zeta} \\
 &\quad + [-m(m-1)\zeta^{m-2} + 2m(n-1-2N)\zeta^{m-1} \\
 &\quad + (-n+1+2N)(n-1)\zeta^m] e^{-(n+1)\zeta} \} \\
 &\equiv a_{m,n} \{ P_{m,n}(\zeta;N) e^{-n\zeta} + Q_{m,n}(\zeta;N) e^{-(n+1)\zeta} \}
 \end{aligned}$$

with the following properties,

$$(A2) \quad R_{0,1}(\zeta;N) = 0$$

$$(A3) \quad R_{0,2N+1}(\zeta;N) = -2a_{0,2N+1}N(2N+1)^2 e^{-(2N+1)\zeta}$$

$$(A4) \quad R_{1,2N+1}(\zeta;N) = a_{1,2N+1} [(2N+1)(6N+1) - 2N(2N+1)^2 \zeta] e^{-(2N+1)\zeta}$$

Equation (A2) indicates that $e^{-\zeta}$ is a homogeneous solution for every N . The construction of the other homogeneous solutions consists of choosing numerical values of $a_{m,n}$ such that

$$\sum_m [a_{m,n+1} P_{m,n+1}(\zeta;N) + a_{m,n} Q_{m,n}(\zeta;N)] = 0$$

for every n . The simplest expressions are obtained when the properties (A3) and (A4) are utilized.

List of references

Riley N. (1965) *Mathematika* 12, 161.

Riley N. (1975) *J. Fluid Mech.* 68, 801.

Stuart J.T. (1966) *J. Fluid Mech.* 24, 673.

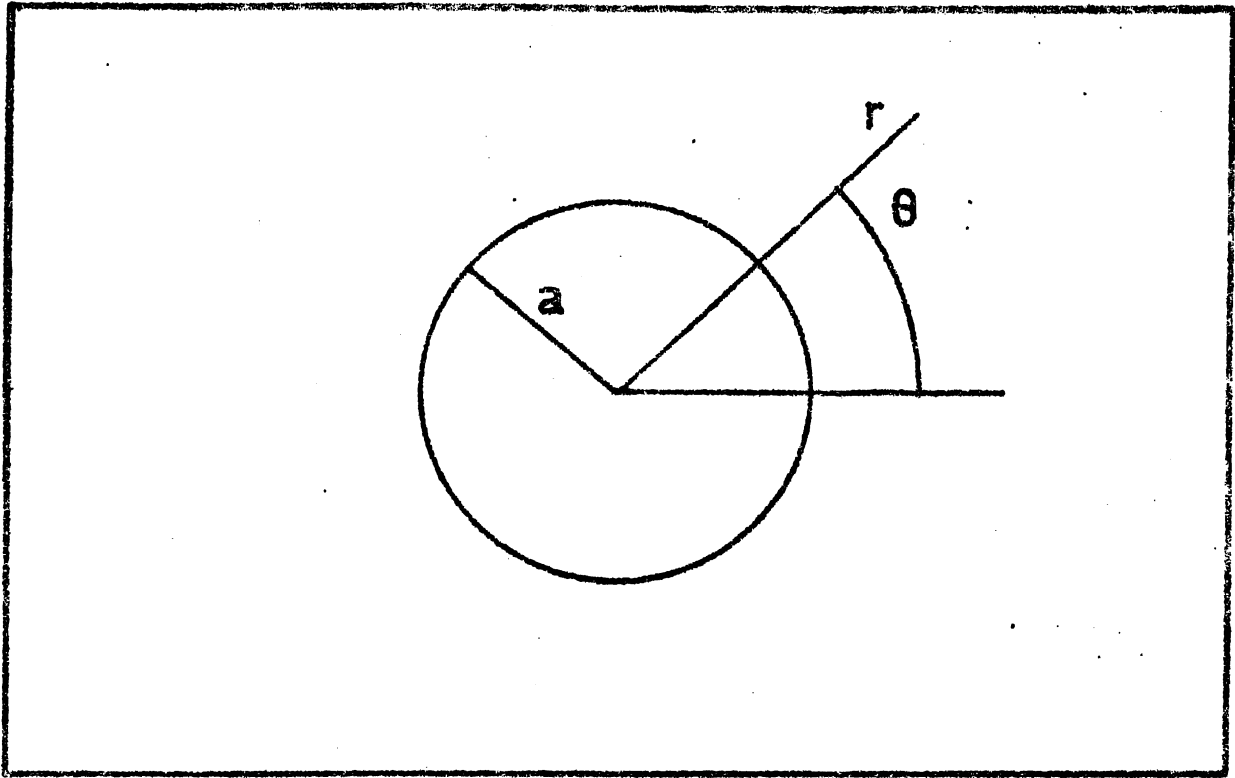


Figure 1. The polar coordinate system (r, θ) referred to in the paper.

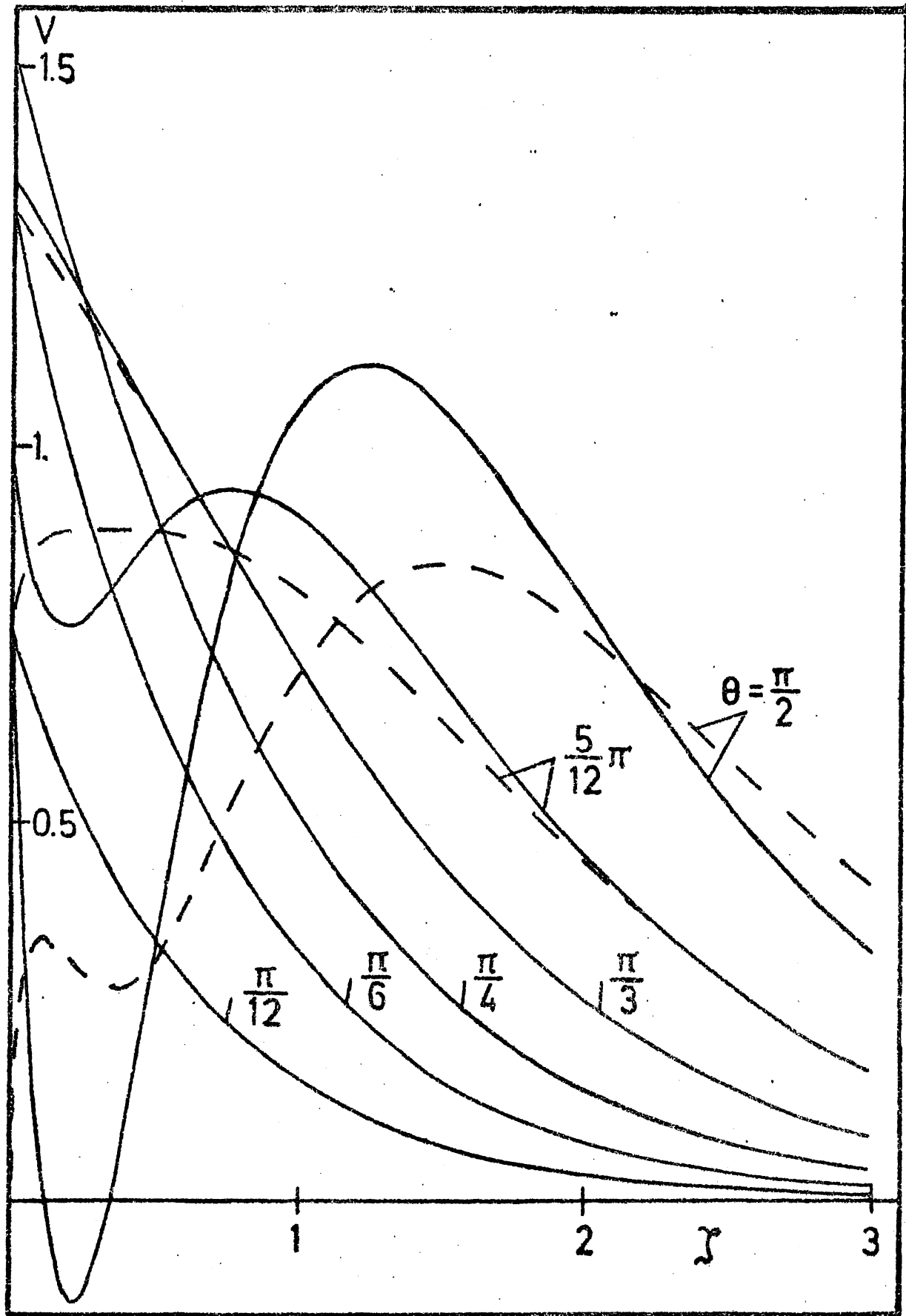


Figure 2. Dimensionless tangential velocity distribution at various angular positions. Full and dashed curves based on three and four terms of the Blasius series, respectively.