A NOTE ON HIGHER ORDER DUAL VARIETIES,

WITH AN APPLICATION TO SCROLLS

by

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1. Introduction.

The dual variety $X^{\vee} \subset \mathbb{P}^{n\vee}$ of a variety $X \subset \mathbb{P}^n$ is the closure of the set of hyperplanes containing the tangent space to X at some smooth point. We define the m-dual variety $X_m^{\vee} \subset \mathbb{P}^{n\vee}$ of X as the closure of the set of hyperplanes containing an m-th osculating space to X, in particular, $X_1^{\vee} \approx X^{\vee}$. More generally, if $G = \mathrm{Grass}_{a+1}(V)$ denotes the Grassmann variety of a-spaces in $\mathbb{P}(V) \approx \mathbb{P}^n$, we can define m-th osculating spaces of a variety $X \subset G$, using the sheaves of principal parts, and hence its m-dual variety $X_m^{\vee} \subset \mathrm{Grass}_{a+1}(V^{\vee})$ as the closure of the set of (n-a-1)-spaces containing an m-th osculating space to X. This is of course closely related to Pohl's associated varieties ([Pohl]).

We show, in Prop. 1, a weak biduality result for m-duals: one always has $X \subset (X_m^{\vee})_m^{\vee}$, and equality holds under a dimension hypothesis, which is always satisfied in the classical case.

It is natural to ask for the degree of $X_m^{\vee}$ in terms of characters of X. Since we are working with "modified" bundles of principal parts, this can be done - at least in principle! - as in the classical case (see e.g. [P2], [U]).

Here we only deal explicitly with the case of a scroll (i.e., a ruled, non-developable surface) $X \subset \mathbb{P}(V)$, or, equivalently, a curve $C \subset \mathrm{Grass}_2(V)$. In general, a scroll has 2nd osculating spaces of dimension 4. We give a formula for $\deg X_2^{\vee}$.

In the case $\mathbb{P}(V) = \mathbb{P}^5$, we call $X^* = X_2^\vee$ the <u>strict dual</u> of X. If $\dim X^* = 2$, then X^* is also a scroll and $X^{**} = X$ holds. Moreover, the dual variety $X^\vee$ (the normal bundle of X) is equal to the osculating developable of X^*, and the dual plane of a tangent plane to X is the tangent plane to X^* at the corresponding point. Hence we get, for scrolls in $\mathbb{P}^5$, a complete parallell to the duality existing between a curve $C \subset \mathbb{P}^3$, its strict dual curve $C^* \subset \overset{\vee}{\mathbb{P}}^3$, and their developables (see e.g. [P1], § 5; [P3], Remark 1 on p. 111).

Scrolls are examples of surfaces with "too small" osculating spaces of higher order, hence are "of type Φ" in the terminology of Corrado Segre. I am grateful to Gianni Sacchiero for bringing these - in particular the scrolls and their strict duals - to my attention.

2. Higher order dual varieties.

Fix the following notations:

V is an $(n+1)$-dimensional vector space over an algebraically closed field k of characteristic 0, G is the Grassmann variety $\mathrm{Grass}_{a+1}(V)$ consisting of $(a+1)$-quotients of V (identified with a-dimensional linear subspaces of $\mathbb{P}(V)$), and $V_G \twoheadrightarrow Q$ is the universal $(a+1)$-quotient on G.

For each integer m there is a natural homomorphism

$$\alpha^m : V_G = H^0(G, Q)_G \rightarrow P_G^m(Q) ,$$

where $P_G^m(Q)$ denotes the bundle of principal parts of order m of Q (see [P1], § 6).

Let $X \subset G$ be a subvariety, of dimension r, and set $E = Q|_X$. The restriction of α^m, composed with the natural map $P_G^m(Q)|_X \to P_X^m(E)$, gives a homomorphism

$$\alpha^m : V_X \to P_X^m(E).$$

A point $x \in X$ is called m-regular if x is smooth and if $a^m(x)$ is surjective; if these points are dense in X, we say that X is generically m-regular. At each m-regular point $x \in X$ there is a well-defined m-th osculating space, of dimension $(a+1)\binom{r+m}{m} - 1$, defined by $a^m(x)$. Hence a generically m-regular X has an m-th associated variety $X^{(m)} \subset \text{Grass}_{(a+1)\binom{r+m}{m}}(V)$, defined as the closure of the set of the m-th osculating spaces (see [Pohl], § IV). We define the m-dual variety $X_m^\vee \subset G^\vee = \text{Grass}_{a+1}(V^\vee) = \text{Grass}_{n-a}(V)$ to be the closure of the set of $(n-a-1)$-spaces containing an m-th osculating space.

Even if X is nowhere m-regular, we can define m-th osculating spaces: let $U \subset X$ be an open dense smooth subscheme such that $K_U = \text{Ker}(a^m)|_U$ is a sub-bundle of V_U, or, equivalently, such that $\text{Im}(a^m)$ is a bundle. If $\text{Im}(a^m) = s+1$, then each point $x \in U$ has an m-th osculating space, of dimension s, defined by $a^m(x)$. The m-dual variety $X_m^\vee \subset G^\vee$ of X is the closure of the set of $(n-a-1)$-spaces containing the m-th osculating spaces. Let $\tilde{X} \subset G \times G^\vee$ be the closure of $\text{Grass}_{a+1}(K_U^\vee) \subset \text{Grass}_{a+1}(V_U^\vee) = U \times G^\vee$; then $X_m^\vee = \text{pr}_2(\tilde{X})$. Let $(X_m^\vee)^\sim \subset G \times G^\vee$ denote the corresponding variety constructed for $X_m^\vee$, so that $(X_m^\vee)_m^\vee = \text{pr}_1((X_m^\vee)^\sim)$.

The following proposition gives a weak biduality for m-dual varieties, generalizing the classical biduality for projective varieties (see [K]).

<u>Proposition 1</u>: If $X_m^\vee \neq \emptyset$, then $\widetilde{X} \subset (X_m^\vee)^\sim$ and $X \subset (X_m^\vee)_m^\vee$. In particular, if $\widetilde{X} = (X_m^\vee)^\sim$ (i.e., if $\dim \widetilde{X} = \dim(X_m^\vee)^\sim$), then $X = (X_m^\vee)_m^\vee$ holds.

<u>Remark</u>: In the classical case $(a = 0, \ m = 1)$, $\dim \widetilde{X} = \dim(X^\vee)^\sim = n-1$ always. An example where the equality does not hold: $X \subset \mathbb{P}^6$ a generically 2-regular surface contained in a hyperplane H. Then $X_2^\vee = \{H\} \in \mathbb{P}^6$ and $(X_2^\vee)_2^\vee = H$.

<u>Proof</u>: It suffices to show the inclusion $\widetilde{X} \subset (X_m^\vee)^\vee$ on an open dense of $\widetilde{X}$. Let $p : \widetilde{X} \to X$ and $q : \widetilde{X} \to X_m^\vee$ denote the projections. Consider a point $(x,y) \in \widetilde{X} \subset G \times G^\vee$ such that $x \in U$, y is in the corresponding $V \subset X_m^\vee$, and q is smooth at (x,y). Let F denote the restriction of the universal $(a+1)$-quotient of $V_{G^\vee}^\vee$ to $X_m^\vee$, and consider the following diagram (restricted to $p^{-1}U$):

$$
\begin{array}{ccccc}
 & 0 \to p^*K_U \to & V_{\widetilde{X}} & \to p^*P_X^m(E) & \searrow \\
 & \nearrow & \| & \downarrow & \\
q^*F^\vee & & V_{\widetilde{X}} & \to P_{\widetilde{X}}^m(p^*E) & \nearrow \quad p^*E \\
 & \searrow & \| & & \\
q^*P_{X_m^\vee}^m(F)^\vee & \to & V_{\widetilde{X}} & &
\end{array}
$$

To show that $(x,y) \in (X_m^\vee)^\vee$ amounts to showing that the composition $q^*P_{X_m^\vee}^m(F) \to p^*E$ is zero (locally at (x,y)). The map $q^*F^\vee \to p^*P_X^m(E)$, and hence also $q^*F^\vee \to P_{\widetilde{X}}^m(p^*E)$, is zero; since the composition $q^*F^\vee \to V_{\widetilde{X}} \to p^*E$ is zero, we obtain, by "differentiating" (i.e. applying the differential operators of order $\leq m$, corresponding to $P_{\widetilde{X}}^m$, to this composition), that $P_{\widetilde{X}}^m(q^*F)^\vee \to p^*E$ is zero. Since $q^*P_{X_m^\vee}^m(F) \to P_{\widetilde{X}}^m(q^*F)$ is locally split at (x,y), we

obtain that $q^* P^m_{X^V_m}(F) \to p^* E$ is zero at (x,y). (This is the same as the argument used for curves in $\mathbb{P}^n$, as in [P1], § 5.)

Suppose X is generically m-regular. Then $\mathrm{rk}\, K_U = n+1 - (a+1)\binom{r+m}{m}$. If $\mathrm{rk}\, K_U \geq a+1$, then $\tilde{X}$ is defined and has dimension $r + (a+1)(n-a-(a+1)\binom{r+m}{m})$. Set $r^V = \dim X^V_m$. If X^V_m is also generically m-regular, then biduality holds if and only if $r - (a+1)^2\binom{r+m}{m} = r^V - (a+1)^2\binom{r^V+m}{m}$.

This is possible only if $a = 0$ and $m = 1$ (the classical case), or if $r = r^V$. In fact, when $\mathrm{rk}\, K_U > a+1$, X^V_m is ruled, and hence should not be generically m-regular. Note that the surjections $P^i_X(E) \to P^{i-1}_X(E)$ give a sequence of inclusions

$$X^V = X^V_1 \supset X^V_2 \supset \ldots \supset X^V_m \supset \ldots \, ,$$

and that one could, instead of $\tilde{X}$, construct an $\tilde{X}_{1,2,\ldots}$ in the product of all the Grassmannians.

As in the case of classical duality, there is an invariance of m-duals under sections and projections: Suppose $W \subset V$ is a subspace, $\dim W \geq a+1$. For $X \subset G = \mathrm{Grass}_{a+1}(V)$, consider the projection

$$X \dashrightarrow \mathrm{Grass}_{a+1}(W) \quad \text{corresponding to} \quad W_G \to E.$$

If the center of projection $\mathbb{P}(V/W)$ is reasonable (i.e., if most of the a-spaces corresponding to points of X are projected to a-spaces in $\mathbb{P}(W)$), this map is rational, and we denote by $\bar{X}$ the closure of its image. From the functorial properties of the sheaves of principal parts, we get:

<u>Proposition 2</u>: The m-dual of a projection is the corresponding
section of the m-dual, i.e.,

$$X_m^V = X_m^V \cap \text{Grass}_{a+1}(W^V) \quad \text{holds.}$$

(The proof is similar to the one in the classical $(a = 0)$
case: See [P2] , p.269, and observe that the genericity assump-
tion made there is unnecessary.)

The <u>degree</u> of $X \subset G$ is its degree in $\mathbb{P}(\wedge^{a+1} V)$ via the Plücke
embedding. Thus we have $\deg X = c_1(E)^r \cap [X]$, and $\deg X_m^V =$
$c_1(F)^{r^V} \cap [X_m^V]$. Whenever we can express F (or q^*F) in terms
of known bundles, we get an expression for $\deg X_m^V$. When X is
generically m-regular, F is determined by $P_X^m(E)$ and the
singularities of a^m; hence we get, at least in principle, an
expression for $\deg X_m^V$ in terms of the degree of X and its Chern
classes (or rather, the Chern classes of a desingularization of X)
and the various singularities of X and a^m. The very simplest
case occurs when X is smooth and m-regular, $n-a = (a+1)\binom{r+m}{m}$
and $r^V = r$. Then $\deg X_m^V = c_1(q^*F)^r = c_1(K^V)^r = c_1(P_X^m(E))^r$.
(For formulas in the classical case, see [P2], [U]; see also [Pohl]
for associated varieties).

In the case of curves, formulas exist: Let $X \subset \mathbb{P}(V)$ be a
curve spanning $\mathbb{P}(V)$. Then X is generically m-regular, for
$m \leq n$, and we have associated curves $X^{(m)} \subset \text{Grass}_{m+1}(V)$ and
corresponding <u>osculating developables</u> $Y_m \subset \mathbb{P}(V)$. We also have
m-dual varieties $X_m^V \subset \mathbb{P}(V^V)$ - these are nothing but the osculating
developables Y_{n-m-1}^* of the <u>strict dual curve</u> $X^* = X^{(n-1)} \subset \mathbb{P}(V^V)$,
and they are also equal to the dual of the osculating developables
of X. More precisely, for each m we have

$$X_m^V = Y_{n-m-1}^* = (Y_{m-1})^V.$$

The first equality follows from the duality of certain exact sequences on X and X^* (see [P1], 5.2), the second holds because the tangent spaces to Y_{m-1} are the m-th osculating spaces to X. Thus we have formulas

$$\deg X_m^V = (m+1)(d + m(g-1)) - \sum_{i=0}^{m-1} (m-i)k_i ,$$

where $d = \deg X$, $g = $ (geometric) genus of X, and k_i is the i-th stationary index of X ([P1], 3.2).

3. Dual varieties of a scroll.

Let $X \subset G$ be as in the preceding section. If m is such that $\tilde{X} \to X$ is birational, i.e., if there is a uniquely determined m-th osculating $(n-a-1)$-space to X at x for most points $x \in X$, we shall call $X^* = X_m^V$ the strict dual variety of X.

For example, if $C \subset \mathbb{P}^n$ is a curve spanning $\mathbb{P}^n$, then $C^* = C_{n-1}^V$. If $X \subset \mathbb{P}^6$ is a surface which is generically 2-regular, then $X^* = X_2^V \subset \mathbb{P}^6$ is the strict dual.

An example of surfaces that are nowhere 2-regular (C. Segre called them "of type Φ"), are the ruled surfaces: scrolls, developables, and cones. The theory of duals of developables and cones reduces to that of curves in projective space; let us now look at the scrolls. By definition, a scroll $X \subset \mathbb{P}(V)$ is a ruled surface such that the tangent planes to X along a (general) generator are non-constant. Suppose $l \subset X$ is a generator, $x \in l$. The 2nd osculating space to X at x, defined by $a^2 : V_X \to P_X^2(1)$, is the space spanned by the tangent planes to X along l (this gives a $\mathbb{P}^3$) and the 2nd osculating space to

a curve on X at x. If X is not contained in a $\mathbb{P}^3$, one expect this space to be of dimension 4; if X is not contained in a $\mathbb{P}^4$, one expects these 4-spaces to vary along l, so that X^* has dimension 2. We shall now generalize to scrolls in $\mathbb{P}^5$ the duality results for curves in $\mathbb{P}^3$([P1], [P3]) : Let $C \subset \mathbb{P}^3 = \mathbb{P}(V)$ be a (non planar) curve, and let $C^* \subset \overset{\vee}{\mathbb{P}}^3$ denote its strict dual. The dual $C^\vee \subset \overset{\vee}{\mathbb{P}}^3$ is the normal bundle to C and the tangent developable of C^*, and similarly for $(C^*)^\vee$. Moreover, the dual line of a tangent line to C is the tangent line to C^* at the corresponding point - in other words, the associated curves $C^{(1)} \subset \mathrm{Grass}_2(V)$ and $C^{*(1)} \subset \mathrm{Grass}_2(V^\vee) = \mathrm{Grass}_2(V)$ are equal.

<u>Proposition 3</u>: Let $X \subset \mathbb{P}(V) = \mathbb{P}^5$ be a scroll which admits a strict dual $X^* = X_2^\vee$, and assume $\dim X^* = 2$. Then X^* is a scroll. The dual $X^\vee \subset \mathbb{P}(V^\vee)$, the normal bundle of X, is equal to the tangent developable of X^*, and vice versa. Moreover, the dual plane of a tangent plane to X is the tangent plane to X^* at the corresponding point - in other words, the associated varieties $X^{(1)} \subset \mathrm{Grass}_3(V)$ and $X^{*(1)} \subset \mathrm{Grass}_3(V^\vee)$ are equal.

<u>Proof</u>: Let X' be a modification of X and of X^* such that $\mathrm{Im}(a^1)$ and $\mathrm{Im}(a^1_*)$ admit quotient bundles P_1 and P_1^* of rank 3. Then $K = \ker(V_{X'} \to P_1)$ and $K^* = \ker(V_{X'}^\vee \to P_1^*)$ are bundles of rank 3, and the sequences $0 \to K \to V_{X'} \to P^1 \to 0$ and $0 \leftarrow P_1^* \leftarrow V_{X'}^\vee \leftarrow K^* \leftarrow 0$ are dual to each other: as in the proof of Prop. 1, one shows that (generically on X') the composition of $(a^1_*)^\vee : P_{X^*}^1(1)^\vee \to V_{X^*}$ with $a^1 : V_X \to P_X^1(1)$ is zero; since a^1_* and a^1 both have rank 3, the result follows. In particular

the existence of the exact sequence

$$0 \longrightarrow (P_1^*)^V \longrightarrow V_{X'} \longrightarrow P^1 \longrightarrow 0$$

shows that the tangent planes to X^* are the dual planes of the tangent planes to X; hence if X is a scroll, so is X^*. The other statements also follow directly from that exact sequence.

There is still another parallell to the curve case, namely to the fact that the strict dual curve $C^* \subset \overset{V}{P}{}^3$ is a <u>cuspidal</u> <u>edge</u> of the dual variety $C^V \subset \overset{V}{P}{}^3$ of a curve $C \subset P^3$.

<u>Proposition 4</u>: If $X \subset P^5$ is a scroll, then its strict dual X^* is a "cuspidal edge" of the dual variety $X^V \subset \overset{V}{P}{}^5$.

<u>Proof</u>: Assume X is smooth, and $\overline{X} \subset P^3$ a generic projection. Scrolls with ordinary singularities in P^3 are numerically self-dual, so $\overline{X}{}^V \subset \overset{V}{P}{}^3$ has a finite number of pinch points, corresponding to the pinch points of $\overline{X}$. If $L \subset P^5$ is the centre of projection, a pinch point of $\overline{X}$ occurs when L intersects a tangent, i.e., when L intersects the P^3 spanned by the tangent planes along a generator. But then L and this P^3 span a P^4 which is necessarily a point in X^* and also in $P^3 = L^V \subset \overset{V}{P}{}^5$. Since $\overline{X}{}^V = X^V \cap \overset{V}{P}{}^3$, it follows that the "ramified singularities" of X^V are precisely the points of X^*. If X is not assumed smooth, there might be other "cuspidal edges", as in the case of curves, where inflectionary tangents are cuspidal edges on the developable.

In order to compute the degree of X^*, it is convenient to consider X as a curve $C \subset G = \text{Grass}_2(V)$. Consider $a^1 : V_C \to P_C^1(E)$, where E is the restriction of the universal 2-quotient of V on G. The subspaces of V defined by a^1 can be interpreted by choosing, locally, a trivialization of E, corresponding to two curve sections of X. Hence a^1 defines, at a generator $1 \in C$ of X, the space spanned by 1 and the tangent to the curves at the points of intersection with 1; hence it is equal to the space spanned by the tangent planes to X along 1. Since X is a scroll, this space has dimension 3, so C is generically 1-regular. It follows that $C^* = C_1^{\vee} \subset \text{Grass}_4(V) = \text{Grass}_2(V^{\vee})$ is the strict dual of C (and $C^* = C^{(1)}$, the 1st associated curve of C). If X^* is a scroll, then C^* is generically 1-regular, and $C^{**} = C$ (by Prop. 1). Moreover, the 2nd osculating spaces to X along a generator 1 are just the 4-spaces containing the 3-space spanned by the tangent planes. In other words, C^* is equal to X^* considered as a curve in $\text{Grass}_2(V^{\vee})$. Thus we have proved:

Proposition 5: If $X \subset \mathbb{P}(V) = \mathbb{P}^5$ is a scroll such that X^* is a scroll, then $X^{**} = X$ holds.

The next proposition gives a formula for the degree of X^*.

Proposition 6: Let $X \subset \mathbb{P}(V) \cong \mathbb{P}^5$ be a scroll of degree d and genus g, and suppose $X^* \subset \mathbb{P}(V^V)$ is a scroll. Then

$$\deg X^* = 2(d+2g-2) - k,$$

where k is the stationary index of $C \subset \mathrm{Grass}_2(V)$.

Remark: Let $\nu : C' \to C$ denote the normalization, then, by definition, $k = \lg(\mathrm{Coker}\, V_{C'} \to P_C^1(\nu^* E))$. By trivializing E one sees that an ordinary cusp of C counts twice in k (which checks with [Edge] § 349). Or, k is the number (counted properly) of singular generators of X ([Pohl], p.208).

Corollary: The stationary index k^* of C^* is given by

$$k^* = 3(d + 2(2g-2)) - 2k.$$

Proof: On C', $P_C^1(E)$ admits a 4-quotient, namely $P_1 = \mathrm{Im}(V_{C'} \to P_C^1(\nu^* E))$. Hence: $\deg X^* = \deg C^* = c_1(P_1) = c_1(P_C^1(\nu^* E)) - k = 2(d+2g-2) - k$. The corollary follows from the duality $X^{**} = X$ of Prop. 5, $d = \deg X = 2(d^* + 2g-2) - k^*$.

Note that if X has no singular generators $(k = 0)$, then $k^* = 3(d + 2(2g-2))$, and hence X^* has no singular generators if and only if $d = d^* = 4$, $g = 0$. (Such scrolls are linearly normal in $\mathbb{P}^5$.)

We shall now look at some other approaches to the degree of X^*.

Because of the following (classical) proposition, the degree of X^* is equal to $\deg(X^*)^V$, hence to the degree of the tangent developable of X (Prop. 3).

<u>Proposition 7</u>: Let $X \subset \mathbb{P}(V) \cong \mathbb{P}^n$ be a scroll, $X^\vee \subset \mathbb{P}(V^\vee)$ its dual. Then:

$$\deg X^\vee = \deg X \,.$$

<u>Proof</u>: The classical proof goes like this: project X to a scroll $\overline{X} \subset \mathbb{P}(W) \cong \mathbb{P}^3$ with $d = \deg \overline{X} = \deg X$. Then $\overline{X}^\vee = X^\vee \cap \mathbb{P}(V/W)$, so $\deg \overline{X}^\vee = \deg X^\vee$ holds. If $L \subset \mathbb{P}(W)$ is a general line, $\deg \overline{X}^\vee = \#\{H \supset L, H \text{ tg. to } X\} = \#\{H \supset L \cup 1 | 1 \text{ generator of } X\} = \# L \cap X = \deg X$. Note that $\overline{X}$ and $\overline{X}^\vee$ are in fact <u>equal</u> considered as curves in $\mathrm{Grass}_2(W) = \mathrm{Grass}_2(W^\vee)$.

For a "modern" proof, one reduces to the case that $X \subset \mathbb{P}(V)$ is smooth, say $X = \mathbb{P}(E) \to C$. Then

$$\deg X^\vee = c_2(P^1_X(1)) = c_2(\Omega^1_X(1)) + c_1(\Omega^1_X(1)) c_1(O_X(1)),$$

which, by standard exact sequences, reduces to

$$\deg X^\vee = 2c_1(E) c_1(O_X(1)) - c_1(O_X(1))^2 = 2d - d = d \,.$$

From the exact sequence given in the proof of Prop. 3 we obtain (using [P2], § 2):

$$\deg X^* = \deg(X^*)^\vee = c_2(P^*_1) = c_1(P_1)^2 - c_2(P_1)$$

$$= c_1(P_1)^2 - \deg X^\vee = c_1(P_1)^2 - d \,.$$

Suppose $X = \mathbb{P}(E) \to C$ is smooth. Then $X' = X$ and $P_1 = P^1_X(1)$, so we get

$$\deg X^* = c_1(P^1_X(1))^2 - d = 3d + 2(2g-2) - d = 2(d+2g-2).$$

In the general case, X is the image of a smooth $Y = \mathbb{P}(E) \to C$, and X' is a blow-up of Y. Then $c_1(P_1) = c_1(P^1_{X'}(1)) - [R]$, where R is the ramification divisor of $X' \to X$, and we obtain

the earlier formula, but with k expressed "in terms of" R.

Two other approaches have been communicated to me by
I. Vainsencher and F. Ronga, respectively.

1. (Vainsencher)

Let $X \subset \mathbb{P}(V) \cong \mathbb{P}^5$ be a smooth scroll,

$Y = \mathbb{P}(N(-1)) = \{(x,H) | H \text{ tg. to } X \text{ at } x\} \subset \mathbb{P}(V) \times \mathbb{P}(V^V)$,

and set $Z = \{(x,H) \in Y | H \cap X = 1_x \cup D \text{ with } D \text{ singular at } x\}$.
Then $X^* = \mathrm{pr}_2(Z)$. One shows that Z is the zeros of a
section of a certain rank 2 bundle on Y; since the class
of Y in $X \times \mathbb{P}(V^V)$ is the 3rd Chern class of a rank 3
bundle, this gives the class of Z in $X \times \mathbb{P}(V^V)$ as a 5th
Chern class, and allows us to compute $\deg X^* = 2(d+2g-2)$,
provided $\dim X^* = 2$.

2. (Ronga)

Assume $X \subset \mathbb{P}(V)$ as above. Now one interprets Z as a modi-
fied $\Sigma^{2,2}$ (again by "forgetting" the generators of X)
of the projection map $X \times \mathbb{P}(V^V) \to \mathbb{P}(V^V)$. By computing all
the normal bundles in sight, one gets an expression for the
class of Z in $X \times \mathbb{P}(V^V)$, which allows one to compute
$\deg X^* = 2(d+2g-2)$.

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