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THE DISCRETE AND THE CONTINUOUS IN MATHEMATICS AND THE NATURAL SCIENCE

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Mathematical intuition is founded upon two sources, the <u>continuous</u> and the <u>discrete</u>, and the tension between the two has always played a central rôle in mathematical thought. This is true not only in the discussion of the foundation of mathematics, but also in the use of mathematics in the modelling of natural phenomena.

The discrete has its root in <u>counting</u> and <u>labeling</u>; one way of connecting it with the continuous is through the act of <u>measurement</u>. Measurements create in the continuum points and intervals, hence also parts of or fractions of intervals. So much for our immediate experience. But do the points we create exhaust the continuum? Is the geometric continuum a point set?

Points and lines

What does geometric intuition tell us? Let us take a brief look at the usual axiomatization of the affine plane, Artin [4]. There are two basic categories of objects, <u>lines</u> and <u>points</u>. There is one basic relation, $P \in \ell$, the point P <u>lies on</u> the line ℓ, and a defined notion, $\ell \parallel m$, the lines ℓ and m are <u>parallel</u>, i.e. either $\ell = m$ or ℓ and m have no point in common.

Basic geometric axioms tell us that two distinct points determinig a unique line; that through a given point there is a unique line parallel to a given line; and - to avoid trivialities - that there are at least three non-collinear points.

A geometry is determined by its symmetries. A <u>dilatation</u> σ is a map from points to points satisfying the following property:

Let two distinct points P and Q be given, and let ℓ be the line determined by P and Q. If ℓ' is the line parallel to ℓ passing through σP, then $\sigma Q \varepsilon \ell'$.

A trace of P with reespect to a nontrivial dilatation σ is a line ℓ such that both P and σP lie on ℓ. A translation τ is either the ideneity map or a dilatation without fixed points, in which case the traces of τ is an equivalence class of parallel lines.

We note that the dilatations D form a group and that the translations T is an invariant subgroup. A tracepreserving homomorphism $\alpha : T \to T$ is a homomorphism satisfying the property that either $\tau^{\alpha} = \alpha(\tau)$ is the identity or τ and τ^{α} have the same equivalence class of parallel lines as their traces. Let k be the set of all trace-preserving homomorphisms of T. We can introduce an addition and a multiplication in k by the equations

$$\tau^{\alpha+\beta} = \tau^{\alpha}\tau^{\beta}$$

$$\tau^{\alpha\beta} = (\tau^{\beta})^{\alpha}$$

If our geometry has enough symmetries, i.e. if it satisfies the theorem of Desargue, then k is a field and we can introduce coordinates from k for points in the plane. This follows from the fact that if τ_1 and τ_2 are translations different from the identity and have different directions, then to any $\tau \in T$ there exist uique $\alpha, \beta \in k$ such that

$$\tau = \tau^{\alpha}\tau^{\beta} = \tau^{\beta}\tau^{\alpha}$$

Choose any point 0 as origin and any two translations τ_1 and τ_2 with different traces. We think of the τ_1-trace and τ_2-trace through 0 as the coordinate axis'. Thus given any point P, let τ_{OP} be the (unique) translation that moves 0 to P, we can write

$\tau_{OP} = \tau_1^{\alpha}\tau_2^{\beta}$ for unique $\alpha, \beta \in k$, and we assign the coordinates (α, β) to P.

So far basic geometric intuition. Any point lies on a line, but a line is not given as a set of points. And if the geometry has enough smmetries, we can introduce coordinates from a field. But this does not yet determine how many points there are on a line.

There are further geometric properties that we may consider. A plane is <u>ordered</u> if the points on each line is linearly ordered (hence gives us a notion of <u>lies between</u> among points on a line), and if parallel projections either preserve or reverse the ordering. The plane is ordered iff the associated field k is (weakly) ordered.

An ordering of the plane is called <u>Archimedian</u> if it has the following property: Let τ_1 and τ_2 be (nonidentity) translations with the same directions and P a point which does not lie between $\tau_1(P)$ and $\tau_2(P)$, then there exists an integer $n > 0$ such that $\tau_2(P)$ lies between P and $\tau_1^n(P)$. It follows that a necessary and sufficient condition for an ordered geometry to come from a field k which is isomorphic to a subfield of the field of real numbers is the Archimedian axiom.

Thus granted the Archimedian axiom we know what possibilities there are for sets of points on the geometric line. But is the Archimedian axiom a "true" geometric fact? What is given in our immediate experience is a <u>limited</u> part of the geometric line with at most a <u>finite</u> number of points marked on it, representing e.g. the results of some physical measurements. The rest is an extension, ideal or real.

This has been forcefully expressed by Abraham Robinson [12]:

> Whatever our outlook and in spite of Leibniz' position, it appears to us today that the infinitely small and the infiniitely large numbers of a nonstandard model of Analysis are neither more nor less real than, for example, the standard

> irrational numbers. This is obvious if we introduce such numbers axiomatically; while in the genetic approach both standard irrational numbers and nonstandard numbers are introduced by certain infinitary processes. This remark is equally true if we approach the problem from the point of view of the empirical scientist. For all measurements are recorded in terms of integers and rational numbers, and if our theoretical framework goes beyond these, then there is no compelling reason why we should stay within an Archimedian number system.

For Robinson, who on several occasions expressed a strong formalist conviction see [11] and [13], this is the end of the story:

> From a formalist point of view we may look at our theory syntactically and consider that what we have done is to introduce <u>new deductive structures</u> rather than new mathematical entities.

Not everyone will agree to this point of view. Gödel in some remarks following a talk by Robinson at the Institute for Advanced Study in March 1973 [10] seems to maintain that there is no ontological difference between the integers, the rationals, the irrationals, and the infinitesimals. By a series of "quite natural steps" we may become familiar with and gain insight into what already exists.

But what are the natural steps, how do we construct points on a line?

<u>How to construct points on a line</u>

Let us for a moment take seriously the intuition that the geometric continuum, space, is not given as a point set.

The standard approach. Through measurements we know how to mark points in the continuum. And having chosen a unit we are then able to construct points corresponding to any rational fraction. But, as the ancient already observed, there must be points not named by the rationals, one example being the incommensurability of the diagonal of the square with its sides. This forces points on us labeled by certain algebraic number fields. But this is not the end.

A Cauchy sequence is a countable sequence of rationals numbers $\langle r_n \rangle_{n \in \mathbb{N}}$ satisfying the condition that given any natural number k there exists a number n_0 such that

$$|r_n - r_m| < \frac{1}{k}, \quad \text{for all } n,m > n_0.$$

The accepted standard point of view is that the geometric line is rich enough in points to contain limits for all Cauchy sequences of rationals. The plane is the "maximal" ordered desarguean geometry satisfying the Archimedian axiom.

Before closing the door, however, let us elaborate on one point. Different Cauchy sequences ought to determine the same point, e.g. $\langle 1,1,1,\ldots \rangle$ and $\langle 5,4,3,2,1,1,1,\ldots \rangle$ should in the limit determine the same point. We need an equality relation in the set of Cauchy sequences; the standard one is

$$\langle r_n \rangle = \langle s_n \rangle \quad \text{iff} \quad \lim_{n\to\infty} (r_n - s_n) = 0.$$

This does not only identify sequences that differ in a finite initial part, but also identifies sequences such as $\langle \frac{1}{n} \rangle$ and $\langle \frac{1}{n^2} \rangle$, which exhibit different convergence behavior. Classically, the set of reals is the set of equivalence classes of Cauchy sequences with respect to the equality relation introduced above. And the set of reals, the arithmetic continuum, is identified with the geometric continuum.

But if we look upon Cauchy sequences as a method of constructing points, is the above equality relation the only permissible one?

A remark on a constructivist point of view. The constructivist is also permitted to construct "rational points" on the line. To obtain further points one introduces constructively given Cauchy sequences $\langle r_n \rangle$. The constructivist also requires a constructive version of the Cauchy criterion, viz. a constructive rate-of-convergence function $\mu: \mathbb{N} \to \mathbb{N}$ such that

$$\forall k > 0 \quad \forall m,n > \mu(k) \; [|r_n - r_m| < \tfrac{1}{k}].$$

A constructive real is a pair $(\langle r_n \rangle, \mu)$ and an equality is defined in the set of constructive reals by

$$(\langle r_n \rangle, \mu) = (\langle s_n \rangle, \nu) \quad \text{iff} \quad (r_n - s_n) \to 0 \;,$$

see Feferman [9].

But this equality relation completely disregards the information given by the rate-of-convergence function. Constructively $(\langle \frac{1}{n} \rangle, \mu)$ and $(\langle \frac{1}{n^2} \rangle, \nu)$ determines the same point on the line even if their rate-of-convergence is not at all the same.

Perhaps, from the point of view of recorded measurements this is adequate. But if we want to preserve in the limit, the difference in convergence bahavior, and if we also want to pay attention to difference in asymptotic behavior, we should consider using a less "crude" equality relation in the set of Cauchy sequences.

The nonstandard approach. Let us reopen the discussion of the limit behavior of sequences $\langle a_n \rangle$. Our interest lies in their "eventual" behavior, i.e. two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ should be identified, if $a_n = b_n$ for all sufficiently large n, or, put in a different way, we identify $\langle a_n \rangle$ and $\langle b_N \rangle$ if there exists some

n_0 such that $a_n = b_n$ for all $n > n_0$. In this way we would distinguish between $\langle \frac{1}{n} \rangle$ and $\langle \frac{1}{n^2} \rangle$. So, perhaps, we have the correct answer.

At this point it is useful to introduce the notion of filter. A filter $\mathcal{F}$ is a family of subsets of $\mathbb{N}$ which satisfies the following conditions: if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$; if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$; the empty set $\emptyset$ does not belong to $\mathcal{F}$. The last point is to avoid the trivial case that $\mathcal{F}$ contains every subset of $\mathbb{N}$. A set is co-finite if its complement in $\mathbb{N}$ is finite, it is easy to see that the family of co-finite sets $\mathcal{F}_{cf}$ is a filter in $\mathbb{N}$.

The condition introduced above, that $\langle a_b \rangle = \langle b_n \rangle$ iff there is some n_0 such that $a_n = b_n$ for all $n > n_0$, can now be rephrased in the following manner: $\langle a_b \rangle = \langle b_n \rangle$ iff $\{n \in \mathbb{N} \mid a_b = b_n\} \in \mathcal{F}_{cf}$.

This is our initial attempt to make precise the notion of "eventually the same behavior". But it leads us into trouble with the algebra. The "natural" way of introducing the algebraic operations in the set of sequences is as follows:

$$\langle a_b \rangle + \langle b_n \rangle = \langle c_n \rangle \quad \text{iff} \quad \{n \in \mathbb{N} \mid a_n + b_n = c_n\} \in \mathcal{F}_{cf}$$

$$\langle a_b \rangle \cdot \langle b_n \rangle = \langle c_n \rangle \quad \text{iff} \quad \{n \in \mathbb{N} \mid a_n \cdot b_n = c_n\} \in \mathcal{F}_{cf}$$

But consider the sequences $\langle 1,0,1,0,\dots \rangle$ and $\langle 0,1,0,1,\dots \rangle$, by the above rules $\langle 1,0,1,0,\dots \rangle \cdot \langle 0,1,0,1,\dots \rangle = \langle 0,0,0,0,\dots \rangle$, but neither $\langle 1,0,1,0,\dots \rangle$ nor $\langle 0,1,0,1,\dots \rangle$ is equal to the zero-sequence $\langle 0,0,0,0,\dots \rangle$. We have introduced zero-divisors, which ruin the standard rules of algebraic manipulations.

We want to preserve the property that if $a \cdot b = 0$ then either $a = 0$ or $b = 0$. Translated into the language of filters this means that if $A \cup B \in \mathcal{F}$ then either $A \in \mathcal{F}$ or $B \in \mathcal{F}$. (In our case let A be the set of even numbers and B the set of odd numbers, then $A \cup B = \mathbb{N} \in \mathcal{F}_{cf}$.) Call a filter a <u>prime filter</u> if it has this property. (This terminology is taken from the characteristic property of prime numbers: if p is a prime and divides $a \cdot b$, the p divedes a or p divides b.)

Prime filters - or ultrafilters - will do the job. Let $\mathcal{U}$ be a prime filter extending $\mathcal{F}_{cf}$; such filters exist. Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be two sequences of real numbers, i.e. $\langle a_n \rangle, \langle b_n \rangle \in \mathbb{R}^{\mathbb{N}}$, we introduce an equality by

$$\langle a_b \rangle = \langle b_n \rangle \quad \text{iff} \quad \{ n \in \mathbb{N} \mid a_n = b_n \} \in \mathcal{U},$$

and let ${}^*\mathbb{R}$ denote the set of equivalence classes with respect to this equality, in standard notation,

$${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}} / \mathcal{U} .$$

In ${}^*\mathbb{R}$ we introduce algebraic operations as above, and we order ${}^*\mathbb{R}$ by the relation

$$a < b \quad \text{iff} \quad \{ n \in \mathbb{N} \mid a_n < b_n \} \in \mathcal{U},$$

where $a = \langle a_n \rangle / \mathcal{U}$ and $b = \langle b_n \rangle / \mathcal{U}$. The set $\mathbb{R}$ of standard reals are imbedded into ${}^*\mathbb{R}$ by mapping $r \in \mathbb{R}$ to the constant sequence $\langle r, r, r, \ldots \rangle$.

An element $a \in {}^*\mathbb{R}$ is called <u>finite</u> if $|a| < n$ for some standard natural number $a \in \mathbb{N}$; let $\mathbb{R}_f$ denote the finite elements in ${}^*\mathbb{R}$. $a \in {}^*\mathbb{R}$ is called <u>infinitesimal</u> if $|a| < \frac{1}{n}$ for all $n \in \mathbb{N}$; let $\mathbb{R}_i$ denote the set of infinitesimals. It is easy to see that

$$\mathbb{R} \simeq \mathbb{R}_f / \mathbb{R}_i \;,$$

thus for every finite $a \in {}^*\mathbb{R}$ there is a (unique) $r \in \mathbb{R}$ such that the difference $(a-r)$ is infinitesimal. We also see that the sequences $\langle \frac{1}{n} \rangle$ and $\langle \frac{1}{n^2} \rangle$ determine two distinct infinitesimals in

${}^{*}\mathbb{R}$, $\delta_1 = \langle\frac{1}{n}\rangle/\mathcal{U}$ and $\delta_2 = \langle\frac{1}{n^2}\rangle/\mathcal{U}$, such that

$$0 < \delta_2 < \delta_1 < \frac{1}{n},$$

all $n \in \mathbb{N}$.

It is now a basic assumption of nonstandard analysis that the construction of ${}^{*}\mathbb{R}$ is a method of constructing new points in the geometric continuum, i.e. ${}^{*}\mathbb{R}$ is an ordered field which can be used to "name" points on a line. ${}^{*}\mathbb{R}$ has strong closure properties and we may therefore "forget" the ambient geometric space and choose one version of ${}^{*}\mathbb{R}$ as the extended geometric line. But this is not correct on our point of view. ${}^{*}\mathbb{R}$ is but one method of constructing points. And we are at liberty to create different point sets on the line for different pruposes. We shall see how this leads to richer possibilities in modelling natural phenomena.

But first two remarks.

Remark 1. A similar point of view was expressed by Charles S. Peirce, see [8] for an exposition. Independently of Dedekind and Cantor he had around 1880 discussed the nature of the geometric continuum and the notion of infinity. He rejected Cantor's and Dedekind's identification of the Archimedian arithmetic continuum with the geometric continuum. He had independently of Cantor proved that the powerset of an infinite set had greater cardinality than the set itself, and he had boldly asserted that the forever increasing multitudes of "points" thus produced could be fitted into the ordered geometric continuum by some process of "interpolation". Peirce never arrived at a precise mathematical theory, he argued for the infinitesimals on the grounds of logical possibility and physical necessity.

Remark 2. The "praxis" of nonstandard theory, as the following examples will show, is often remarkably concrete, even "constructive", with explicit calculations replacing abstract developments. There has been attempts within constructive mathematics to make space for infinitesimals on the line; see e.g. the brief note by R. Vesley [16]. The clue here is to let "very small" mean "not equal to 0 and not known to be separated from 0 ".

Modelling natural phenomena

Having a richer pointset on the geometric line gives us a frame for a geometric analysis of physical phenomena on many scales and of physical phenomena that are too singular to fit in a direct way into the standard frame.

Canards. Our first example concerns a new type of limit cycles in the van der Pol equation

$$\varepsilon\ddot{x}+(x^2-1)\dot{x}+x-a = 0.$$

We are interested in what happens when $\varepsilon \to 0$ and $a \to 1$. The usual Lienard substitution

$$u = F(x) + \varepsilon\cdot\dot{x}$$

$$F(x) = x^3/3 - x$$

transforms the equation to the system

$$\varepsilon\dot{x} = u-F(x)$$

$$\dot{u} = a-x.$$

The standard approach is to use asymptotic expansions in powers of ε. Working on the extended line we choose ε to be a positive infinitesimal and $a \approx 1$. It turns out that for certain values of a, viz.

$$a = 1 - \varepsilon/8 - \varepsilon \cdot \eta,$$

for some $\eta \approx 0$, a new type of limit cycle, a <u>canard</u> is observed. Notice that $st(a) = 1$, so that the condition on a has no immediate arithmetic expression in $\mathbb{R}$. To "see" what is happening, we magnify the immediate or infinitesimal neighbourhood of the slow manifold $u = F(x)$ by the substitution $y = (u-F(x))/\varepsilon$, i.e. we transform to an infinitesemal scale. This gives us a vector field

$$(\star) \qquad \begin{aligned} \dot{x} &= y \\ \varepsilon\dot{y} &= a-x - (x^2-1)y \end{aligned}$$

It is easily seen that the standard part in the x,y-plane of the slow manifold of the vector field $(\star)$ is the union of the line $x = 1$ and $y = -1/(x+1)$. The derivative of $(\star)$ at $(1,-\frac{1}{2})$ is

$$\frac{dy}{dx} = \frac{1}{\varepsilon}\left(\frac{a-x}{y} - (x^2-1)\right)\Big|_{\substack{x=1 \\ y=-\frac{1}{2}}} = 2(1-a)/\varepsilon .$$

A geometric analysis tells us that the new type of limit cycle, the "canard", should be parallel (modulo an infinitesimal) to $y = -1/(x+1)$ as we approach $(1,-\frac{1}{2})$, i.e.

$$2(1-a)/\varepsilon - 1/4 \approx 0,$$

which gives the relationship

$$a = 1 - \varepsilon/8 - \varepsilon \cdot \eta,$$

for some infinitesimal η.

<u>Remark</u>. Canards were discovered by a group of French mathematicians, J.L. Callot, F. Diener and M. Diener. For an exposition of their work, see Benoit, Callot, F. Diener, M. Diener [5], and also Cartier [7].

The relationship between a and ε also has a <u>numerical content</u> and tells us how to arrange an approximative calculation of canards. It is, of course, a well-known fact in numerical analysis that various approximations, step-lengths, often must be chosen to depend on each other in quite specific ways in order to exhibit a particular phenomenon. It could be that the nonstandard theory is the "right" way to discuss this.

Singular perturbations. Let $H_0 = -\Delta$ be the free Schrödinger operator, where Δ is the Laplacian in $\mathbb{R}^d$. It corresponds to a quadratic form

$$E_0(f,g) = \int(-\Delta f)g\,dx.$$

We want to perturb the operator on a "small" set, i.e. a set of measure 0 in $\mathbb{R}^d$. This means that we want the "free" particle to feel a force concentrated on a small set, e.g. a point lattice in $\mathbb{R}^d$ as in solid state physics, or a "thin" submanifold of $\mathbb{R}^d$, or a Brownian path $C_\omega = \{b(\omega,t) \mid t \in [0,1]\}$ as in polymer science or quantum field theory.

Formally, this means to add to H_0 a singular perturbation of the form

$$H = H_0 + V,$$

where V in the Brownian path case would be

$$V(x) = -\int \lambda_\omega(x)\delta(x-b(\omega,t))dt,$$

δ being the δ-function in $\mathbb{R}^d$.

In terms of quadratic forms this can be written as

$$E(f,g) = E_0(f,g) + \int_{C_\omega} \lambda fg\,d\rho,$$

where $\rho(A) = m\{t \in [0,1] \mid b(\omega,t) \in A\}$, m the Lebesgue measure on the real line.

This case is studied in detail in [1], see also the anouncement [2]. Our approach is to lift the problem to a hyperfinite setting and use the theory of hyperfinite quadratic forms that we develop in [1]. The result is that H exists as a nontrivial, self-adjoint, lower bounded perturbation of $-\Delta$ for $d \leq 5$. We must choose λ to be an infinitesimal, positive function for $d = 4,5$; for $d \leq 3$ any bounded real-valued function λ will do. The connections with polymer models as well as with quantum fields are discussed in [1].

The space-inhomogeneous Boltzmann equation. The Boltzmann equation describes a gas of identical point molecules which are interacting by a potential of finite range. The molecules move with unrestricted speed in some region Λ of $\mathbb{R}^3$. Assuming periodic boundary conditions we have as phase space $M = \Lambda \times \mathbb{R}^3$, where $\Lambda = \mathbb{R}^3/\mathbb{Z}^3$. Letting $F(x,v,t)$ denote the density of the molecules the Boltzmann equation is

$$\frac{\partial F(x,v,t)}{\partial t} + v \cdot \nabla_x F(x,v,t) = (QF)(x,v,t),$$

where Q denotes the usual collision operator.

The first step of the analysis is to consier a "truncated" version by replacing Q by some suitable "cut-off" Q_n, $n \in \mathbb{N}$. It is then standard to prove that for suitable initial data $F_0(x,v) = F(x,v,0)$ there exists a unique non-negative L^∞-solution of the equation

$$F(x+tv,v,t) = F_0(x,v) + \int_0^t Q_n F(x+sv,v,s)ds.$$

By transfer we obtain for $n \in {}^*\mathbb{N}-\mathbb{N}$ an internal solution f of the extended equation. The main analytic part now comes in showing that f has a standard part ${}^0 f$. One can prove that with the (internal) initial condition

$$f_0(x,v) = \min({}^*F_0(x,v),n) + n^{-1}\exp(-v^2),$$

where $n \in {}^*\mathbb{N}-\mathbb{N}$ and ${}^*F_0$ is the standard extension of F_0, the standard part ${}^0f$ of the internal solution f is nearstandard for Loeb-almost all $(x,v) \in {}^*\Lambda \times ns({}^*\mathbb{R}^3)$, where $ns({}^*\mathbb{R}^3)$ is the set of nearstandard points of ${}^*\mathbb{R}^3$. Furthermore, the standard part ${}^0f$ satisfies the correct integral equation; for details see the exposition in [1].

This result was proved by L. Arkeryd [3] and is the first general result of its kind in the space-inhomogeneous case. Notice that ${}^0f$ has everywhere standard values, but it lives on an enriched or denser phase-space ${}^*\Lambda \times ns({}^*\mathbb{R}^3)$. But this, we claim, is perfectly acceptabsle from a physical point of view.

Quantum fields. Constructing models for quantum fields is no small task. A favored approach is to start with free lattice fields. Let $\delta > 0$ be a fixed positive real number and define the lattice L_δ with spacing δ to be the set $L_\delta = \{n\delta \mid n \in \mathbb{Z}^d\}$. Let Δ_δ be the standard discretization of the Laplacian and introduce the covariance matrix

$$C_{n,n'} = \delta^{-d}(-\Delta_\delta + m^2)^{-1}_{n,n'} ,$$

where m is a positive real number, the "mass". Let $\Lambda \subseteq \mathbb{R}^d$ be a bounded region and let $\Lambda_\delta = \Lambda \cap L_\sigma$. We let $\partial\Lambda_\delta$ denote the boundary of Λ_δ. Λ_δ is a finite set, let $\ell = |\Lambda_\delta|$ be the number of points in Λ_δ. Our measure space will be the finite product $Q_{\Lambda_\delta} = \mathbb{R}^{\Lambda_\delta}$ equipped with the gaussian measure

$$d\mu_{0,\Lambda_\sigma}(q) = (2\pi)^{-\ell/2}[\det(C^\Lambda)]^{-\frac{1}{2}}\exp(-\tfrac{1}{2} \sum_{\substack{n\delta, \\ n'\delta \in \Lambda_\sigma}} (C^\Lambda)^{-1}_{n,n'} q_{n\delta} q_{n'\delta})dq,$$

where $q \in Q_{\Lambda_\delta}$, $dq = \Pi dq_{n\delta}$ and C^Λ is the restriction of the ccovriance matrix to Λ_δ.

We may now introduce the random field Φ_δ indexed by Λ_δ as the map $\Phi_\delta : \Lambda_\delta \times Q_{\Lambda_\delta} \to \mathbb{R}$ given by

$$\Phi_\delta(n)(q) = q_{n\delta}\ , \qquad n\delta \in \Lambda_\delta .$$

It is called <u>the free lattice field of mass m</u> in Λ_δ; for an exposition see Simon [15].

One may now show that as $\delta \to 0$ the fields Φ_δ "converges" to a gaussian random field. "Convergence" is here somewhat problematic, changing δ not only gives us a new random variable, it also gives us a new measure space. And what is more awkward, in the limit the lattice disappears!

In the hyperfinite picture there is an easy way to overcome these difficulties. In addition to the standard finite approximations "from the inside", we can also approximate "from the outside" Let $\delta > 0$ be infinitesimal and let Λ_δ be a hyperfinite lattice in ${}^*\mathbb{R}$ with spacing δ. By transfer we have <u>a hyperfinite lattice field</u> which we can easily show to be a model of the free Euclidean field of quantum field theory. What is noteworthy here is that the hyperfinite field is pointwise defined. We still have infinities, but they can be controlled through a consistent algebra, hence lead to unambiguous and meaningful results; see [1].

The free field is of rather limited physical interest, the goal is to construct fields which model various forms of particle interaction. One way of doing this is to construct suitable "local additive functionals" of the free field. Let $\delta > 0$ be a standard real and consider the lattice Λ_δ obtained from a bounded domain $\Lambda \subseteq \mathbb{R}^d$. Let g be a positive function with support in Λ and let u_δ be any continuous real function. We will study <u>interactions</u> of the form

$$U_g^\delta = \lambda_\delta \sum \delta^d g(n\delta) u_\delta(\Phi_\delta(n)),$$

where λ_δ is a real constant, the "coupling constant". We may introduce the perturbed measure

$$d\mu_{g,\Lambda_\delta} = \frac{\exp(-U_g^\delta)\, d\mu_{0,\Lambda_\delta}}{\int \exp(-U_g^\delta)\, d\mu_{0,\Lambda_\delta}}$$

on the space Ω_{Λ_δ}, but keep the field $\Phi_\delta(n)(q) = q_{n\delta}$. We would hope that under suitable conditions $d\mu_{g,\Lambda_\sigma}$ leads to a nongaussian measure (either in the limit or in the hyperfinite picture), hence to a model of an interacting field.

The function g in U_g^δ represents a kind of "space cut-off", it could e.g. be the characteristic function of some domain Λ_δ in Λ. In order to obtain a nontrivial field we now let δ tend to zero while at the same time letting $\Lambda_\delta \uparrow \mathbb{R}^d$. To remove the "space cut-off" we let g converge to the constant function 1 or $\mathbb{R}^d$.

This is the standard program. In the hyperfinite version it means that we want to choose $\delta > 0$ infinitesimal, Λ_δ a hyperfinite lattice and g an internal function such that $g(n\delta) = 1$ for all nearstandard $n\delta \in \Lambda_\delta$. We would then like to extract from the internal construct μ_{g,Λ_δ} a non-gaussian measure satisfying the field-theoretic axioms; see [1].

We cannot carry through this in detail in this exposition, the reader is referred to [1] for a full discussion. We indicate a few steps to exhibit some of the parts that come into play. Let us for simplicity choose an exponential interaction $\mu_\delta(y) = \exp(\alpha y)$. Let us make the following calculation to see what we have to do in order to control the possible infinities. We assume that $\delta > 0$ is standard, that Λ_δ is a finite lattice and that $g > 0$ has support in Λ:

$$(+) \qquad \int (U_g^\delta)^2 d\mu_{0,\Lambda_\delta} = \lambda_\delta^2 (A_\alpha^\delta)^2 \sum_{\substack{n\delta, \\ n'\delta \in \Lambda_\delta}} \delta^{2d} g(n\delta) g(n'\delta) e^{\alpha^2 C^\Lambda_{nn'}},$$

where

$$A_\alpha^\delta = \exp\left(\frac{\alpha^2}{2}(2\pi)^{-d} \int_{T_\delta} \mu_\delta(k)^{-2} dk\right),$$

where $T_\delta = [-\pi/\delta, \pi/\delta]^d$ and $\mu_\delta(k)^2 = \delta^{-d}[2d-2\sum_{i=1}^{d} \cos(k_i\delta)]+m^2$.

Let $d = 2$, choose $\delta > 0$ infinitesimal and Λ_δ hyperfinite, but keep a cut-off g of compact support. If $\alpha^2 < 4\pi$ we easily see that the sum on the right hand side of (+) is finite, but the integral in A_α^δ diverges. However, one infinity can be balanced against another. The coupling constant is so far left unspecified, so let us choose

$$\lambda_\delta = \lambda \cdot (A_\alpha^\delta)^{-1},$$

where $\lambda > 0$ is a finite real number idenpendent of δ and α. With this choice the right hand side of (+) is nearstandard, and it is not difficult to see that the standard part is

$$\lambda^2 \int_{\mathbb{R}^d} g(x)g(y)e^{\alpha^2 G(x-y)}dxdy,$$

where $G(x,y)$ is the kernel of the operator $(-\Delta+m^2)^{-1}$.

Remark. For the informed reader we note that the interaction U_g^δ is more commonly written

$$U_g^\delta = \lambda \sum \delta^d g(n\delta) :e^{\alpha\Phi_\delta(n)}: ,$$

where $:e^{\alpha\Phi_\delta(n)}: = (A_\alpha^\delta)^{-1} e^{\alpha\Phi_\delta(n)}$ is the so-called Wick renormalization of $e^{\alpha\Phi_\delta(n)}$.

Thus one important stage in the program is completed. It remains to remove the space cut-off g. Let once more $\delta > 0$ be a standard real, Λ_δ a finite lattice, and let g have support in Λ. The Schwinger functions associated with the measure μ_{g,Λ_δ} are given by

$$S_g^\delta(n_1\delta, \ldots, n_k\delta) = \int \Phi_\delta(n_1)\ldots\Phi_\delta(n_k)d\mu_{g,\Lambda_\delta}.$$

Let g, g' both have support in Λ, assume that the support of g is contained in the support of g', and that $g = g'$ on the support of g. Then one may show that

$$0 \leqslant S^{\delta}_{g'} \leqslant S^{\delta}_{g} \leqslant S^{\delta}_{0} \ ,$$

where S_0^δ is the Schwinger function of the free field.

In the final stage of the program we pass to the hyperfinite picture and remove the finite space cut-off by choosing an internal function g_ω such that $g_\omega(n\delta) = 1$ for all finite $n\delta$. Then we may prove that $L(\mu_{g_\omega, \Lambda_\delta})$, the Loeb-measure associated with $\mu_{g_\omega \Lambda_\delta}$ is a non-gaussian probability measure. And using the inequalities established above, we may verify that this measure for $d = 2$ gives us a model for a quantum field with non-trivial interaction.

Remark. The book [1] contains further examples, e.g. a nonstandard version of the ϕ_3^4 model due to Brydges, Frölich, Sokal [6]; a construction of gauge fields in dimension 2; and a discussion of polymer measure and quantum fields.

On nonstandard praxis

It has often been held that nonstandard analysis is highly nonconstructive, thus somewhat suspect, depending as it does upon the ultrapower construction to produce a model. On the other hand, as we already remarked above, nonstandard praxis is remarkably constructive; having the extended number system we can proceed with explicit calculations. A case in point is the Cauchy-Peano existence theorem for ordinary differential equations. In the standard approach one uses in the final step the Ascoli lemma which asserts that every bounded equicontinuous sequence of functions on an interval I has a uniformly bounded subsequence. This part of the argument is lacking in the nonstandard proof, which makes it more direct. And indeed it is in the following precise sense. It is possible to recast the nonstandard proof to give a proof of the Cauchy-Peano existence theorem where the only non-recursive element is the Weak König's Lemma which asserts that every infinite binary tree, i.e. infinite tree of sequences of 0's and 1's, has an infinite path. And this is a principle which is provably weaker, i.e. more constructive, than the Ascoli lemma [4].

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