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§1. Introduction

Let U be an open subset of $\mathbf{R}^n$, $n \geq 2$, and $\phi : U \to \mathbf{R}^n$ be continuous. Then we say that ϕ is *quasiregular* if ϕ is absolutely continuous on almost every straight line segment in U with partial derivatives which are locally L^n-integrable wrt. Lebesque measure (i.e. $\phi \in ACL^n$) and there exists a constant $K < \infty$ such that

$$|\phi'(x)|^n \leq K \cdot J_\phi(x) \quad \text{for all } x \in U , \tag{1.1}$$

where $|\phi'(x)|$ denotes the norm of the linear map $\phi'(x)$ given by the matrix

$$\phi'(x) = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1} & \cdots & \frac{\partial \phi_n}{\partial x_n} \end{bmatrix} = \left[\frac{\partial \phi_i}{\partial x_j} \right]_{ij} \tag{1.2}$$

and $J_\phi(x) = \det(\phi'(x))$ is the Jacobian of ϕ at x. The smallest K such that (1.1) holds is called the *outer dilation* of ϕ and denoted by $K_0(\phi)$ or just K_0. If we put

$$l(\phi'(x)) = \inf\{|\phi'(x)h|; \ |h| = 1\}$$

then there exists K such that

$$J_\phi(x) \leq K[l(\phi'(x))]^n \quad \text{for all } x \in U , \tag{1.3}$$

and the smallest $K \geq 1$ such that (1.3) holds is called the *inner dilation* of ϕ

and denoted by $K_I(\phi)$ or K_I. We define $K(\phi) = \max(K_0(\phi), K_I(\phi))$. Note that for $k \in \mathbf{R}^n$ we have

$$K_I^{-1} J_\phi(x) |k|^n \leq |\phi'(x) k|^n \leq K_0 J_\phi(x) |k|^n ,$$

so if $J_\phi(x) > 0$ for some x then $\phi'(x)$ is invertible and if we put $k = (\phi'(x))^{-1} h$ we have $\phi'(x) k = h$ and so

$$K_I^{-1} J_\phi(x) |(\phi'(x))^{-1} h|^n \leq |h|^n \leq K_0 J_\phi(x) |(\phi'(x))^{-1} h|^n ,$$

or

$$K_0^{-1/n} J_{\phi(x)}^{-1/n} |h| \leq |(\phi'(x))^{-1} h| \leq K_I^{1/n} J_{\phi(x)}^{-1/n} |h| . \tag{1.4}$$

We refer to Martio, Rickman & Vaisala [19] or Vaisala [24] for more information about quasiregular functions.

If $n = 2$ and we identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ then a C^1-function $\phi : U \to \mathbf{C}$ is analytic if and only if

$$|\phi'(x)|^2 = J_{\phi(x)} \text{ for all } x \in U . \tag{1.5}$$

Thus in this case the quasiregular functions may be regarded as generalizations of the analytic functions. In view of the fact that the analytic functions are Brownian path preserving, i.e. they map Brownian motion into Brownian motion except for a change of time scale, (see [2] or [18]) it is natural to ask if there is also a connection between quasiregular functions and Brownian motion. The purpose of this paper is to establish such a connection, valid for all dimensions $n \geq 2$. More precisely, we will prove in §2 that if $\phi : U \to \mathbf{R}^n$ is quasiregular then there exists a Markov process X_t in U such that ϕ is $X_t - B_t$ path preserving, i.e. ϕ maps X_t into Brownian motion B_t in $\mathbf{R}^n$ (Theorem 2.3).

For $n > 2$ a weak growth condition has to be imposed on ϕ. The process X_t is obtained as the Markov process associated to a regular Dirichlet form $\mathcal{E}(\cdot,\cdot)$ which can be described explicity.

It is now well known that many important properties of analytic functions can be proved by using that they are Brownian-path-preserving. Similarly, when the relation in Theorem 2.3 between a quasiregular map ϕ and the process X_t and B_t is established, it gives a number of results about ϕ. For example, we give a new proof of the Picard theorem $(n = 2)$, we establish a Rado type theorem about removable singularities $(n = 2)$ and we prove results about the existence of boundary values $(n \geq 2)$.

§2. The main result

If $U \subset \mathbb{R}^n$ is open we let $C_0^\infty(U)$ denote the infinitely differentiable real functions with compact support in U. Let $\mathcal{E}(u,v) : C_0^\infty(U) \times C_0^\infty(U) \to \mathbf{R}$ be a *regular Dirichlet form* on $L^2(U,dm)$, where m is a Radon measure on U. (See Fukushima [10].)

Let $\mathcal{D}[\mathcal{E}]$ denote the closure of $C_0^\infty(U)$ in the norm whose square is $\mathcal{E}_1(u,u) = (u,u) + \mathcal{E}(u,u)$, where

$$(u,v) = \int_U uv\, dm; \quad u,v \in L^2(U,dm) . \tag{2.1}$$

Define a selfadjoint nonpositive operator A on $\mathcal{D}[A] \subset \mathcal{D}[\mathcal{E}]$ by

$$\mathcal{E}(u,v) = (-Au,v), \quad u \in \mathcal{D}[A],\ v \in \mathcal{D}[\mathcal{E}] . \tag{2.2}$$

Then there exists a Hunt process $(X_t,\Omega,\mathcal{M},P^x)$ in U whose generator is A ([10], Theorem 6.2.1). The process X_t is unique up to *equivalence*, i.e. if X_t,X'_t

are two such processes then we can find a common *properly exceptional set* $N \subset U$ such that the transition functions of X_t

$$p_t(x,f) = E^x[f(X_t)], \ f \in C_0^\infty(U), \quad t \geq 0$$

coincide with those of X'_t for all $x \in U \setminus N$. (N is called a properly exceptional set for X_t if $m(N) = 0$ and

$$P^x[\exists t \geq 0; X_t \in N] = 0 \text{ for all } x \in U \setminus N) .$$

We will use the term *quasi-everywhere* for "except of a properly exceptional set".

We refer to Fukushima [10] for more information about Dirichlet forms and associated Hunt processes.

First we establish two useful auxiliary results:

LEMMA 2.1 (The Dynkin formula). *Let* $\mathcal{E}(u,v)$ *be a regular Dirichlet form on* $L^2(U,dm)$ *with associated Hunt process* (X_t,Ω,P^x) *whose generator is* $A : \mathcal{D}[A] \to L^2(U,dm)$. *Choose* $g \in C_0(U) \cap \mathcal{D}[A]$ *and let* τ *be a stopping time for* X_t. *Then there exists a properly exceptional set* N *for* X_t *such that*

$$E^x[g(X_{t\wedge\tau})] = g(x) + E^x\left[\int_0^{t\wedge\tau} (Ag)(X_s)\, ds\right] \tag{2.3}$$

for all $t \geq 0$ and all $x \in U \setminus N$.

Proof. Define the transition function of X_t by

$$p_t(x,f) = E^x[f(X_t)]; \quad x \in U, \ t \geq 0, \ f \in C_0(U) \tag{2.4}$$

and the resolvent of X_t by

$$R_\lambda(x,b) = \int_0^\infty e^{-\lambda t}\, p_t(x,f)\, dt \ ; \quad \lambda > 0 . \tag{2.5}$$

Then $p_t(\cdot,f)$ and $R_\lambda(\cdot,f)$ are quasicontinuous versions of the semigroup $\{T_t\}$ and resolvent $\{G_\lambda\}$ associated with A.

By Theorem 5.1 p. 132 in Dynkin [8] we have for any $h \in L^2(U,dm)$, $\lambda > 0$

$$E^x[e^{-\lambda(t\wedge\tau)}R_\lambda h(X_{t\wedge\tau})] = (R_\lambda h)(x) - E^x\left[\int_0^{t\wedge\tau} e^{-\lambda s}h(X_s)\,ds\right]. \tag{2.6}$$

In particular, if we choose $h = \lambda g - Ag$ we have $R_\lambda h = g$ as elements of $L^2(U,dm)$ and so by quasicontinuity

$$R_\lambda h = g \text{ quasi-everywhere in } U.$$

Hence there exists a properly exceptional set N such that if $x \in U \setminus N$ we have

$$R_\lambda h(x) = g(x) \tag{2.7}$$

and

$$E^x[e^{-\lambda(t\wedge\tau)}R_\lambda h(X_{t\wedge\tau})] = E^x[e^{-\lambda(t\wedge\tau)}g(X_{t\wedge\tau})]. \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.6) and letting $\lambda \to 0$ we obtain (2.3).

In the following dx,dy etc. will denote Lebesque measure in $\mathbf{R}^n$ and, unless otherwise stated, a.e. will mean with respect to Lebesque measure. If M is a matrix then M^T denotes the transposed of M. The notation $W \subset\subset U$ will mean that W is an open subset of U, the closure $\overline{W}$ is compact and $\overline{W} \subset U$. The boundary of U is denoted by ∂U.

LEMMA 2.2. Let $\phi : U \to \mathbf{R}^n$ be a (non-constant) quasiregular function. Suppose that

for a.a. $y \in U$ there exists $r > 0$ such that $\int_{|x-y|<r} J_{\phi(x)}^{2/n-1}\,dx < \infty$. (2.9)

Let $a = [a_{ij}]$ be the $n \times n$ matrix

$$a = J_\phi(\phi')^{-1}((\phi')^{-1})^T \tag{2.10}$$

and define

$$\mathcal{E}(u,v) = \frac{1}{2}\int_U (\nabla u)^T a(\nabla v)\, dx, \quad u,v \in C_0^\infty(U) . \tag{2.11}$$

Then $\mathcal{E}$ is a regular Dirichlet form on $L^2(U,dm)$, where $dm = J_\phi dx$..

Remark. Since $J_\phi > 0$ a.e. in U (see [18], Theorem 8.2) the matrix a in (2.10) is defined a.e. in U.

Proof. We must establish that the symmetric bilinear form $\mathcal{E}(\cdot,\cdot) : C_0^\infty(U) \times C_0^\infty(U) \to \mathbb{R}$ given by (2.11) is *Markovian*, *regular*, and *closable*. First note that

$$2\mathcal{E}(u,v) = \int_U [((\phi')^{-1})^T \nabla u]^T [((\phi')^{-1})^T \nabla u] J_\phi dx \tag{2.12}$$

so that, by (1.4)

$$2\mathcal{E}(u,u) = \int_U |((\phi')^{-1})^T \nabla u|^2 J\phi\, dx \tag{2.13}$$

$$\leq \int_U |\nabla u|^2 K_I^{2/n} J_\phi^{1-2/n}\, dx < \infty \text{ if } u \in C_0^\infty(U) .$$

Therefore $\mathcal{E}$ is regular and Markovian, by a general result proved in [10] (Example 1.2.1, p. 6). It remains to prove that $\mathcal{E}$ is closable. To this end, let $\{u_k\}$ be a sequence in $C_0^\infty(U)$ such that

$$\mathcal{E}(u_k - u_l, u_k - u_l) \to 0 \text{ and } u_k \to 0 \text{ in } L^2(U,dm) \text{ as } k,l \to \infty .$$

We must show that $\mathcal{E}(u_k,u_k) \to 0$ as $k \to \infty$.

First note that, by (2.12) and (1.4), for $v \in C_0^\infty(U)$,

$$\int_U |\nabla v|^2 K_0^{-2/n} J_\phi^{1-2/n}\, dx \le 2\mathcal{E}(v,v) \le \int_U |\nabla v|^2 K_I^{2/n} J_\phi^{1-2/n}\, dx \tag{2.14}$$

or,

$$K_0^{-2/n} \mathcal{E}_0(v,v) \le \mathcal{E}(v,v) \le K_I^{2/n} \mathcal{E}_0(v,v) \ , \tag{2.15}$$

where

$$\mathcal{E}_0(u,v) = \int_U (\nabla u)^T \nabla u \cdot J_0\, dx \ , \ \text{ with } J_0(x) = J_\phi^{1-2/n}(x) \ . \tag{2.16}$$

Therefore $\mathcal{E}$ is closable if and only if $\mathcal{E}_0$ is closable. Since

$$\int_G |\nabla(u_k - u_l)|^2 J_0\, dx \to 0 \ \text{ as } \ k,l \to \infty$$

there exist $f_1,\ldots,f_n \in L^2(J_0 dx)$ such that

$$\frac{\partial u_k}{\partial x_i} \to f_i \ \text{ in } \ L^2(J_0 dx) \ . \tag{2.17}$$

Put $\vec{f} = (f_1,\ldots,f_n)$ and let $H = H(y,r)$ be a cube in U of the form $H = \{(x_1,\ldots,x_n);\ |x_i - y_i| \le r;\ 1 \le i \le n\}$, $r > 0$. Then

$$\begin{aligned} |\int_H \vec{f}\, dx|^2 &= |\int_H (\vec{f} - \nabla u_k) dx + \int_H \nabla u_k\, dx|^2 \\ &\le 2|\int_H (\vec{f} - \nabla u_k)\, dx|^2 + 2|\int_H \nabla u_k\, dx|^2 \ . \end{aligned} \tag{2.18}$$

Since $u_k \to 0$ in $L^2(dm)$ we see that, by taking a subsequence

$$\int_H \nabla u_k\, dx \to 0 \ \text{ as } \ k \to \infty \ ,$$

for a.a. $r > 0$. The first term in (2.18) is estimated by

$$\left| \int_H (\vec{f} - \nabla u_k) \sqrt{J_0} \frac{1}{\sqrt{J_0}}\, dx \right|^2 \le \left(\int_H |\vec{f} - \nabla u_k|^2 J_0\, dx \right) \cdot \left(\int_H \frac{dx}{J_0} \right) .$$

We conclude that $\vec{f} = 0$ a.e. outside the set of points y s.t. $\int_{H(y,r)} dx/J_0 = \infty$ for all $r > 0$. So from assumption (2.9) we conclude that $\vec{f} = 0$ a.e. And then from (2.18)

$$2\mathcal{E}_0(u_k,u_k) = \int_U |\nabla u_k|^2 J_0 \, dx \to 0 \text{ as } k \to \infty ,$$

which shows that $\mathcal{E}_0$, and hence $\mathcal{E}$, is closable.

Remark. Condition (2.9) is satisfied if, for example, $\phi \in C^1(U)$ or, more generally, if J_ϕ is locally bounded away from 0 a.e. in U. It is natural to ask if (2.9) holds for *all* quasiregular functions ϕ. The following argument shows that $J_\phi^{2/n-1}$ need not be locally in L^1 *everywhere*:

Let B_ϕ denote the branch set of ϕ, i.e. the set of points where ϕ is not a local homeomorphism. Then B_ϕ is a closed set of Lebesque measure 0 ([19], Theorem 8.3). Choose $z \in U \setminus B_\phi$ and let r be so small that ϕ is a homeomorphism on $H(z,r)$. Then, with $\psi = \phi^{-1}, H = H(z,r)$

$$\int_H \frac{dx}{J_0} = \int_H J_\phi^{(2/n)-1} \, dx = \int_H J_\phi^{(2/n)-2} \cdot J_\phi \, dx = \int_{\phi(H)} J_\phi^{(2/n)-2}(\psi(y)) \, dy$$

$$= \int_{\phi(H)} J_\psi^{2-2/n}(y) \, dy \, . \tag{2.19}$$

Gehring [11] has given an example of a quasiconformal function ψ such that $J_\psi \notin L^p_{loc}$ everywhere if

$$p = \frac{1}{K_{(\psi)}^{1/(n-1)} - 1} \, . \tag{2.20}$$

Thus for any $n > 2$ there exists a K and a z such that (2.19) diverges if $K(\psi) \geq K$. Hence the integral in (2.9) need not converge *everywhere*.

Moreover, (2.9) is actually also a *necessary* condition that the form (2.11) is closable. This follows by the argument used by Hamza [12] to characterize the closable 1-dimensional forms.

Before we formulate the main result we describe a weak extension of the concept of a Markovian *path preserving function*, which was introduced in [5] (see also [22]).

Let (X_t,Ω,P^x), $(Y_t,\hat{\Omega},\hat{P}^y)$ be Hunt processes associated to regular Dirichlet forms $\mathcal{E}(\cdot,\cdot)$, $\hat{\mathcal{E}}(\cdot,\cdot)$ on $L^2(U,dm_1)$ and $L^2(V,dm_2)$, respectively, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets. The *time changes* $\beta_t = \beta_t(\omega)$ we will consider are of the following form:

Let $c(x) \geq 0$ be a Borel measurable function on U and put

$$\beta_t = \int_0^t c(X_s)\, ds\ . \tag{2.21}$$

We will say that β_t is a *time change* (for X_t) with *time change rate* c.

For each $\omega \in \Omega$ the function $t \to \beta_t(\omega)$ is non-decreasing. Let $\alpha_t = \beta_t^{-1}$ be its right-continuous inverse:

$$\alpha_t = \inf\{s; \beta_s > t\} \quad (\alpha_t = \infty \text{ if } \beta_s \leq t \text{ for all } s) \tag{2.22}$$

We say that a continuous function $\phi : U \to V$ is (quasi) $X_t - Y_t$ *path-preserving* if there exists a time change β_t for X_t as above such that if we define, for any choice of function ψ s.t. $\phi(\psi(y)) = y$, $y \in \phi(U)$, any $W \subset\subset U$ with

$$\tau = \tau_W = \inf\{t > 0; X_t \notin W\} \text{ (the first exit time from } W \text{ for } X_t)\ , \tag{2.23}$$

$$Z_t = Z_t(\omega,\hat{\omega}) = \begin{cases} \phi(X_{\alpha_t}); & t < \beta_\tau \\ Y_{t-\beta_\tau}; & t \geq \beta_\tau \end{cases} \tag{2.24}$$

with probability law $\tilde{P}^y$ given by ($\tilde{E}^y$ is expectation wrt. $\tilde{P}^y$ etc.)

$$\tilde{E}^y[f_1(Z_{t_1}) \cdots f_k(Z_{t_k})\chi_{\{t_j \le \beta_\tau < t_{j+1}\}}] = \tag{2.25}$$

$$E^x[f_1(\phi(X_{\alpha_{t_1}})) \cdots f_k(\phi(X_{\alpha_{t_j}}))\chi_{\{t_j \le \beta_\tau < t_{j+1}\}} \hat{E}^{\phi(X_\tau)}[f_{j+1}(Y_{t_{j+1}-\beta_\tau}) \cdots f_k(Y_{t_k-\beta_\tau})]] ,$$

where $x = \psi(y)$ if $y \in \phi(U)$, then $(Z_t,\tilde{P}^{\phi(x)})$ coincide in law with $(Y_t,\hat{P}^{\phi(x)})$, for all $x \in U \setminus N$, where N is a properly exceptional set for X_t.

If X_t, Y_t are Browian motions, then (by Feller continuity) this definition is equivalent to the definition of a Brownian pathpreserving function, introduced in [2].

We are now ready for the main result:

THEOREM 2.3. *Let $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a quasiregular function satisfying (2.9). Let X_t be the Hunt process associated to the Dirichlet form $\mathcal{E}$ given by (2.11) and let $(B_t,\hat{\Omega},\hat{P}^y)$ be n-dimensional Brownian motion. Then ϕ is $X_t - B_t$ path preserving, without time change. In other words, if we define as in (2.5),*

$$Z_t(\omega,\hat{\omega}) = \begin{cases} \phi(X_t); & t < \tau \\ B_{t-\tau}; & t \ge \tau \end{cases} \tag{2.26}$$

where $\omega \in \Omega$, $\hat{\omega} \in \hat{\Omega}$ and $\tau = \tau_W$, $W \subset\subset U$, with probability law $\tilde{P}^{\phi(x)}$ given by (2.24), then $(Z_t,\tilde{P}^{\phi(x)})$ is n-dimensional Brownian motion, for quasi-all x.

Remark. From the expression for $\mathcal{E}$ we know that X_t has continuous paths and no killing occurs inside U. See [10], Theorem 4.5.3.

Proof of Theorem 2.3. Choose $W \subset\subset U$. For each $y \in \phi(W)$ there exists a neighbourhood V_y of y such that each component W_j of $\phi^{-1}(V_y)$ which intersects W is a normal domain ([19]). Fix such a W_j and let $f \in C_0^\infty(V_y)$. Then we claim that

$$(f \circ \phi) \cdot \chi_{W_j} \in \mathcal{D}[A] \text{ and } A[(f \circ \phi)\chi_{W_j}] = (\hat{A}[f] \circ \phi) \cdot \chi_{W_j} , \qquad (2.28)$$

where $\hat{A}[f] = 1/2\ \Delta$ is the selfadjoint nonpositive operator corresponding to the classical Dirichlet form

$$\hat{\mathcal{E}}(u,v) = \frac{1}{2} \int (\nabla u)^T \nabla v dx .$$

To prove (2.28) we first note that for each $x \in W_j \backslash B_\phi$ there exists a neighbourhood D_x of x such that $\phi | D_x$ is a homeomorphism. Let $\{D_i\} = \{D_{x_i}\}$ be a countable family of such neighbourhoods covering $W_j \backslash B_\phi$. Then by partition of unity on $W_j \backslash B_\phi$ any $h \in C_0^\infty(U)$ can be written

$$h = \Sigma h_i \text{ on } W_j \backslash B_\phi ,$$

where $h_i \in C_0^\infty(D_i)$. Thus there exists $g_i \in C_0(\phi(D_i))$ such that $h_i = g_i \circ \phi$ on D_i. Hence

$$\mathcal{E}((f \circ \phi) \cdot \chi_{W_j}, h) = \frac{1}{2} \int_{W_j} \nabla (f \circ \phi)^T \cdot a \cdot \nabla h \, dx$$

$$= \frac{1}{2} \sum_i \int_{D_i} \nabla (f \circ \phi)^T \cdot (\phi')^{-1} \cdot ((\phi')^{-1})^T \cdot \nabla (g_i \circ \phi) \cdot J_\phi \, dx$$

$$= \frac{1}{2} \sum_i \int_{D_i} ((\nabla f)^T \cdot \nabla g_i) \circ \phi \cdot J_\phi \, dx = \frac{1}{2} \sum_i \int_{\phi(D_i)} (\nabla f)^T \cdot \nabla g_i \, dy$$

$$= \frac{1}{2} \sum_i \int_{\phi(D_i)} (-\Delta f) \cdot g_i \, dy = \frac{1}{2} \sum_i \int_{D_i} ((-\Delta f) \cdot g_i) \circ \phi J_\phi \, dx$$

$$= \int_{W_j} \left(-\frac{1}{2}\Delta f\right) \circ \phi \cdot h \cdot J_\phi \, dx = (-(\hat{A}f) \circ \phi, h) ,$$

where (,) denotes inner product in $L^2(U, dm)$, with $dm = J_\phi dx$ as before. Since this holds for all h we have proved (2.28).

The proof that (2.28) implies that ϕ is $X_t - B_t$ pathpreserving is a slight variation of the argument given in [22]. For completeness we give the details.

Let $\tau = \tau_W$. Choose $g \in C_0^\infty(\mathbf{R}^n)$. On $\overline{W}$ we may write $g = \Sigma f_i$, where $f_i \in C_0^\infty(V_{y_i})$ as above. Then by Dynkin's formula (Lemma 2.1) we can find a properly exceptional set N_1 such that if $x \in \overline{W} \backslash N_1$, $y = \phi(x)$ we have for all i and all $t \geq 0$

$$\tilde{E}^y[f_i(Z_{t \wedge \tau})] = E^x[(f_i \circ \phi)(X_{t \wedge \tau})] \tag{2.29}$$

$$= \sum_j E^x[(f_i \circ \phi) \cdot \chi_{W_j}(X_{t \wedge \tau})]$$

$$= \sum_j (f_i \circ \phi)\chi_{W_j}(x) + \sum_j E^x[\int_0^{t \wedge \tau} \hat{A}[f_i] \circ \phi \cdot \chi_{W_j}(X_s) ds]$$

$$= (f_i \circ \phi)(x) + \sum_j E^x[\int_0^{t \wedge \tau} \hat{A}[f_i] \circ \phi \cdot \chi_{W_j}(X_s) ds] \qquad \text{(by (2.28))}$$

$$= f_i(\phi(x)) + E^x[\int_0^{t \wedge \tau} \hat{A}[f_i](\phi(X_s)) \, ds]$$

$$= f_i(\phi(x)) + \tilde{E}^y[\int_0^t (\hat{A}f_i)(Z_s) \, ds \cdot \chi_{\{t \leq \tau\}}] + \tilde{E}^y[\int_0^\tau (\hat{A}f_i)(Z_s) \, ds \cdot \chi_{\{t > \tau\}}] .$$

Adding over all i we see that this holds with f_i replaced by f. Similarly, Dynkin's formula applied to B_t gives

$$\tilde{E}^y\left[f(Z_t) \cdot \chi_{\{t > \tau\}}\right] = E^x\left[\hat{E}^{\phi(X_\tau)}[f(B_{t-\tau})] \cdot \chi_{\{t > \tau\}}\right]\tau\} \tag{2.30}$$

$$= E^x[f(X_\tau) \cdot \chi_{\{t>\tau\}}] + E^x\left[\hat{E}^{\phi(X_t)}\left[\int_0^{t-\tau} (\hat{A}f)(B_r)\,dr\right]\chi_{\{t>\tau\}}\right]$$

$$= E^x[f(X_\tau)\chi_{\{t>\tau\}}] + E^x\left[\hat{E}^{\phi(X_\tau)}\left[\int_\tau^{t} (\hat{A}f)(B_{s-\tau})\,ds\right]\chi_{\{t>\tau\}}\right] .$$

Since

$$\tilde{E}^y[f(Z_{t\wedge\tau})] = \tilde{E}^y[f(Z_t) \cdot \chi_{\{t\le\tau\}}] + E^x[f(\phi(X_\tau)) \cdot \chi_{\{t>\tau\}}]$$

we get by adding (2.29) and (2.30)

$$\tilde{E}^y[f(Z_t)] = f(\phi(x)) + \int_0^t \tilde{E}^y[(\hat{A}f)(Z_s)]\,ds \,. \tag{2.31}$$

Similarly, Dynkin's formula applied to B_t gives

$$\hat{E}^y[f(B_t)] = f(y) + \int_0^t \hat{E}^y[(\hat{A}f)(B_s)]\,ds \,. \tag{2.32}$$

So by uniqueness we conclude (see Lemma 2.5 in [22]) that

$$\tilde{E}^y[f(Z_t)] = \hat{E}^y[f(B_t)] \quad \text{for all } t \ge 0 \,.$$

As in [5] we now proceed by induction to show that if $f_1,\dots,f_k \in C_0^\infty(\mathbb{R}^n)$ there exists a properly exceptional set $N_k \subset G$ such that

$$\tilde{E}^y[f_1(Z_{t_1}) \cdots f_k(Z_{t_k})] = \hat{E}^y[f_1(B_{t_1}) \cdots f_k(B_{t_k})] \tag{2.33}$$

for all $t_i \ge 0$, $x \in U \setminus N_k$.

By choosing $\{f_k\}_{k=1}^\infty$ to be a dense sequence in $C_0(\mathbb{R}^n)$ and putting $N = \bigcup_{k=1}^\infty N_k$ we obtain Theorem 2.3.

Just as in Theorem 2 in [5] we may now obtain the following extension of Theorem 2.3 (notation as in Theorem 2.3):

THEOREM 2.4. *Let* ϕ *be as in Theorem 2.3. Then*

$$\phi^*(\omega) = \lim_{t\to\tau} \phi(X_t) \text{ exists a.s. on } \{\tau < \infty\} \tag{2.34}$$

wrt. P^x, *for quasi-all* $x \in U$.

Moreover, if we define $(Z_t, \tilde{P}^y)$ *as in (2.24), (2.25) but with* τ_W *replaced by* τ_U *and* $\phi(X_\tau)$ *replaced by* ϕ^* *then* Z_t *is identical in law to n-dimensional Brownian motion* B_t.

Remark. Theorem 2.4 may be regarded as a result about the existence of boundary values of ϕ. From (2.24) we see that if $t < \tau = \tau_U$ then $Z_t \in \phi(U)$. Therefore

$$\tau \leq \tau_{\phi(U)}\ , \tag{2.35}$$

where $\tau_{\phi(U)} = \tau^B_{\phi(U)}$ is the first exit time from $\phi(U)$ for B_t. In particular, if ϕ is bounded then $\tau < \infty$ a.s. and therefore

$$\phi^*(\omega) = \lim_{t\to\tau} \phi(X_t) \text{ exists a.s.} \tag{2.36}$$

Remark. Since we know that no killing of X_t occurs inside U we know that if $\tau < \infty$ then X_t must approach ∂U as $t \to \tau$. Therefore Theorem 2.4 is a genuine boundary value result, valid for all quasiregular functions satisfying (2.9). Note however, that it does not immediately give the existence of asymptotic values, since we do not know in general if $\lim_{t\to\tau} X_t$ exists. But in the case when $n = 2$ we have additional information. See §3 below.

An immediate consequence of Theorem 2.4 is the following:

COROLLARY 2.5. *Let* ϕ, X_t *be as in Theorem 2.3. Let* $F \subset \mathbb{R}^n$ *be a* *polar set for Brownian motion (i.e.* $P^x[\exists t > 0; B_t \in F] = 0$ *for all* $x \in \mathbb{R}^n$). *Then* $\phi^{-1}(F)$ *is a properly exceptional set for* X_t. *This raises the question how to describe the properly exceptional sets for* X_t. *They coincide with the sets* H *such that* $\mathrm{Cap}(H) = 0$, *where* Cap *is the capacity associated to the Dirichlet form* $\mathcal{E}$ *([10], Theorem 4.3.1). Again we refer to §3 for the special case* $n = 2$.

A biproduct of Theorem 2.3 of independent interest is the following:

THEOREM 2.6. *Let* $\phi: U \subset \mathbb{R}^n \to V = \phi(U)$ *be a homemorphism and assume that* $\phi \in \mathrm{ACL}^n$ *and* $J_\phi > 0$ *a.e. in* U. *Let* $\mathcal{E}, \hat{\mathcal{E}}$ *be regular Dirichlet forms on* $L^2(U, J_\phi dx)$ *and* $L^2(V, dy)$ *such that* $C_0(U) \subset \mathcal{D}[\mathcal{E}]$, $C_0(V) \subset \mathcal{D}[\hat{\mathcal{E}}]$, *with associated Hunt processes* (X_t, Ω, P^x) *and* $(Y_t, \hat{\Omega}, \hat{P}^y)$ *whose generators are* $A, \hat{A}$ *respectively. Then the following are equivalent:*

(i) $\mathcal{E}(f \circ \phi, g \circ \phi) = \hat{\mathcal{E}}(f,g)$ *for all* $f, g \in C_0(V)$

(ii) $f \in C_0(V) \cap \mathcal{D}[\hat{A}] \Rightarrow f \circ \phi \in \mathcal{D}[A]$ and $A[f \circ \phi] = \hat{A}[f] \circ \phi$

(iii) ϕ *is* $X_t - Y_t$ *pathpreserving, without time change.*

Proof. (i) $\Rightarrow$ *(ii)*: Let $f \in C_0(V) \cap \mathcal{D}[\hat{A}]$. Then

$$\mathcal{E}(f \circ \phi, g \circ \phi) = \hat{\mathcal{E}}(f,g) = (-\hat{A}f, g) = \int_V (-\hat{A}f) \cdot g \, dy$$

$$= \int_U (-(\hat{A}f) \circ \phi)(g \circ \phi) J_\phi \, dx = (-(\hat{A}f) \circ \phi, g \circ \phi)$$

for all $g \in C_0(V)$. This proves that $f \circ \phi \in \mathcal{D}[A]$ and $A[f \circ \phi] = \hat{A}[f] \circ \phi$.

(ii) $\Rightarrow$ *(i)* is proved by reversing the above argument.

(ii) $\Rightarrow$ *(iii)*: This proof is similar to the proof of Theorem 2.3, after (2.28) is established.

(iii) $\Rightarrow$ *(ii)*: Assume (iii) holds, Let $f \in C_0(V) \cap \mathcal{D}[\hat{A}]$.

Then if $y = \phi(x)$

$$\frac{E^x[(f \circ \phi)(X_t)] - f(\phi(x))}{t} = \frac{\hat{E}^y[f(Y_t)] - f(y)}{t}$$

$$\to \hat{A}[f](y) \text{ in } L^2(V,dy) \text{ as } t \to 0 .$$

Therefore

$$\frac{E^x[(f \circ \phi)(X_t)] - f(\phi(x))}{t} \to \hat{A}[f](\phi(x)) \text{ in } L^2(U,J_\phi dx)$$

as $t \to 0$, which proves (ii).

§3. The case when n = 2

If $n = 2$ we have much additional information. The reasons for this are:

a) Condition (2.9) is trivially satisfied for all quasiregular ϕ.

b) Recall that the Dirichlet form $\mathcal{E}$ associated to a quasiregular ϕ satisfying (2.9) (and with corresponding Hunt process (X_t,Ω,P^x)) is given by

$$\mathcal{E}(u,v) = \frac{1}{2} \int_U (\nabla u)^T \cdot a \cdot \nabla v \cdot dx \text{ on } L^2(U,J_\phi dx) , \tag{3.1}$$

where $a = J_\phi \cdot (\phi')^{-1} \cdot ((\phi')^{-1})^T$.

Now define

$$\bar{\mathcal{E}}(u,v) = \frac{1}{2} \int_U (\nabla u)^T a \nabla v \cdot dx \text{ on } L^2(U,dx) \tag{3.2}$$

and let $(\bar{X}_t,\bar{\Omega},\bar{P}^x)$ be the associated Hunt process. Then X_t can be obtained from $\bar{X}_t$ by the following time change:

Put

$$\beta_t = \int_0^t J_\phi(\bar{X}_s)\, ds$$

and let $\alpha_t = \inf\{s; \beta_s > t\}$. By the connection between $\mathcal{E}$ and $\bar{\mathcal{E}}$ it follows that

$$X_t = \bar{X}_{\alpha_t} \tag{3.3}$$

(see [10], (5.5.17) p. 169). Thus from Theorem 2.3 we conclude that ϕ is $\bar{X}_t - B_t$ pathpreserving, with time change rate J_ϕ. The advantage with this formulation is that when $n = 2$ we can say more about the process $\bar{X}_t$:

The generator $\bar{A}$ of $\bar{X}_t$ is given by

$$A = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j}\right), \quad \text{(in the sense of distributions)}$$

where $a = (a_{ij}) = J_\phi(\phi')^{-1}((\phi')^{-1})^T$, and this operator is *uniformly elliptic* in $U \subset \mathbb{R}^2$, because by (1.4)

$$K_0^{-1}|\xi|^2 \leq J_\phi|(\phi')^{-1}\xi|^2 = \sum_{i,j=1}^{2} a_{ij}\, \xi_i\, \xi_j \leq K_I|\xi|^2$$

for all $\xi \in \mathbb{R}^2$.

Therefore, if B_t denotes Brownian motion in $\mathbb{R}^2$ we see that the following holds:

For all subsets H of U we have

$$K_0^{-1}\, C_{\bar{X}}(H) \leq C_B(H) \leq K_I\, C_{\bar{X}}(H)\ ,$$

where $C_{\bar{X}}(W) = \inf\{\bar{\mathcal{E}}(f,f); f \in C_0^\infty(G); f \geq 1 \text{ on } W\}$ is the *capacity* of W w.r.t. $\bar{X}_t$ if $W \subset U$ is open and

$$C_{\bar{X}}(H) = \inf\{C_{\bar{X}}(W); W \text{ open}, \ W \supset H\}$$

for general H (and similarly for C_B) [10].

In particular,

$$X_t \text{ and } B_t \text{ have the same properly exceptional sets .} \tag{3.4}$$

Moreover, by uniform ellipticity (only the right hand side inequality is needed here) we know (see [13] Theorem A or Comparison Theorem in the survey article [7]) the following:

If ϕ is (non-constant) quasiregular on the whole of $\mathbf{R}^2$ then $\bar{X}_t$ is *recurrent*, i.e. for all non-empty open sets $W \subset \mathbf{R}^2$ we have

$$P^x[\exists t > 0; \bar{X}_t \in W] = 1 \tag{3.5}$$

for quasi-all $x \in \mathbf{R}^2$.

We first illustrate Theorem 2.4 by using it to give a proof of the following well known result:

COROLLARY 3.1. *(The Picard theorem for quasiregular functions.) Let* ϕ *be a non-constant quasiregular function on* $\mathbf{R}^2$. Then $\mathbf{R}^2 \setminus \phi(\mathbf{R}^2)$ *contains at most one point.*

Proof. The proof follows the proof of Davis [6] of the Picard theorem for analytic functions using Brownian motion. We only have to check that his proof extends to our case:

First note that in this case $\tau = \tau_U^{\bar{X}} = \infty$. So by Theorem 2.4 we have that

$$\phi^*(\omega) = \lim_{t\to\infty} \phi(X_t) \text{ exists a.s. on } \{\beta_\infty < \infty\} .$$

Since $\bar{X}_t$ is recurrent and ϕ is non-constant we know that a.s. this limit does not exist. Therefore

$$\beta_\infty = \infty \quad \text{a.s. } P^x \ , \qquad \text{for quasi-all } x \ .$$

So by Theorem 2.3 and the definition (2.24) we know that

$$Z_t = \phi(\bar{X}_{\alpha_t}); \quad 0 \leq t < \infty$$

is 2-dimensional Brownian motion. In particular, since $\phi(\bar{X}_{\alpha_t})$ of course never hits $\mathbb{R}^2 \setminus \phi(\mathbb{R}^2)$ the same must be true a.s. for Z_t.

Suppose $\mathbb{R}^2 \setminus \phi(\mathbb{R}^2)$ contains at least two point y_1, y_2. Then we know that Z_t – and hence $\phi(\bar{X}_{\alpha_t})$ - gets more and more tangled up in its winding about these two points (Ito & McKean [14]). So by the recurrence of $\bar{X}_t$ and the fact that in $\mathbb{R}^2$ every closed curve is homotopic to 0, we get a contradiction just as in [6].

We proceed to prove some apparently new results about quasiregular functions. First we recall some useful properties of the process $\bar{X}_t$ on $U \subset \mathbb{R}^2$:

As explained in [13] one may combine local existence results by Kanda [15] and Kunita [16] with the globalization method of Courrege and Priouret [4] to construct a *minimal* diffusion process whose generator coincide with the uniformly elliptic generator A in (3.1) of X_t. (That the process is minimal means that its transition semigroup $\tilde{T}_t$ satisfies $\tilde{T}_t f \leq T_t f$ for all f and all semigroups T_t with generator A.) From now on we will assume that $\bar{X}_t$ is chosen to be this minimal diffusion (as before killed when it leaves U). Then we know:

$$\text{If } U \text{ is bounded then } \tau_U < \infty \ \text{ a.s. } P^x \text{ for all } x \in U \ . \tag{3.6}$$

Suppose $U \subset \mathbb{R}^2$ has non-polar complement, i.e. $C_B(\mathbb{R}^2 \backslash U) > 0$. The Green function $\bar{G}(x,y)$ of $\bar{X}_t$ defined by

$$\int_U f(y)\bar{G}(x,y)\, dy = E^x\left[\int_0^{\tau_U} f(\bar{X}_t)\, dt\right]; \qquad f \in C_0(U) \tag{3.7}$$

satisfies the following property:

For all $x \in U$ there exists a neighbourhood W of x and constants c_1, c_2 such that

$$c_1 \log \frac{1}{|x-y|} \leq \bar{G}(x,y) \leq c_2 \log \frac{1}{|x-y|} \tag{3.8}$$

for all $y \in W$. (See Aronson [1], Theorem 1) (The communication property) For all non-empty $W \subset U$ and $x \in U$

$$P^x[\exists t < \tau_U,\ \bar{X}_t \in W] > 0 \tag{3.9}$$

$$\bar{X}_t \text{ is a Feller process, } i.e. \tag{3.10}$$

$$x \to E^x[f(\bar{X}_t)]$$

is continuous for all $f \in C_0(U)$. This allows us to replace "quasi-all x" by "all x" in Theorem 2.3.

From (2.35) and (3.3) we see that

$$\beta_\tau \leq \tau^B_{\phi(U)}$$

and since by (3.10)

$$E^x[\beta_\tau] = E^x\left[\int_0^\tau J_\phi(\bar{X}_s)\, dx\right] = \int_U J_\phi(y)\bar{G}(x,y)\, dy \text{ we obtain}$$

COROLLARY 3.2. *Suppose* $\mathbb{R}^2 \backslash U$ *is non-polar and* ϕ: $U \to \mathbb{R}^2$ *is* *quasiregular such that*

$$E^z[\tau^B_{\phi(U)}] < \infty$$

for all $z \in \phi(U)$. *(This occurs for example if* ϕ *is bounded.) Then*

$$\int_U J_\phi(y)\bar{G}(x,y)\, dy < \infty \quad \text{for all } x \in U .$$

Combining this with the local estimate (3.8) for the Green function, we obtain:

COROLLARY 3.3. *Suppose* ϕ: $U \subset \mathbb{R}^2 \to \mathbb{R}^2$ *is quasiregular. Then*

$$J_\phi(y) \log \frac{1}{|x-y|} \in L^L_{\text{loc}}(dy) \quad \textit{for all } x \in U .$$

We may of course extend the operator A to a uniformly elliptic operator on $\mathbb{R}^2$ by putting $a_{ij} = \delta_{ij}$ outside U. This gives a corresponding extension of $\bar{X}_t$ to the whole of $\mathbb{R}^2$. Thus we see that

$$\bar{X}_\tau = \lim_{t \to \tau_U} \bar{X}_t \text{ exists a.s.}$$

We define the *X-harmonic measure* $\lambda^x = \lambda^x_{\bar{X}}$ of $\bar{X}$ (wrt. U) by

$$\lambda^x(F) = P^x[\bar{X}_{\tau_U} \in F] \text{ for } F \subset U, \quad x \in U . \tag{3.11}$$

By Moser's Harnack inequality [20] we see that for every compact $M \subset U$ there exists $c < \infty$ s.t.

$$\frac{d\lambda^x}{d\lambda^y} \le c \tag{3.12}$$

for all $y \in M$.

LEMMA 3.4. *If* g *and* U *are bounded then*

$$\tilde{g}(x) = E^x[g(\bar{X}_\tau)]$$

is a continuous function of x.

Proof. First assume that g is continuous and U is a Lipschitz domain. Then there exists $u \in C(\bar{U})$ such that

$$\bar{A}u = 0 \text{ in } U$$

$$u = g \text{ on } \partial U$$

(see [17]).

By Dynkin's formula we have

$$u(x) = E^x[g(\bar{X}_\tau)] ,$$

which proves the Lemma in this case.

If g is just assumed to be bounded choose $W \subset\subset U$ and continuous functions g_n such that

$$g_n \to g$$

boundedly, pointwise a.e. wrt. λ^x, for $x \in W$. Then by (3.12)

$$E^x[g_n(\bar{X}_\tau)] \to E^x[g(\bar{X}_\tau)]$$

uniformly in W. So $E^x[g(\bar{X}_\tau)]$ is continuous for all bounded g.

If U is not a Lipschitz domain choose a Lipschitz domain $V \subset\subset U$. Then by the strong Markov property ($\mathcal{M}_{\tau_V}$ is the σ-algebra generated by $X_{s \wedge \tau_V}; s \geq 0$)

$$\tilde{g}(x) = E^x[g(\bar{X}_\tau)] = E^x[E^x[g(\bar{X}_\tau)|\mathcal{M}_{\tau_V}]] \,. = E^x[E^{\bar{X}_{\tau_V}}[g(\bar{X}_\tau)]]$$

So $\tilde{g} = E^x[\tilde{g}(\bar{X}_{\tau_V})]$ and by the above $\tilde{g}$ is continuous in V. That completes the proof.

COROLLARY 3.5. *(A Rado theorem for quasiregular functions.) Let* $U \subset \mathbf{R}^2$ *be open and* F *a relatively closed subset of* U. *Suppose* ϕ *is a bounded quasiregular function on* $U \backslash F$ *such that*

$$\mathrm{cap}(\mathrm{Cl}(\phi, F)) = 0 , \tag{3.13}$$

where cap denotes logarithmic capacity and $\mathrm{Cl}(\phi, F)$ *is the cluster set of* ϕ *at* F. *Then* ϕ *extends to a quasiregular function on* U.

Proof. We adopt the proof in [21]. Condition (3.13) says that $\mathrm{Cl}(\phi, F)$ is a.s. never hit by 2-dimensional Brownian motion. Therefore

$$P^x[\bar{X}_{\tau_{U \backslash F}} \in F] = 0 \text{ for all } x \in U \setminus F ,$$

i.e. F has $\bar{X}$-harmonic measure 0 wrt. $U \backslash F$. Define as in Theorem 2.4

$$\phi^*(\omega) = \lim_{t \to \tau_{U \backslash F}} \phi(\bar{X}_t) = \lim_{t \to \tau_U} \phi(\bar{X}_t)$$

By Dynkin's formula we have

$$\phi(x) = E^x[\phi^*] \text{ for } x \in U \backslash F .$$

Define

$$\tilde{\phi}(x) = E^x[\phi^*] \text{ for } x \in U .$$

Then by the strong Markov property we have, for $x \in W \subset\subset U$,

$$\tilde{\phi}(x) = E^x[\tilde{\phi}(\bar{X}_{\tau_W})] .$$

So by Lemma 3.4 $\tilde{\phi}$ is continuous in U. Therefore $\tilde{\phi}$ is quasiregular in U, since F has zero area.

Finally we consider the question of boundary values for quasiregular functions:

COROLLARY 3.6. *Let* $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ *be quasiregular. Assume that*

$$\operatorname{cap}(\mathbb{R}^2 \setminus \phi(U)) > 0 .$$

Then

$$\lim_{t \to \tau} \phi(\bar{X}_t) \text{ exists a.s. } P^x \text{ for all } x \in U . \tag{3.14}$$

a) In particular, if U is bounded then ϕ has asymptotic values a.e. on ∂U wrt. $\bar{X}_t$-harmonic measure $\lambda_{\bar{X}}$.

b) If in addition U is a Lipschitz domain then ϕ has asymptotic values on a dense set of points in ∂U.

c) If U is a C^1-domain and the matrix $[a_{ij}]$ in (3.1) extend continuously to $\bar{U}$ such that its normal modulus of continuity $\eta(t)$ satisfies the Dini-type condition

$$\int_0 \frac{\eta^2(t)}{t}\, dt < \infty \tag{3.15}$$

then ϕ has asymptotic values a.e. on ∂U wrt. arc length.

Remark. If $\nu(y)$ is the outer normal direction to $y \in \partial U$ then η is defined by

$$\eta(t) = \sup\{|a_{ij}(y - r\nu(y)) - a_{ij}(y)|; y \in \partial U, 1 \le i,j \le 2 , \ r > 0\} .$$

Proof. The statement (3.14) is just (2.36). Using known properties of $\bar{X}$-harmonic measure $\lambda_{\bar{X}}$ (see Lemma 2.1 in [3]) we obtain b), and c) follows from the condition in [9] that $\lambda_{\bar{X}}$ is absolutely continuous wrt. arc length on ∂U.

Some open problems

This paper raises some interesting questions about the behaviour of the processes X_t in $U \subset \mathbf{R}^n$ for $n \geq 3$. For example:

1) If $X_t \to \partial U$ as $t \to \tau$ a.s. P^x, when will the limit

$$X_\tau = \lim_{t \to \tau_U} X_t$$

exist a.s. P^x?

2) If the limit X_τ in 2) exists a.s. P^x we can define the X-harmonic measure λ_X^x on ∂U as in (3.11). What are the metric properties of λ_X^x? For example, under reasonable conditions on ∂U can one relate λ_X^x to Hausdorff measures?

3) What are the properly exceptional sets of X_t? Can they be described by metric conditions?

4) If we define $J_\phi(x)$ pointwise as in [11] by

$$J_\phi(x) = \limsup_{r \to 0} \frac{\mathrm{Vol}(\phi(D(x,r)))}{\mathrm{Vol}(D(x,r))}$$

where $D(x,r) = \{y \in \mathbf{R}^n; |y - x| < r\}$, is the set

$$N = \{x; J_\phi(x) = 0\}$$

a properly exceptional set for X_t?

5) Can one prove an n-dimensional version of Corollary 3.3, for example by replacing $\log 1/|x - y|$ by $|x - y|^{2-n}$?

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