SOME INTRINSIC AND EXTRINSIC CHARACTERIZATIONS OF THE PROJECTIVE SPACE

by

W. FULTON*, S. KLEIMAN**, R. PIENE and H. TAI

It is surprising in view of the work of the Italian school of algebraic geometry 70 years ago that the following theorem is new - new even in the special case characterizing the Veronese surface in the complex projective 5-space.

THEOREM. *Over an algebraically closed field of any characteristic, the nth Veronese embedding of* $\mathbb{P}^r$ *is the one and only immersion* $f:X \to \mathbb{P}^N$ *where* X *is a smooth, irreducible* r-*fold and* $N = \binom{n+r}{r}-1$, *such that the nth osculating space at every point* x *of* X *is all of* $\mathbb{P}^N$.

Recall that the nth osculating space at x is, by definition, the linear subspace of $\mathbb{P}^N$ determined by the first N+1 partial derivatives of f taken with respect to a system of local parameters for X at x and evaluated at x.

Suppose that $X = \mathbb{P}^r$. The nth Veronese embedding is given in affine coordinates centered at x by

$$f(x_1,\ldots,x_r) = (x_1,\ldots,x_r,x_1^2,x_1x_2,\ldots,x_r^2,x_1^3,\ldots,x_r^n).$$

Hence the first N+1 derivatives evaluated at x form the standard frame for $\mathbb{P}^N$. Thus the nth osculating space is all of $\mathbb{P}^N$.

For any X and f the nth osculating space at x is, in other words, the space determined by the fiber at x of the following natural map of sheaves on X:

$$a : O_X^{N+1} \to P_X^n(L)$$

where $L = f^*O(1)$ and where the target is the twisted sheaf of principal parts (see Piene [1977], §2 and §6). Hence the osculating space is all of $\mathbb{P}^N$ if and only if a is surjective at x, or equivalently, an isomorphism at x.

Recall [EGA IV, 16.10.1] that there is a natural exact sequence,

$$(1) \quad 0 \to (S^n\Omega_X^1)\otimes L \xrightarrow{b} P_X^n(L) \to P_X^{n-1}(L) \to 0.$$

It yields via a straightforward calculation the following relation among the first Chern classes:

$$(2) \quad c_1(P_X^n(L)) = \binom{n+r}{r+1}c_1(\Omega_X^1)+\binom{n+r}{r}c_1(L).$$

Assume now that a is an isomorphism at every point x. Form the composition $a^{-1}b$, dualize it and tensor with L. The result is a surjection,

$$L^{N+1} \twoheadrightarrow S^nT_X.$$

Since L is ample, therefore S^nT_X is ample and hence T_X is ample, by Propositions 2.2 and 2.4 of Hartshorne [1966]. Consequently, by Theorem 8 of Mori [1979], $X = \mathbb{P}^r$.

Finally, since a is an isomorphism, (2) yields the relation:

$$(3) \quad nc_1(T_X) = (r+1)c_1(L).$$

Since $X = \mathbb{P}^r$, therefore $L = O_X(n)$. Hence f is the nth Veronese embedding, possibly followed by a projection and then an inclusion. However, each nth osculating space is all of $\mathbb{P}^N$. So f is simply the Veronese embedding.

There is another proof that $X = \mathbb{P}^r$ when a is an isomorphism, which works in the following three cases:
(i) $r = 1, 2$; (ii) $r = 3$ and the characteristic is 0;
(iii) n and $r+1$ are relatively prime and the characteristic is 0. In any case, this proof reduces the problem to establishing the following conjecture, which may be of interest in its own right (for other, related, conjectures, see Fujita [1980]).

CONJECTURE: Over an algebraically closed field of any characteristic, a smooth, irreducible r-fold X is isomorphic to $\mathbb{P}^r$ if an anticanonical divisor $-K$ is ample and if either one of the following hypotheses is satisfied:

(i) there exists a divisor H such that $(r+1)H$ is numerically equivalent to $-K$;
(ii) $\int c_1(O_X(-K))^r = (r+1)^r$.

The other proof is this. First, (3) says that $-K$ is ample. So, if $r = 1$, then $X = \mathbb{P}^r$. Second, (3) implies that, if $H = v(-K)+uD$ where $un+v(r+1) = 1$ and $O_X(D) = L$, then $(r+1)H$ is linearly equivalent to $-K$. Third, the following argument shows that all the Chern numbers of X are the same as those of $\mathbb{P}^r$ and that

$$d = \int c_1(L)^r$$

is equal to n^r for $r = 2$ in any characteristic and for any r in characteristic 0.

Since a is an isomorphism, the following numerical relations hold:

$$\int c_1^{i_1} \cdots c_j^{i_j} \cdot c_j(P_X^n(L)) = 0$$

where $c_i = c_i(T_X)$ and where $i_1+2i_2+\cdots+ji_j+j = r$. There is one relation for each Chern number $\int c_1^{i_1} \cdots c_r^{i_r}$ (where $i_1+2i_2+\cdots+ri_r = r$). Use (3) to simplify the relations so that they involve only the Chern numbers and d. Then the relations are independent, because, ordered via the lexicographical order on the r-tuples $(i_1,\ldots,i_r)$, each one involves a Chern number that does not appear in any of the following ones.

Now the Riemann-Roch theorem gives another relation, which expresses $\chi(O_X)$ in terms of the Chern numbers. It is independent of the other relations, because it does not involve d. Hence the relations determine the Chern numbers and d. So these numbers are those of $\mathbb{P}^r$ and $d = n^r$, provided $\chi(O_X) = 1$.

Suppose that X is a surface. Then (1) yields the formula

$$c_2(P^n_X(L)) = \binom{n+3}{4}(c_2 + \frac{n^2-1}{3} c_1^2 - 2nc_1 \cdot c_1(L) + 3c_1(L)^2).$$

Hence the relations are as follows:

$$3c_2+(n^2-1)c_1^2 = 9d$$
$$n^2c_1^2 = 9d$$
$$c_2+c_1^2 = 12\chi(O_X).$$

These relations imply that $\chi(O_X) \geqslant 1$. On the other hand,

$$h^2(O_X) = h^0(K) \quad \text{and} \quad h^0(K) = 0$$

by duality and by the ampleness of $-K$. Since $h^0(O_X) = 1$, it follows that $\chi(O_X) = 1$.

In characteristic 0, Kodaira's vanishing theorem yields $h^i(O_X) = 0$ for $i \geqslant 1$ for any r. Since $h^0(O_X) = 1$, therefore $\chi(O_X) = 1$.

It now remains to prove the following result.

PROPOSITION. (1) <u>(Fujita) Under hypothesis</u> (i), <u>the conjecture holds in characteristic</u> 0.
(2) <u>Under hypothesis</u> (ii), <u>the conjecture holds for</u> $r = 1,2$ <u>in any characteristic and for</u> $r = 3$ <u>in characteristic</u> 0.

Indeed, (2) holds trivially for $r = 1$, by the classification of Del Pezzo surfaces for $r = 2$ (e.g., Theorem 24.4 (i) of Manin [1974]) and by the classification of Fano 3-folds for $r = 3$ (Iskovskih [1977, 1978] and Mori-Mukai [1981]). As to (1), $h^i(K+jH) = 0$ for $i \geqslant 1$ and $j \geqslant 1$ by Kodaira's vanishing theorem and for $i = 0$ and $j \leqslant r$ because $-K$ is ample and $(K+jH).(-K)^{r-1} < 0$. Furthermore, $\chi(O_X) = 1$. Since $\chi(mH)$ is of degree r, therefore it must be $\binom{m+r}{r}$. Therefore $H^r = 1$ and

$h^0(H) = r+1$. By Theorem 1 of Goren [1968], $X = \mathbb{P}^r$. (While Goren does not assume X to be Cohen-Macaulay, he uses this hypothesis implicitly in the last line of the proof of Lemma 2. This interesting characterization of $\mathbb{P}^n$ has been rediscovered at least twice after Goren, by Kobayashi and Ochiai [1973] and by Fujita [1975]. Earlier Hirzebruch and Kodaira [1957], Theorem 6, gave a weaker form of Goren's result, which is insufficient for our purposes.) Fujita [1975] proved (1), although he used linear equivalence in place of numerical equivalence. He determined the polynomial $\chi(mH)$ much as above, and he reproved Goren's theorem. Kollár [1981] gave a similar proof, using the stronger form of Mori's theorem instead of Goren's theorem.

REMARK. The Hilbert polynomial of the nth Veronese embedding of $\mathbb{P}^r$ does not always suffice to characterize the embedding. For example, the surface $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^5$ embedded by $O(2,2)$ has the same Hilbert polynomial (in fact the same Hilbert function) as the Veronese surface in $\mathbb{P}^5$. Conceivably, the case $n = 2$, $r = 2$ is the only case in which the embedding is not characterized by its Hilbert polynomial.

References.

Fujita, T. [1975]: "On the structure of polarized varieties with Δ-genera zero", J. Fac. Sci. Univ. Tokyo Sect. IA Math. 22 (1975), 103-115.

Fujita, T. [1980]: "On topological characterizations of complex projective spaces and affine linear spaces", Proc. Japan Acad., 56, Ser. A (1980), 231-234.

Goren, R. [1968]: "Characterization and algebraic deformations of projective space", J. Math. Kyoto Univ. 8-1 (1968), 41-47.

Grothendieck, A. [EGA IV]: Eléments de géométrie algébrique IV, Publ. Math. I.H.E.S. No. 32 (1967).

Hartshorne, R. [1966]: "Ample vector bundles", Publ. Math. I.H.E.S. 29 (1966), 63-94.

Hirzebruch, F. and Kodaira, K. [1957]: "On the complex projective spaces", J. Math. pures et appl. Ser. 9, 36 (1957), 201-216.

Iskovskih, V. A. [1977]: "Fano 3-folds I", Izv. Akad. Nauk. 41 (1977) = Engl. transl. Math. USSR, Izv. 11 (1977), 485-527.

Iskovskih, V. A. [1978]: "Fano 3-folds II", Izv. Akad. Nauk. 42 (1978) = Engl. transl. Math. USSR, Izv. 12 (1978), 469-506.

Kobayashi, S. and Ochiai, T. [1973]: "Characterizations of complex projective spaces and hyperquadrics", J. Math. Kyoto Univ. 13-1 (1973), 31-47.

Kollár, J. [1981]: "Fano varieties of large dimension and large index", Vestnik Moskovskogo Univ. No. 3 (1981), 31-34.

Manin, Yu. I. [1974]: <u>Cubic Forms</u>, North Holland Publishing Company, Amsterdam-London 1974.

Mori, S. [1979]: "Projective manifolds with ample tangent bundles", Ann. of Math. 110 (1979), 593-606.

Mori, S. and Mukai, S. [1981]: "Classification of Fano 3-folds with $B_2 \geq 2$", manuscr. math. 36 (1981), 147-162.

Piene, R. [1977]: "Numerical characters of curves in projective n-space", Real and complex singularities, Oslo 1976, Ed. P. Holm, Sijthoff & Noordhoff, Alpen aan den Rijn 1977, 475-495.

William Fulton
Dept. of Math.
Brown University
Providence, RI 02912
U.S.A.

Steven L. Kleiman
Dept. of Math.
M.I.T.
Cambridge, MA 02139
U.S.A.

Ragni Piene
Matematisk institutt
PB 1053 Blindern
Oslo 3
Norway

Hsin-sheng Tai
Dept. of Math.
University of Notre Dame
Notre Dame, IN 46556
U.S.A.