O N   S U B T R A N S V E R S A L I T Y

by

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The notion of subtransversality is due to Aldo Andreotti and was introduced in [1]. The definition is algebraic rather than geometric and goes well with certain standard operations in analytic geometry such as blowing up. However, it was felt that in the smooth case and in particular in the situation studied in [1] the notion of subtransversality or rather σ-subtransversality should have a simple geometric meaning. The present paper shows that this is the case (theorem 1.1 and theorem 3.1). In particular it sharpens and elucidates theorem (19.3) in [1].

1. Preliminaries and statements. We recall a few concepts from [1]. Let X and Y be smooth (i.e. C^∞-) manifolds, $\dim X > 0$, and let A and B be closed submanifolds of X and Y. We denote by $C^\infty(X,A;Y,B)$ the set of smooth maps $g:X \to Y$ such that $g(A) \subseteq B$. This is a closed subset of $C^\infty(X,Y)$ in the Whitney topology (= the fine C^∞-topology).

Furthermore, denote by $C_x^\infty(X)$ the local ring of germs of smooth functions at $x \in X$. An ideal $I \subseteq C_a^\infty(X)$ is regular of codimension k if I has k generators $h_1, h_2, \ldots, h_k$ such that $dh_1 \wedge \ldots \wedge dh_k \neq 0$. This requires I to be a proper ideal of $C_a^\infty(X)$. In addition we consider $I = C_a^\infty(X)$ to be a regular ideal of codimension k for any integer k. Then $V(I) = \{x \in (X,a) \mid h(x) = 0 \ \forall h \in I\}$ is the germ of a smooth submanifold of X at a of codimension k (empty if $I = C_a^\infty(X)$). Clearly a mapping $g:X \to Y$ is transverse to B at $a \in X$ if and only if $C_a^\infty(X)\cdot g^*I(B)_{g(a)}$ is a regular ideal of codimension k, where k is the codimension of B at $g(a)$ and $I(B)_{g(a)} \subseteq C_{g(a)}^\infty(Y)$ is the ideal of smooth germs at $g(a)$ vanishing on B.

Next, let $g \in C^\infty(X,A;Y,B)$ and let $a \in A$; then $C_a^\infty(X)\cdot g^*I(B)_{g(a)} \subseteq I(A)_a$. Consider the conductor ideal $c_g(I(A)_a, I(B)_{g(a)}) \subseteq C_a^\infty(X)$. By definition $h \in c_g(I(A)_a, I(B)_{g(a)})$ if and only if $h \cdot I(A)_a \subseteq C_a^\infty(X)\cdot g^*I(B)_{g(a)}$. We say that g is subtransverse to B at a if $c_g(I(A)_a, I(B)_{g(a)})$ is regular of codimension equal the codimension of B at $g(a)$, and strongly subtransverse to B at a if $c_g(I(A)_a, I(B)_{g(a)}) + I(A)_a$ is regular of codimension equal the sum of the codimensions of A and B at a and b.

Finally, let $\tilde{X}$ be the blow-up of X along A and $\sigma:\tilde{X} \to X$ the collapse mapping. Then $\tilde{X}$ is canonically a smooth manifold with $\tilde{A} = \sigma^{-1}(A)$ a codimension one submanifold, [2], §3. A mapping $g \in C^{\infty}(X,A;Y,B)$ is <u>(strongly) σ-subtransverse</u> to B at a if $g \circ \sigma$ is (strongly) subtransverse to B at any point of $\sigma^{-1}\{a\}$.

We shall look at the case where g is a product mapping $f \times f : N \times N \to P \times P$ and A and B the diagonals Δ_N and Δ_P, respectively. In this case the geometric content of the definitions is given by the following.

<u>Theorem</u> 1.1. <u>Let</u> $f:N \to P$ <u>be a smooth mapping. Then the statements</u>

(i) $f \times f$ <u>is σ-subtransverse to</u> Δ_P <u>at all points of</u> Δ_N

(ii) $f \times f$ <u>is strictly σ-subtransverse to</u> Δ_P <u>at all points of</u> Δ_N

(iii) Tf <u>is transverse to</u> O_P <u>outside</u> O_N

<u>are equivalent.</u>

Here $Tf:TN \to TP$ is the tangent bundle mapping, and O_N and O_P are the zero-sections of TN and TP.

The theorem is essentially a corollary of theorems 2.2 and 3.1 of section 2 and 3. Theorem 3.1 gives yet another characterization of σ-subtransversality.

2. <u>Double points and residual singularities</u>. Let $W = W(N)$ be the blow-up of $N \times N$ along the diagonal Δ_N. Then W is obtained from $N \times N$ by suitably replacing Δ_N with PTN, the projectivized tangent bundle of N, see for instance [2], §4. Set $N \times N - \Delta_N = W_1$ and $PTN = W_2$ so that $W = W_1 \cup W_2$.

We construct a smooth manifold $E = E(N,P)$ over W. First, set $E = E_1 \cup E_2$ where

$$E_1 = \{(x,x',y,y') \mid x,x' \in N,\ y,y' \in P,\ x \neq x'\}$$
$$E_2 = \{(x,l,y,\phi) \mid x \in N,\ y \in P,\ l \in PT_xN,\ \phi \in \mathrm{Hom}(l,T_yP)\}.$$

Then there is a natural projection π of E onto W defined by

$$\pi(x,x',y,y') = (x,x') \qquad \text{(on } E_1\text{)}$$

$$\pi(x,l,y,\phi) = (x,l) \qquad \text{(on } E_2\text{)}$$

Secondly, for every smooth mapping $f:X \to Y$ there is an induced mapping $\hat{f}:W \to E$, which is a section of π, defined by

$$\hat{f}(x,x') = (x,x',f(x),f(x')) \qquad \text{(on } W_1\text{)}$$

$$\hat{f}(x,l) = (x,l,f(x),Tf|l) \qquad \text{(on } W_2\text{)}$$

When P is a point, then $E(N,P) = W(N)$ (as a set), and π is the identity mapping.

We need a smooth structure on E. First notice that E_1 and E_2 are naturally smooth manifolds of dimensions $2n+2p$ and $(2n-1)+2p$ over the smooth manifolds W_1 and W_2. In fact $E_1 = (N \times N - \Delta_N) \times P \times P$. As for E_2 let LTN be the tautological line bundle over PTN, and $\mathrm{Hom}\,(LTN,TP)$ the corresponding vector bundle over $PTN \times P$; then $E_2 = \mathrm{Hom}\,(LTN,TP)$.

__Lemma__ 2.1. $E = E(N,P)$ __has a canonical smooth structure compatible with that of__ E_1 __and__ E_2, __such that__ $\pi \in C^\infty(E,W)$ __and__ $\hat{f} \in C^\infty(W,E)$ __for any__ $f \in C^\infty(N,P)$.

__In particular__ $E(N,P) = W(N)$ __(as a manifold) when__ P __is a point__.

__Proof__. Consider the case $N = \underline{\underline{R}}^n$, $P = \underline{\underline{R}}^p$. Define $A_k \subset E$, $1 \leq k \leq n$, by $A_k = A_{k1} \cup A_{k2}$ where

$$A_{k1} = \{(x,x',y,y') \in E_1 \,|\, x_k \neq x'_k\}$$

$$A_{k2} = \{(x,l,y,\phi) \in E_2 \,|\, l_k \neq 0\}$$

and $(l_1,\dots,l_n)$ are homogeneous coordinates for l. Evidently $E = A_1 \cup \dots \cup A_n$.

Next, define mappings $\alpha_k : A_k \to \underline{\underline{R}}^n \times \underline{\underline{R}}^n \times \underline{\underline{P}}^{n-1} \times \underline{\underline{R}}^p \times \underline{\underline{R}}^p$ $(1 \leq k \leq n)$ by

$$\alpha_k(x,x',y,y') = (x,x',\underline{\underline{R}}(x'-x),y,(y'-y)/(x'_k-x_k)) \qquad (\text{on } A_{k1})$$

$$\alpha_k(x,l,y,\phi) = (x,x,l,y,\phi(l_1/l_k,\dots,l_n/l_k)) \qquad (\text{on } A_{k2}).$$

Clearly α_k is injective for all k. We topologize A_k so that α_k is a homeomorphism onto its image. Then $A_k \cap A_l$ is an open subset of A_k and A_l for each k and l, as is quickly checked, and the topology induced by A_k on $A_k \cap A_l$ coincides with the topology induced by A_l since the mappings $\alpha_l \circ \alpha_k^{-1}$ are continuous and therefore homeomorphisms. Consequently there is a unique topology on E such that each space A_k occurs as an open subspace of E. It is easy to see that E is a Hausdorff space.

We show that $\alpha_k(A_k)$ is a $(2n+2p)$-dimensional smooth submanifold of $\underline{\underline{R}}^{2n} \times \underline{\underline{P}}^{n-1} \times \underline{\underline{R}}^{2p}$. Set $U_k = \underline{\underline{R}}^{2n} \times \underline{\underline{P}}_k^{n-1} \times \underline{\underline{R}}^{2p}$ where $\underline{\underline{P}}_k^{n-1}$ is the affine open coordinate set $\{L \in \underline{\underline{P}}^{n-1} \mid L_k \neq 0\}$ in $\underline{\underline{P}}^{n-1}$. Then $\alpha_k(A_k) \subset U_k$ for $k = 1,\dots,n$; in fact (X,X',L,Y,Y') is in $\alpha_k(A_k)$ if and only if $L_k \neq 0$ and $(X'_i-X_i)L_k = L_i(X'_k-X_k)$ for all $i \neq k$.

Define $\theta_k : U_k \to \underline{\underline{R}}^{n-1}$ by $\theta_k(X,X',L,Y,Y') =$
$\frac{1}{L_k}(L_k(X'_1-X_1)-L_1(X'_k-X_k), L_k(X'_2-X_2)-L_2(X'_k-X_k),\dots)$ where the k-th component $(\equiv 0)$ is omitted. Then θ_k is a submersion onto $\underline{\underline{R}}^{n-1}$. Since $\alpha_k(A_k) = \theta_k^{-1}\{o\}$, it follows that $\alpha_k(A_k)$ is a smooth submanifold of U_k, hence of $\underline{\underline{R}}^{2n} \times \underline{\underline{P}}^{n-1} \times \underline{\underline{R}}^{2p}$, of codimension $n-1$.

By means of α_k we pull back the smooth structure on $\alpha_k(A_k)$ to A_k. We now need to show that A_k and A_l induce the same smooth structure on the open set $A_k \cap A_l$ for any two k and l.

But this holds since the mappings $\alpha_1 \circ \alpha_k^{-1}$ are smooth and therefore diffeomorphisms. Thus $E = A_1 \cup \ldots \cup A_n$ receives a smooth structure in which $A_1, \ldots, A_n$ are open submanifolds.

For $p = 0$, i.e. $P = \{o\}$, we clearly get $E = W$. (Alternatively define the smooth structure on $W(\underline{\underline{R}}^n)$ as that of $E(\underline{\underline{R}}^n, o)$.) Throughout the paper we use primed letters A_k', α_k', ... in the particular case $E = W$, i.e. primed letters refer to W. Then we have a commutative diagram:

$$\begin{array}{ccc} A_k & \xrightarrow{\alpha_k} & \underline{\underline{R}}^{2n} \times \underline{\underline{P}}^{n-1} \times \underline{\underline{R}}^{2p} \\ \pi\downarrow & & \downarrow pr \\ A_k' & \xrightarrow{\alpha_k'} & \underline{\underline{R}}^{2n} \times \underline{\underline{P}}^{n-1} \end{array}$$

showing that π is smooth on A_k, $a \leqslant k \leqslant n$. Thus π is smooth (on E).

Finally we need to check that $\hat{f}:W \to E$ is smooth for smooth f. Obviously it suffices to check this at a point $(x,1) \in W_2$. Let k be such that $(x,1) \in A_k'$. We have $\hat{f}(A_k') \subset A_k$ and therefore a map $\tau_k:\alpha_k'(A_k') \to \alpha_k(A_k)$ defined by the commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow{\alpha_k} & \alpha_k(A_k) \\ \hat{f}\uparrow & & \uparrow \tau_k \\ A_k' & \xrightarrow{\alpha_k'} & \alpha_k'(A_k') \end{array}$$

Extend τ_k to a mapping $T_k:U_k' \to U_k$ in the following way: Write

$$f(X')-f(X) = \sum_{j=1}^{n} (X_j'-X_j)F_j(X,X')$$

with the $F_j(X,X') = \int_0^1 \frac{\partial f}{\partial x_j}(X+t(X'-X))dt$, so that $F_j(X,X) = \frac{\partial f}{\partial x}(X)$, $1 \leq j \leq n$. Now set

$$T_k(X,X',L) = (X,X',L,f(X),\frac{1}{L_k}\sum_{j=1}^{n} L_j F_j(X,X')).$$

Then T_k extends τ_k as claimed. Since T_k is smooth, so is τ_k. Consequently $\hat{f}$ is smooth.

This concludes the proof in the affine case $N = \mathbb{R}^n$, $P = \mathbb{R}^p$. The extension to the flat case, where N and P are diffeomorphic to $\mathbb{R}^n$ and $\mathbb{R}^p$, is by transport of structure; the result is easily seen to be independent of the choice of diffeomorphisms. The extension to the general case is then by patching over coordinate neighbourhoods in N and P, thereby constructing the germ of E along E_2 compatible with E_1, and joining the result to E_1. The procedure is straightforward. We omit further details.

<u>Remark 1</u>. By construction E_1 and E_2 are built in as submanifolds of E. Since E_1 is an open submanifold, E_2 is a <u>closed</u> submanifold of E.

2. There is also a smooth projection $\pi_2 : E \to P \times P$ defined by

$$\pi_2(x,x',y,y') = (y,y') \qquad (\text{on } E_1)$$
$$\pi_2(x,1,y,\phi) = (y,y) \qquad (\text{on } E_2).$$

More symmetrically we have the smooth projections

$$N \times N \xleftarrow{\pi_1} E \xrightarrow{\pi_2} P \times P$$

where $\pi_1 = \sigma \circ \pi$. Thus the extension $\hat{f}$ of f fits into the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\hat{f}} & E \\ \sigma\downarrow & & \downarrow\pi_2 \\ N\times N & \xrightarrow{f\times f} & P\times P \end{array}$$

We next define a special submanifold Z of E. Let $Z = Z_1 \cup Z_2$, where

$$Z_1 = \{(x,x',y,y') \in E_1 | y = y'\}$$
$$Z_2 = \{(x,1,y,\phi) \in E_2 | \phi = 0\}.$$

Then $Z \subset E$; we claim that Z is a closed submanifold of E. First notice that $Z \cap E_1 = Z_1$ is certainly a closed submanifold of E_1. Thus if $a \in E$ is in the closure of Z, then $a \in E(U,V)$ for suitable coordinate patches $U \subset N$, $V \subset P$. Thus $a \in Z$ if $Z \cap E(U,V)$ is closed in $E(U,V)$. Moreover, Z is a submanifold of E locally around a if $Z \cap E(U,V)$ is a submanifold of $E(U,V)$. Consequently we are reduced to substanciating our claim in the affine case $N = \underline{\underline{R}}^n$, $P = \underline{\underline{R}}^p$. Again, in the affine case it suffices to show that $Z \cap A_k$ is a closed submanifold of A_k for $k = 1,\ldots,n$. Let $\rho:\underline{\underline{R}}^{2n}\times\underline{\underline{P}}^{n-1}\times\underline{\underline{R}}^{2p} \to \underline{\underline{R}}^p$ be the projection to the last p coordinates. It is quickly checked that $\rho|\alpha_k(A_k)$ has constant rank p, i.e. that $\rho\circ\alpha_k$ has constant rank p. But $Z \cap A_k = (\rho\circ\alpha_k)^{-1}\{o\}$, and so $Z \cap A_k$ is indeed a closed submanifold of A_k.

Notice that Z_2 is a closed submanifold of Z. This follows by the construction of Z, or by the fact that Z_2 is a closed submanifold of E_2 and therefore of E.

We shall devote the rest of this section to characterizing the smooth maps $f:N \to P$ such that $\hat{f}$ is transverse to Z.

<u>Proposition</u> 2.2. <u>Let</u> $f:N \to P$ <u>be a smooth mapping and</u> w <u>a point of</u> W. <u>Then</u> $\hat{f} \pitchfork Z$ <u>at</u> w <u>if and only if</u>

(i) $f\times f \pitchfork \Delta_P$ <u>at</u> w, <u>in case</u> $w = (a,a') \in W_1$.

(ii) $Tf \pitchfork O_P$ <u>at</u> l, <u>in case</u> $w = (a,l) \in W_2$.

The second statement means that $Tf:TN \to TP$ is transverse to the zero-section $O_P \subset TP$ at $\nu \in TN$ for some (hence any) non-zero vector ν in $l \subset T_aN$.

<u>Proof</u>. The case $w \in W_1$ is trivial. Assume $w = (a,l) \in W_2$. By restricting to suitable coordinate patches around a and $f(a)$, it suffices to consider the case $N = \mathbb{R}^n$, $P = \mathbb{R}^p$, $a = o$, $f(a) = o$. Then the tangent bundles TN and TP are trivial, and we can write $\nu = (x,v)$, $Tf(\nu) = (f(x),Df(x)v)$. In fact we may assume the coordinatization at a and $f(a)$ performed such that f has the form

$$f(x) = (x_1,\ldots,x_r;\phi(x))$$

with $\phi:\mathbb{R}^n \to \mathbb{R}^{p-r}$ a smooth mapping such that $\phi(o) = o$, $D\phi(o) = o$. (Thus r is the rank of f at the origin.)

Now, let $v = (v',v'') \in \mathbb{R}^r\times\mathbb{R}^{n-r}$ be a non-zero vector and $l \in \mathbb{P}^{n-1} = PT_o\mathbb{R}^n$ the line spanned by v. We have $\hat{f}(o,l) = (o,l,f(o),Df(o)|l)$ with

$$Df(o) = \left[\begin{array}{c|c} I_r & o \\ \hline o & o \end{array}\right]$$

Thus $Df(o)v = v'$ and so

(i) $\hat{f}(o,l) \notin Z$ if and only if $v' \neq o$.

Suppose $v' = o$. With notations as before choose k such that $(o,l) \in A'_k$; then $\hat{f}(o,l) \in A_k$. Recall that $\rho\circ\alpha_k:A_k \to \mathbb{R}^p$ is a submersion and that $Z \cap A_k = (\rho o \alpha_k)^{-1}\{o\}$. Thus

$$\begin{aligned} &\hat{f} \pitchfork Z \quad \text{at} \quad (o,1) \\ \iff \quad &\rho o \alpha_k o \hat{f}: A_k' \to \underline{\underline{R}}^p \quad \text{is submersive at } (o,1) \\ \iff \quad &\rho o \tau_k: \alpha_k'(A_k') \to \underline{\underline{R}}^p \quad \text{is submersive at } \alpha_k'(o,1) \\ \iff \quad &\rho o T_k o i_k': \alpha_k'(A_k') \to \underline{\underline{R}}^p \quad \text{is submersive at } \alpha_k'(o,1) \end{aligned}$$

Here $i_k': \alpha_K'(A_k') \to U_k'$ is the inclusion mapping,

$$\begin{array}{ccccccc} \alpha_k'(A_k') & \xrightarrow{i_k'} & U_k' & \xrightarrow{T_k} & U_k & \xrightarrow{\rho} & \underline{\underline{R}}^p \\ & & \downarrow \theta_k' & & & & \\ & & \underline{\underline{R}}^{n-1} & & & & \end{array}$$

Consequently we want to determine the range of $D(\rho o T_k o i_k')(\alpha_k'(o,1))$. Since $i_k'(\alpha_k'(A_k')) = \theta_k'^{-1}\{o\}$, we have range $Di_k'(\alpha_k'(o,1)) = \ker D\theta_k'(\alpha_k'(o,1))$, with $\alpha_k'(o,1) = (o,o,(v_1,\dots,v_n))$. Now $D\theta_k'(\alpha_k'(o,1))$ has the matrix block form $[M \;\; -M \;\; O]$ with

$$M = \begin{bmatrix} -I & V_k' & O \\ O & V_k'' & -I \end{bmatrix}$$

where as usual I means an identity matrix and O a zero matrix. V_k' and V_k'' are the column matrices

$$\begin{bmatrix} V_{1k} \\ \vdots \\ V_{k-1,k} \end{bmatrix} \text{ and } \begin{bmatrix} V_{k+1,k} \\ \vdots \\ V_{n,k} \end{bmatrix}$$

where $V_{ik} = v_i/v_k$. (Recall that $v_k \neq 0$ since $(o,1) = (o,(v_1,\dots,v_n)) \in A_k'$.) In particular $V_{1k} = \dots = V_{rk} = 0$ since $v' = o$.

It now follows by straightforward computation that range $D(\rho \circ T_k \circ i_k')(\alpha_k'(o,1))$ is spanned by the r first standard basis vectors $e_1,\dots,e_r$ in $\underline{\underline{R}}^p$ together with the n vectors $(1 \leq i \leq n)$

$$\left(0,\dots,0,\ \sum_{j=r+1}^{n} V_{jk}\frac{\partial^2\phi_1}{\partial x_i \partial x_j}(0),\dots,\ \sum_{j=r+1}^{n} V_{jk}\frac{\partial^2\phi_{p-r}}{\partial x_i \partial x_j}(0)\right)$$

Finally, upon multiplication with the non-zero constant v_k the coefficient V_{jk} in the last n vectors is replaced by v_j. Thus

(ii) $\hat{f}(o,1) \in Z$ and $\hat{f} \pitchfork Z$ at $(o,1)$ if and only if the vectors

$$\left(0,\dots,0,\ \sum_{j=r+1}^{n} v_j\frac{\partial^2\phi_1}{\partial x_i \partial x_j}(0),\dots,\ \sum_{j=r+1}^{n} v_j\frac{\partial^2\phi_{p-r}}{\partial x_i \partial x_j}(0)\right)$$

form a set of rank $p-r$ in $\underline{\underline{R}}^p$.

To complete the proof of proposition 2.2 we now appeal to the following elementary

Lemma 2.3. *Let* $f:\underline{\underline{R}}^n \to \underline{\underline{R}}^p$ *be a smooth mapping of the form* $f(x) = (x_1,\dots,x_r,\phi(x))$ *with* $\phi(o) = o$ *and* $D\phi(o) = 0$. *Let* $v = (v',v'')$ *be a non-zero vector of* $T_o\underline{\underline{R}}^n = \underline{\underline{R}}^r \times \underline{\underline{R}}^{n-r}$.

Then $Tf \pitchfork O_{\underline{\underline{R}}^p}$ at $(o,v) \in T\underline{\underline{R}}^n$ *if and only if either*

(i) $v' \neq 0$ (*then* $Tf(o,v) \notin O_{\underline{\underline{R}}^p}$)

or

(ii) $v' = 0$ (*then* $Tf(o,v) \in O_{\underline{\underline{R}}^p}$) *and the matrix*

$$\begin{bmatrix} \sum_{j=r+1}^{n} v_j \frac{\partial^2\phi_1}{\partial x_1 \partial x_j}(0) & \cdots & \sum_{j=r+1}^{n} v_j \frac{\partial^2\phi_1}{\partial x_n \partial x_j}(0) \\ \vdots & & \\ \sum_{j=r+1}^{n} v_j \frac{\partial^2\phi_{p-r}}{\partial x_1 \partial x_j}(0) & \cdots & \sum_{j=r+1}^{n} v_j \frac{\partial^2\phi_{p-r}}{\partial x_n \partial x_j}(0) \end{bmatrix}$$

has rank $p-r$.

The proof of lemma 2.3 is left to the discretion of the reader.

3. Subtransversality. The purpose of this section is to prove the following result.

Theorem 3.1. *Let* $f:N \to P$ *be a smooth mapping. Then* $f \times f$ *is σ-subtransverse to* Δ_P *at all points of* Δ_N *if and only if* $\hat{f} \pitchfork Z$ *on* W_2, *and strongly σ-subtransverse if and only if* $\hat{f} \pitchfork Z_2$ *on* W_2.

Proof. Let $(a,l) \in W_2$ and $b = f(a)$. Let r be the rank of f at a. With respect to suitable coordinate systems at a and b f has the form $f(x) = (x_1,\ldots,x_r,\phi(x))$ with $\phi(o) = o$, $D\phi(o) = 0$, $a = o \in \mathbb{R}^n$, $f(a) = o \in \mathbb{R}^p$.

Let $(l_1,\ldots,l_n)$ be homogenous coordinates for l and assume $l_k \neq 0$, i.e. $l \in \mathbb{P}_k^{n-1}$. Define $s_k:\mathbb{R}^{2n} \times \mathbb{P}^{n-1} \times \mathbb{R}^{2p} \to \mathbb{R}$ by $s_k(X,X',L,Y,Y') = X_k'-X_k$ and let $s_k':\mathbb{R}^{2n} \times \mathbb{P}^{n-1} \to \mathbb{R}$ be equal s_k when $p = 0$.

Then $s_k' \circ i_k' \circ \alpha_k' : A_k' \to \mathbb{R}$ is a submersion, and $W_2 \cap A_k' = (s_k' \circ i_k' \circ \alpha_k')^{-1}\{0\}$. Therefore $I(W_2)_{(o,l)}$ is the principal ideal generated by the germ of $s_k' \circ i_k' \circ \alpha_k'$ at (o,l).

Now let $\psi:\mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ be the difference mapping $\psi(Y,Y') = Y'-Y$, and as before let $\rho:\mathbb{R}^{2n} \times \mathbb{P}^{n-1} \times \mathbb{R}^{2p} \to \mathbb{R}^p$ be the projection to the least p coordinates. Recall the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\hat{f}} & E \\ \sigma\downarrow & & \downarrow\pi_2 \\ N\times N & \xrightarrow{f\times f} & P\times P \end{array}$$

The ideal $I(\Delta_{\underline{\underline{R}}}p)_{(o,o)}$ is generated by the germs of $\phi_1,\dots,\phi_p$ at (o,o). The pullback by the mapping $(f\times f)\circ\sigma$ is therefore generated by the germs of $\phi_j\circ\pi_2\circ\hat{f}$ at $(o,1)$, $j = 1,\dots,p$.

Let $r_k:\underline{\underline{R}}^{2n}\times\underline{\underline{P}}^{n-1}\times\underline{\underline{R}}^{2p}\to\underline{\underline{R}}^{2p}$ be the mapping $r_k(X,X',L,Y,Y') = (Y,Y+(X_k'-X_k)Y')$, $1 \leq k \leq n$. Since $\pi_2|A_k = r_k\circ i_k\circ\alpha_k$, we have $\phi\circ\pi_2\circ\hat{f}|A_k' = (s_k\rho)\circ i_k\circ\alpha_k\circ\hat{f} = (s_k\rho)\circ T_k\circ i_k'\circ\alpha_k' = (s_k'\circ i_k'\circ\alpha_k')(\rho\circ T_k\circ i_k'\circ\alpha_k')$ with T_k as before. The conductor $c_f(I(W_2)_{(o,1)}, I(\Delta_{\underline{\underline{R}}}p)_{(o,o)})$ is therefore the ideal generated by the germs of $\rho_j\circ T_j\circ i_k'\circ\alpha_k'$ at $(o,1)$, $j = 1,\dots,p$.

Finally, let $\nu_k:\underline{\underline{R}}^{2n}\times\underline{\underline{P}}^{n-1}\times\underline{\underline{R}}^{2p}\to\underline{\underline{R}}^{p+1}$ be the mapping $\nu_k(X,X',L,Y,Y') = (X_k'-X_k,Y')$. Then $c_f(I(W_2)_{(o,1)}, I(\Delta_{\underline{\underline{R}}}p)_{(o,o)})+I(W_2)_{(o,1)}$ is the ideal generated by the germs of $\nu_{kj}\circ T_k\circ i_k'\circ\alpha_k'$ at $(o,1)$, $j = 1,\dots,p+1$.

For the first part of the theorem: Suppose $l_k \neq 0$ for some $k \leq r$. On U_k we have

$$\rho_k(T_k(X,X',L)) = \frac{1}{L_k}\sum_{j=1}^{n} L_j\int_0^1 \frac{\partial f_k}{\partial x_j}(X+t(X'-X))dt = 1.$$

Thus $c_f(I(W_2)_{(o,1)}, I(\Delta_{\underline{\underline{R}}}p)_{(o,o)})$ contains the unit element in $C^\infty_{(o,1)}(W)$, and so by our convention is regular of codimension p at $(o,1)$. But we have also $\hat{f}(o,1) \notin Z$ (p. 8 statement (i)).

Suppose on the other hand $l_1 = \dots = l_r = 0$. Then $c_f(I(W_2)_{(o,1)}, I(\Delta_{\underline{\underline{R}}}p)_{(o,o)})$ is regular of codimension p if and only if $\rho\circ T_k\circ i_k'$ is a submersion at $\alpha_k'(o,1)$. But this is equivalent to $\hat{f} \pitchfork Z$ at $(o,1)$ (p. 9).

For the second part of the theorem: Suppose again $l_k \neq 0$ for some $k \leq r$. Then $c_f(I(W_2)_{(o,1)}, I(\Delta_P)_{(o,o)})+I(W_2)_{(o,1)} = C^\infty_{(o,1)}(W)$ and so is regular of codimension $p+1$, and $\hat{f} \pitchfork Z_2$ at $(o,1)$ since $\hat{f}(o,1) \notin Z_2$.

Suppose on the other hand $l_1 = \ldots = l_r = 0$. Then $c_f(I(W_2)_{(o,1)}, I(\Delta_{\mathbb{R}^p})_{(o,o)}) + I(W_2)_{(o,1)}$ is regular of codimension $p+1$ if and only if $\nu_k \circ T_k \circ i'_k$ is a submersion at $\alpha'_k(o,1)$. But the last condition is equivalent to $\hat{f} \cap Z_2$ at $(o,1)$; this follows by an argument analogous to that for the case $f \pitchfork Z$ on p. 9.

It follows that $f \times f$ is strongly σ-subtransverse to Δ_P at all points of Δ_N if and only if $\hat{f} \cap Z_2$ on W_2. This completes the proof of theorem 3.1.

We close section 3 by giving the one piece of information which together with theorem 3.1 yields theorem 1.1.

<u>Lemma</u> 3.2. <u>Let</u> $f: N \to P$ <u>be a smooth mapping</u>. <u>Then</u> $\hat{f} \pitchfork Z$ <u>on</u> W_2 <u>if and only if</u> $\hat{f} \pitchfork Z_2$ <u>on</u> W_2. <u>Moreover</u> $\hat{f} \pitchfork Z_2$ <u>on</u> W_2 <u>if and only if</u> $\hat{f} \pitchfork Z_2$ (<u>on</u> W).

<u>Proof</u>. The last claim is obvious since $\hat{f}(W_1) \cap Z_2 = \emptyset$.

Let $(a,l) \in W_2$ and assume that $\hat{f}(a,l) \in Z_2$. Again, by suitable coordinatizations we may assume $N = \mathbb{R}^n$, $a = o$, $P = \mathbb{R}^p$, $f(a) = o$.

We know that $\hat{f} \pitchfork Z$ at (o,l) if and only if $\rho \circ T_k \circ i'_k : \alpha'(A'_k) \to \mathbb{R}^p$ is a submersion at $\alpha'_k(o,l)$. Since $\rho = pr_2 \circ \nu_k$ where $pr_2 : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}^p$ is the projection, this is equivalent to $\nu_k \circ T_k \circ i'_k$ being transverse to $K = \mathbb{R} \times \{o\} \subset \mathbb{R} \times \mathbb{R}^p$ at $\alpha'_k(o,l)$.

We show that $T_o K \subset \text{range}\, D(\nu_k \circ T_k \circ i'_k)(\alpha'_k(o,l))$. Thus if $\hat{f} \cap Z$ at (o,l), then $\nu_k \circ T_k \circ i'_k$ is a submersion at $\alpha'_k(o,l)$ and so $\hat{f} \pitchfork Z_2$ at (o,l).

As usual let $(l_1,\ldots,l_n)$ be homogenous coordinates for l and set $l_{jk} = l_j/l_k$ when $l_k \neq 0$, $j = 1,\ldots,n$. Define the smooth curve $c:\langle-\varepsilon,\varepsilon\rangle \to \alpha_k'(A_k')$ by $c(t) = (-tl_{1k},\dots,-tl_{nk},tl_{1k},\dots,tl_{nk},1)$; then $c(0) = \alpha_k'(o,1)$. Since

$$\nu_k \circ T_k(X,X',L) = (X_k'-X_k, \sum_{j=1}^{n} L_{jk} \int_0^1 \frac{\partial f}{\partial x_j} (X+s(X'-X))ds)$$

on U_k', we find

$$\nu_k \circ T_k \circ i_k' \circ c(t) = (2t, \sum_{j=1}^{n} l_{jk} \int_0^1 \frac{\partial f}{\partial x_j} (t(2s-1)(l_{1k},\dots,l_{nk}))ds).$$

From this we get

$$\frac{d}{dt} (\nu_k \circ T_k \circ i_k' \circ c)(0) = (2,0,\dots,0) \in T_o K$$

which confirms that $T_o K$ sits in the range of $D(\nu_k \circ T_k \circ i_k')(\alpha_k'(o,1))$. Thus $\hat{f} \pitchfork Z_2$ on W_2 if $\hat{f} \pitchfork Z$ on W_2. The converse is of course trivial.

4. <u>Complements</u>. The following is an easy consequence of theorems 1.1, 2.2 and 3.2.

<u>Proposition</u> 4.1. <u>The smooth mappings</u> $f:N \to P$ <u>such that</u> $\hat{f} \pitchfork Z_2$ <u>form a dense open subset of</u> $C^\infty(N,P)$.

For the condition $\hat{f} \pitchfork Z_2$ is equivalent to $Tf \pitchfork O_P$ outside O_N, and the latter condition is satisfied for an open dense set of mappings f by a standard transversality argument.

The construction E is tailored to the study of the generic double points of f, as indicated by proposition 2.2. Let $D_f \subseteq N$ be the locus of genuine double points of f and $S_f \subseteq N$ the singular locus of f. Thus $x \in D_f$ if $f(x) = f(x')$ for some point $x' \neq x$, and $x \in S_f$ if $\ker Tf_x \neq \{o\}$.

<u>Proposition</u> 4.2. <u>If</u> $f:N \to P$ <u>is a proper smooth mapping such that</u>

$\hat{f} \pitchfork Z_2$, then $\bar{D}_f = D_f \cup S_f$.

Proof. Let $\sigma_1 : W \to N$ be the smooth mapping $pr_1 \circ \sigma$, where $pr_1 : N \times N \to N$ is the projection to the first factor. Then $\sigma_1(x,\xi) = x$ for arbitrary $(x,\xi) \in W_1 \cup W_2$, and so $D_f = \sigma_1(\hat{f}^{-1}(Z_1)), S_f = \sigma_1(\hat{f}^{-1}(Z_2))$. Consequently

$$D_f \cup S_f = \sigma_1(\hat{f}^{-1}(Z)).$$

Since f is proper, $\sigma_1 | \hat{f}^{-1}(Z)$ is also proper. Hence $D_f \cup S_f$ is a closed subset of N; in particular $\bar{D}_f \subseteq D_f \cup S_f$.

It remains to show that $S_f \subseteq \bar{D}_f$. Let $a \in S_f$, so that $(a,l) \in \hat{f}^{-1}(Z_2)$ for a suitable $l \subseteq T_1 N$. Again, by means of coordinate systems at a and $f(a)$ we are reduced to the affine case $a = o \in R^n$, $f(a) = o \in \underline{R}^p$. Choose $k \leq n$ such that $(o,l) \in A'_k$. Since $\hat{f} \cap Z_2$, $\nu_k \circ \alpha_k \circ \hat{f} : A'_k \to \underline{R}^{p+1}$ is a submersion at (o,l), and we may choose a local coordinate system around (o,l) in W in which $\nu_k \circ \alpha_k \circ \hat{f}$ is presented as the standard projection $\nu_k \circ \alpha_k \circ \hat{f}(w_1,\ldots,w_{2n}) = (w_1,\ldots,w_{p+1})$. In this coordinate system, which flattens $W(\underline{R}^n)$ into $\underline{R}^{2n}$ around (o,l), we have $\hat{f}^{-1}(Z_2) = \{w \in R^{2n} | w_1 = \ldots = w_{p+1} = 0\}$ and $\hat{f}^{-1}(Z_1) = \{w \in \underline{R}^{2n} | w_2 = \ldots = w_{p+1} = 0$ and $w_1 \neq 0\}$. Obviously then the origin $o \in \hat{F}^{-1}(Z_2)$ belongs to the closure of $\hat{f}^{-1}(Z_1)$. Backtracking this means that (a,l) belongs to the closure of $\hat{f}^{-1}(Z_1)$. By continuity this implies that $a = \sigma_1(a,l)$ belongs to the closure of $\sigma_1(\hat{f}^{-1}(Z_1))$, i.e. to $\bar{D}_f$. Thus $S_f \subseteq \bar{D}_f$.

This gives at neat proof that $\bar{D}_f = D_f \cup S_f$ is a generic property for proper mappings, satisfied by those mappings $f \in C^\infty_{pr}(N,P)$ such that $Tf \cap O_P$ outside O_N.

One can also prove a general transversality result.

Proposition 4.3. *Let* M *be a smooth submanifold of* E. *The smooth mappings* $f:N \to P$ *such that* $\hat{f} \pitchfork M$ *form a dense subset of* $C^{\infty}(N,P)$. *If* M *or* N *is compact, this subset is open*.

In general the openess property fails unless there is a compactness condition. E.g. proposition 4.1 holds because of the special character of the submanifold Z_2.

We omit the proof of proposition 4.3.

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