

Hypercausal Linear Operators *

by

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Abstract

Several different classes of hypercausal operators are useful in linear system theory. The relationships amongst these classes have not, in all instances, been clarified. It is the purpose of this note to clarify these relationships and to provide, for each pair of classes of hypercausal operators, necessary and sufficient conditions on the Hilbert resolution space to guarantee equality of the classes. In addition, the effect of similarity transforms on each class is discussed.

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In linear system theory the concept of physical realizability, or causality, of an operator corresponds to the mathematical concept of a nest algebra. The reader is referred to [3] and to the bibliography cited therein for a detailed account of the rationale behind the identification of the causal operators as the operators in a nest algebra. Three separate hypercausality concepts are discussed in [3], each to express in some fashion the notion that the present output of a system does not depend upon the present input. The strongest, strict causality, coincides with the Jacobson radical of the nest algebra. The other two are, in order of strength, strong causality (introduced in [2]) and strong strict causality. In between these two lies Larson's ideal R_N^∞ . We shall define all four of these concepts below, using a single coherent scheme, and give necessary and sufficient conditions on the nest for each pair of concepts to coincide.

Throughout this paper, H will denote a separable Hilbert space. A nest (or resolution of the identity) is a subset of the set of orthogonal projections on H which contains 0 and I , is totally ordered under the usual ordering for projections, and is closed in the strong operator topology. The pair (H, N) is called a Hilbert resolution space and the causal operators are, by definition, just the operators in nest algebra, $\text{Alg}N = \{T \in B(H) \mid TP = PTP, \text{ for all } P \in N\}$.

A projection E in $B(N)$ is called an interval from N if E can be written as $E = P - Q$, where $P, Q \in N$ and $Q < P$. If E is an interval then the projections P and Q are uniquely determined. They are called the upper and lower endpoints of E . There is a natural partial order \ll on the set of intervals from N : we say that $E \ll F$ if the upper endpoint of E is a subprojection of (or equal to) the lower endpoint of F .

A partition $P = \{E_i\}_{i \in I}$ is a family of pairwise orthogonal intervals from N such that $\sum_{i \in I} E_i = I$. (The sum converges in the strong operator topology over the net of finite subsets of the index set I .) Since the Hilbert space is separable, the index set I is always finite or countably infinite. If E and F are two orthogonal intervals from N , then either $E \ll F$ or $F \ll E$; consequently, each partition P is totally ordered by \ll . It is easy to construct an example of a partition with any given countable order type. If (P, \ll) is order isomorphic to a subset of the integers, with the usual ordering, then we say that P is an integer ordered partition. If $P' = \{F_j\}_{j \in J}$ and $P = \{E_i\}_{i \in I}$ are partitions, we say that P' is a refinement of P and write $P < P'$ if each F_j is a subprojection of some E_i . This gives a partial order on the family of all partitions. Each of the three families, the set of all partitions of N , the set of integer ordered partitions, and the set of finite partitions becomes a directed set under ordering by refinement. Each of these directed sets will serve as the index set for convergent nets used in the definition of distinct notions of hypercausality.

For finite partitions, the more customary definition of partition can be obtained by replacing the intervals in the partition by the endpoints of the intervals. For integer ordered partitions, the endpoints of the intervals form a generalized partition, as defined in [3, Chapter 2, section C]. In each case the two approaches are equivalent; it is more convenient for us to define partitions in terms of intervals so that we can accommodate arbitrary partitions without change of notation.

If $P = \{E_i\}_{i \in I}$ is a partition of N and $A \in \text{Alg} N$ is a causal operator, let $A_P = \sum_{i \in I} E_i A E_i$. (When infinite, the sum converges in the strong operator topology over the net of finite partial sums.)

For each causal operator A we thereby obtain three distinct nets of operators, depending on whether we take as the index set the finite partitions, the integer ordered partitions, or the arbitrary partitions. A class of hypercausal operators is obtained by considering all causal operators A such that $A_p \rightarrow 0$ with respect to one of these index sets with convergence in one of the five natural topologies on $\text{Alg}N$. Fortunately (at least from the point of view of reducing the tedium), the a priori possibility that there are 15 distinct notions of hypercausality does not, in fact, occur. Indeed, there are at most five (and at least four) separate notions.

In what follows, \lim will denote convergence with respect to the norm topology and (fin)-, (int)-, or (arb)- preceding the word \lim will indicate whether the index set is the directed set of finite partitions, integer ordered partitions or arbitrary partitions. Convergence in the strong operator topology will be denoted by $s\text{-}\lim$ and in the weak operator topology (which we shall have little cause to discuss) by $w\text{-}\lim$. The remaining two topologies, the ultrastrong and the ultraweak, yield nothing new: indeed, the strong and ultrastrong (respectively, weak and ultraweak) topologies agree on bounded sets and each of the nets A_p is bounded. The following proposition further limits the number of hypercausality concepts:

Proposition 1. Let $A \in \text{Alg}N$. Then the following are equivalent:

- (i) $(\text{fin})\text{-}s\text{-}\lim_p A_p = 0$
- (ii) $(\text{int})\text{-}s\text{-}\lim_p A_p = 0$
- (iii) $(\text{arb})\text{-}s\text{-}\lim_p A_p = 0.$

Proof. Let \mathcal{D} denote either the directed set of integer ordered partitions or the directed set of arbitrary partitions. We shall show (i) \Rightarrow (ii) and (i) \Rightarrow (iii) simultaneously; the two arguments are identical and each implication is obtained by giving \mathcal{D} the appropriate interpretation. Assume that (i) holds and let $\varepsilon > 0$ and $x \in H$ be given. We must prove that there exists a partition P in \mathcal{D} such that if Q is a partition in \mathcal{D} which refines P then $\|A_Q x\|^2 < \varepsilon$.

Let $\delta = \varepsilon(1 + \|A\|^2)^{-1}$. Let $P = \{E_1, \dots, E_n\}$ be a finite partition such that for any finite refinement Q' of P , we have $\|A_{Q'} x\|^2 < \varepsilon$. Now suppose that $Q = \{F_j\}_{j \in J}$ is a partition in \mathcal{D} which refines P . We will show that $\|A_Q x\|^2 < \varepsilon$. Since $\sum_{j \in J} F_j = I$, there is a finite subset $J_0 \subseteq J$ such that $\|(\sum_{j \notin J_0} F_j)x\|^2 < \delta$. For each $j \in J_0$, F_j is a subprojection of some E_i in P , hence there exists a finite partition $G = \{G_1, \dots, G_n\}$ such that $\{F_j | j \in J_0\} \subseteq G$ and G refines P . Therefore, $\sum_{j \in J_0} \|F_j A F_j x\|^2 < \sum_{i=1}^n \|G_i A G_i x\|^2 = \|A_G x\|^2 < \delta$. Consequently, we have

$$\begin{aligned} \|A_Q x\|^2 &= \sum_{j \in J} \|F_j A F_j x\|^2 = \sum_{j \in J_0} \|F_j A F_j x\|^2 + \sum_{j \notin J_0} \|F_j A F_j x\|^2 \\ &< \delta + \sum_{j \notin J_0} \|A\|^2 \|F_j x\|^2 = \delta + \|A\|^2 \sum_{j \notin J_0} \|F_j x\|^2 \\ &< \delta + \|A\|^2 \delta = \varepsilon. \end{aligned}$$

This completes the proof that (i) \Rightarrow (ii) and (i) \Rightarrow (iii).

The converses, (ii) \Rightarrow (i) and (iii) \Rightarrow (i) will also be proven simultaneously. So assume either (ii) or (iii), i.e. assume $s\text{-}\lim_P A_P = 0$, where $P \in \mathcal{D}$. We must prove that $(\text{fin})\text{-}s\text{-}\lim_P A_P = 0$. Let $\varepsilon > 0$ and $x \in H$ be given. We must find a finite partition P such that $\|A_P x\|^2 < \varepsilon$ for any finite partition P' which refines P .

Let $Q = \{E_i\}_{i \in I}$ be a partition in \mathcal{D} such that for any refinement Q' in \mathcal{D} , $\|A_{Q', x}\|^2 < \delta$. There exists a finite subset $I_0 \subseteq I$ such that $\left\| \left(\sum_{i \notin I_0} E_i \right) x \right\|^2 = \sum_{i \notin I_0} \|E_i x\|^2 < \delta$. Let P be the finite partition obtained by arranging in order the right and left endpoints of the intervals $E_i, i \in I_0$ and taking successive differences. P is, in fact, the smallest finite partition such that $\{E_i | i \in I_0\} \subseteq P$. Let $P' = \{G_1, \dots, G_n\}$ be any finite partition which refines P . We shall prove that $\|A_{P', x}\|^2 < \epsilon$.

Now, every projection G_j is either a subprojection of some E_i with $i \in I_0$ or is orthogonal to each E_i with $i \in I_0$. Let

$$J_0 = \{j | G_j \leq E_i, \text{ for some } i \in I_0\} \text{ and}$$

$$J_1 = \{j | G_j E_i = 0, \text{ for all } i \in I_0\} = \{j | G_j \leq \sum_{i \notin I_0} E_i\}$$

Then $J_0 \cap J_1 = \emptyset$ and $J_0 \cup J_1 = \{1, 2, \dots, n\}$. Let Q' be a partition in \mathcal{D} which is a common refinement of Q and P' and has the property that $\{G_j | j \in J_0\} \subseteq Q'$. (Such refinements exist since every G_j with $j \in J_0$ is a subprojection of some E_i in P .)

Since Q' refines Q , we have $\|A_{Q', x}\|^2 < \delta$. But $\{G_j | j \in J_0\} \subseteq Q'$; so we obtain $\sum_{j \in J_0} \|G_j A_{G_j} x\|^2 < \delta$. Therefore,

$$\begin{aligned} \|A_{P', x}\|^2 &= \sum_{j=1}^n \|G_j A_{G_j} x\|^2 \\ &= \sum_{j \in J_0} \|G_j A_{G_j} x\|^2 + \sum_{j \in J_1} \|G_j A_{G_j} x\|^2 \\ &< \delta + \|A\|^2 \sum_{j \in J_1} \|G_j x\|^2 \\ &= \delta + \|A\|^2 \left\| \left(\sum_{j \in J_1} G_j \right) x \right\|^2 \\ &= \delta + \|A\|^2 \left\| \left(\sum_{i \notin I_0} E_i \right) x \right\|^2 \\ &< \delta + \|A\|^2 \delta = \epsilon. \end{aligned}$$

This ends the proof of the proposition.

Remark. Essentially the same argument as the one given above, with a few fairly routine modifications, proves the equivalence of the three conditions $(\text{fin})\text{-w-}\lim_P A_p = 0$, $(\text{int})\text{-w-}\lim_P A_p = 0$, and $(\text{arb})\text{-w-}\lim_P A_p = 0$, for $A \in \text{Alg } N$. It is also immediate that, for $A \in \text{Alg } N$, $s\text{-}\lim_P A_p = 0$ implies $w\text{-}\lim_P A_p = 0$. It is not known if, and perhaps not likely that, the converse holds. The three possible limits in the uniform topology are, in general, distinct. They are, however, easier to work with than the strong or weak topology limits. The reason is that if P is any partition which refines Q , then $\|A_p\| \leq \|A_Q\|$. Thus, for each of the three nets, to prove $\lim_P A_p = 0$, it is sufficient to find, for $\epsilon > 0$ given, an appropriate partition P such that $\|A_p\| < \epsilon$.

Definition. Let N be a nest. Define the following families of causal operators:

$$\begin{aligned} R_N &= \{A \in \text{Alg } N \mid (\text{fin})\text{-}\lim_P A_p = 0\}, \\ R_N^{\text{int}} &= \{A \in \text{Alg } N \mid (\text{int})\text{-}\lim_P A_p = 0\}, \\ R_N^\infty &= \{A \in \text{Alg } N \mid (\text{arb})\text{-}\lim_P A_p = 0\}, \\ S_N &= \{A \in \text{Alg } N \mid (\text{fin})\text{-}s\text{-}\lim_P A_p = 0\}. \end{aligned}$$

In view of the remark above, R_N consists of exactly those causal operators which satisfy Ringrose's criterion for membership in the radical of $\text{Alg } N$ [5], thus R_N is precisely the Jacobson radical of $\text{Alg } N$. R_N^{int} is exactly the set of strongly strictly causal operators, as defined in [3, Chapter 2, section B]. As shown in [3], R_N^{int} is a uniformly closed two-sided ideal in $\text{Alg } N$. R_N^∞ , another uniformly closed two-sided ideal, was introduced by Larson in [4] and plays an important role in the study of similari-

ties of nest algebras. S_N is the uniformly closed left ideal of strongly causal operators, as defined in [2] or [3]. With the aid of Proposition 1, the relation between the strongly strictly causal operators and the strongly causal operators now becomes clear:

$R_N^{int} \subseteq S$. Indeed, in view of the remark above, the following relations are evident.:

$$R_N \subseteq R_N^{int} \subseteq R_N^\infty \subseteq S_N.$$

Propositions 2, 3 and 5 below will provide appropriate necessary and sufficient conditions on the nest N to ensure that each containment is, in fact, an equality.

Each of the four ideals above can be viewed as the operators which have, in an appropriate sense, zero diagonal part. The diagonal of a nest algebra is the subalgebra $\text{Alg}N \cap (\text{Alg}N)^*$; the operators in the diagonal are the memoryless operators. If $A \in \text{Alg}N$ and the net A_p is convergent in any of the senses above, then the limit, D , commutes with each projection in N . Thus the limit, when it exists, is in the diagonal and may be thought of as the diagonal (or memoryless) part of A . In this case, of course, $A-D$ belongs to the ideal which corresponds to the sense in which the net converges.

It is instructive to look, in particular, at the behavior "at atoms". An atom is an interval $E=P-Q$ from N where Q is the immediate predecessor of P in the order of the nest. Suppose that E is an atom and that $A \in S_N$. Let x be any vector in E . If P is any partition which contains E , then $A_p x = EAEx$. Since every partition has a refinement which contains E , we see that $\|EAEx\| < \epsilon$, for every $\epsilon > 0$. Thus $EAE = 0$. We may view EAE as the part of the diagonal of A corresponding to the atom E . In particular, suppose that N is purely atomic, i.e. that $I = \bigcup_{i \in I} E_i$, where $P = \{E_i\}_{i \in I}$ is the set of atoms from N . In the net of arbi-

bitrary refinements of N , P is the terminal element. Therefore, $(\text{arb})\text{-}\lim_P A_P$ and $(\text{arb})\text{-s-}\lim_P A_P$ always exist and both are equal to $\sum_{i \in I} E_i A E_i$. Thus, for a totally atomic nest N , $R_N^\infty = S_N$ and each consists of the causal operators with diagonal part zero, i.e. $A \in R_N^\infty = S_N$ if, and only if, $E A E = 0$ for every atom E from N . We now proceed to the propositions which classify when the various ideals of hypercausal operators are equal.

Proposition 2. Let N be a nest. The following are equivalent:

(i) 0 has an immediate successor and I has an immediate predecessor.

(ii) $R_N = R_N^{\text{int}}$.

Proof. Assume (i) holds. It is then clear that any partition P of N must have a first and a last element with respect to the order \ll . If P is integer ordered, then P is necessarily finite. Thus the directed set of finite partitions coincides with the directed set of integer ordered partitions and so $R_N = R_N^{\text{int}}$.

Now assume that (i) does not hold. Suppose, for example, that I has no immediate predecessor (the argument is essentially the same if 0 has no immediate successor). Then there is an increasing sequence $0 < P_1 < P_2 < \dots$ of projections in N which converges strongly to I . Let $E_1 = P_1$ and $E_i = P_i - P_{i-1}$, for $i > 2$. Then $P = \{E_i\}_{i \in \mathbb{N}}$ is an integer ordered partition. Let x_i be a unit vector in E_i , for each i , and let $A = \sum_{i=2}^{\infty} x_i \otimes x_{i-1}$. (The rank one operator $x_i \otimes x_{i-1}$ is defined by $(x_i \otimes x_{i-1})y = \langle y, x_i \rangle x_{i-1}$. It is easy to check that the infinite sum converges in the strong operator topology.) Since any integer ordered partition possesses an integer ordered refinement which is also a refinement of P and

since $A_p=0$, it is clear that $A \in R_N^{\text{int}}$. On the other hand, if Q is a finite refinement and if E is the last interval in Q (namely, the interval which has I as its upper endpoint), then $EE_i=E_i$ for all i greater than some integer i_0 . Therefore $\|EAE\|=1$ and so $\|A_Q\|=1$. Thus $A \notin R_N$ and so $R_N \neq R_N^{\text{int}}$.

Proposition 3. Let N be a nest. The following are equivalent:

(i) Each element of N excepting 0 and I has an immediate predecessor and an immediate successor.

(ii) $R_N^{\text{int}}=R_N^\infty$.

Proof. Condition (i) is equivalent to the statement that N is order isomorphic to a subset of the extended integers, $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$. When this holds N is totally atomic and the set of atoms, P , forms an integer ordered partition. Furthermore, P is the terminal element in the directed net of arbitrary partitions; in particular, every partition is integer ordered. Therefore $R_N^{\text{int}}=R_N^\infty$ whenever (i) holds.

The proof that (ii) implies (i) is, in spirit, similar to the proof of the preceding proposition. Suppose that $P \neq 0, I$ is an element of N with no immediate successor. (An analogous argument works if P has no immediate predecessor.) Then there is a sequence P_n of projections in N such that $P_1=I$, $P_n > P_{n+1}$, for all n , and $\lim_{n \rightarrow \infty} P_n = P$. Let $E_n = P - P_{n+1}$, for all n . Let x_n be a unit vector in E_n and let $A = \sum_{n=1}^{\infty} x_n \otimes x_{n+1}$. Then $A \in \text{Alg} N$ and $\|A\|=1$. Also note that if Q is any projection in N which is greater than P , then $\|(Q-P)A(Q-P)\|=1$.

The set of intervals $P = \{E_n \mid n=1, 2, \dots\} \cup \{P\}$ is a partition of N and it is easy to check that $A_p=0$. Thus $A \in R_N^\infty$. On the other hand, if $Q = \{F_n\}$ is an integer ordered partition, then there is an

integer k such that $F_k = Q_k - R_k$ with $Q_k, R_k \in N$ and $R_k \leq P < Q_k$.
 Therefore $\|F_k A F_k\| = 1$, hence $\|A_{Q_k}\| = 1$. Since Q is an arbitrary
 integer ordered partition, we see that $A \notin R_N^{\text{int}}$. Thus (ii) \Rightarrow (i).

Remark The known fact that $R_N = R_N^\infty$ if, and only if N is a finite
 nest also follows from Propositions 2 and 3.

Proposition 5 will characterize the nests for which $R_N^\infty = S_N$.
 The most essential ingredient is contained in the lemma below. A
continuous nest is a nest which has no atoms. Every continuous
 nest is order isomorphic to the interval $[0,1]$. (Indeed, if x is
 a separating vector for the abelian von Neumann algebra generated
 by N then the mapping $P \rightarrow \langle Px, x \rangle$ is an order isomorphism of N
 onto $[0,1]$.) Thus when N is continuous, we may use $[0,1]$ as an
 index set for the elements of N .

Lemma 4. If N is a continuous nest then R_N^∞ is a proper subset
 of S_N .

Proof. Let $N = \{P_r \mid r \in [0,1]\}$ be a continuous nest. Enumerate the
 rational numbers in $(0,1)$, i.e. write $Q \cap (0,1)$ as a sequence
 $\{r_k \mid k=1,2,3,\dots\}$. We will choose by induction two sequences,
 $(t_n)_{n=1,2,\dots}$ and $(\epsilon_n)_{n=1,2,\dots}$ with the following properties:

(i) $0 < \epsilon_n < \frac{1}{2^{n+1}}$, for all n

(2) The intervals $[t_n - \epsilon_n, t_n + \epsilon_n]$, $n=1,2,3,\dots$ are pairwise
 disjoint subintervals of $[0,1]$.

(3) Each $t_n \in Q \cap (0,1)$. If $t_n = r_h$ and $j < h$ then

$$r_j \in \bigcup_{i < n} [t_i - \epsilon_i, t_i + \epsilon_i].$$

Indeed, let $t_1 = r_1$ and let $\epsilon_1 < 1/4$ be sufficiently small that

$[t_1 - \varepsilon_1, t_1 + \varepsilon_1] \subseteq [0, 1]$. Suppose t_1, \dots, t_{n-1} and $\varepsilon_1, \dots, \varepsilon_{n-1}$ have been chosen satisfying (1)-(3). Let h be the smallest integer such that $r_h \notin \bigcup_{i=1}^{n-1} [t_i - \varepsilon_i, t_i + \varepsilon_i]$ and let $t_n = r_h$. Since the complement of $\bigcup_{i=1}^{n-1} [t_i - \varepsilon_i, t_i + \varepsilon_i]$ in $[0, 1]$ is open and $t_n \neq 0, 1$, there is a number ε_n such that $0 < \varepsilon_n < \frac{1}{2^{n+1}}$ and $[t_n - \varepsilon_n, t_n + \varepsilon_n]$ is disjoint from $\bigcup_{i=1}^{n-1} [t_i - \varepsilon_i, t_i + \varepsilon_i]$.

For each pair of numbers $r, s \in [0, 1]$ with $s < r$, let $E[r, s] = P_s - P_r$. For each $n=1, 2, \dots$, let x_n be a unit vector in $E[t_n, t_n + \varepsilon_n]$ and let y_n be a unit vector in $E[t_n - \varepsilon_n, t_n]$. Let $A = \sum_{n=1}^{\infty} x_n \otimes y_n$. It is easy to check that the sum converges in the strong operator topology, that $A \in \text{Alg} N$ and that $\|A\| = 1$. We shall finish the proof by showing that $A \in S_N$ and $A \notin R_N^{\infty}$.

To prove that $A \in S_N$, let $w \in H$ and $\varepsilon > 0$ be given. Since the projections $E[t_n - \varepsilon_n, t_n + \varepsilon_n]$ are pairwise orthogonal, there is an integer m such that $\sum_{n>m} \|E[t_n - \varepsilon_n, t_n + \varepsilon_n]w\|^2 < \varepsilon$. Let $P = \sum_{n>m} E[t_n - \varepsilon_n, t_n + \varepsilon_n]$ and $Q = \sum_{n \leq m} E[t_n - \varepsilon_n, t_n + \varepsilon_n]$. Then P and Q are disjoint projections and $A = A(P+Q)$. Now let p' be a partition which contains the intervals $E[t_n - \varepsilon_n, t_n]$ and $E[t_n, t_n + \varepsilon_n]$, $n=1, 2, \dots, m$ among its elements. Let $p = \{F_i\}_{i \in I}$ be any refinement of p' . We need merely show that $\|A_p w\|^2 < \varepsilon$.

If $F_i \leq E[t_n - \varepsilon_n, t_n]$ or $F_i \leq E[t_n, t_n + \varepsilon_n]$ then $F_i A F_i = 0$. Let $J = \{i \in I \mid F_i \leq I - Q\}$. Then we have:

$$\begin{aligned} \|A_p w\|^2 &= \sum_{i \in J} \|F_i A F_i w\|^2 = \sum_{i \in J} \|F_i A (P+Q) (I-Q) F_i w\|^2 \\ &= \sum_{i \in J} \|F_i A F_i P w\|^2 \leq \sum_{i \in J} \|F_i P w\|^2 \\ &\leq \|P w\|^2 < \varepsilon. \end{aligned}$$

To show that $A \notin R_N^{\infty}$, we shall prove that $\|A_p\| = 1$ for any partition p . So let $p = \{F_i\}$ be a partition. Each element F_k

in \mathcal{P} is of the form $E[l_k, h_k]$, for uniquely determined elements l_k, h_k in $[0, 1]$. By the choice of the ε_n , the set

$\bigcup_{n=1}^{\infty} [t_n - \varepsilon_n, t_n + \varepsilon_n]$ has Lebesgue measure strictly smaller than 1. On

the other hand, since $\sum_{k=1}^{\infty} F_k = I$, the set $\bigcup_{k=1}^{\infty} (l_k, h_k)$ has measure 1.

Therefore, there exists a number $q \in [0, 1]$ and an index k such that q belongs to the open interval (l_k, h_k) , but does not belong

to $\bigcup_{n=1}^{\infty} [t_n - \varepsilon_n, t_n + \varepsilon_n]$. Since $q \neq h_k$, there is a $\delta > 0$ so that

$(q, q + \delta) \subseteq [l_k, h_k]$. Let r be a rational number in the interval

$(q, q + \delta)$. Since $Q \cap (0, 1) \subseteq \bigcup_{n=1}^{\infty} [t_n - \varepsilon_n, t_n + \varepsilon_n]$, r lies in some inter-

val $[t_n - \varepsilon_n, t_n + \varepsilon_n]$. But $q \notin [t_n - \varepsilon_n, t_n + \varepsilon_n]$, so we must have

$q < t_n - \varepsilon_n < r < q + \delta$. We would like to have $t_n + \varepsilon_n < q + \delta$, but this may not

be true. The situation is easily rectified by repeating the procedure once again: let s be a rational number in the interval

$(q, t_n - \varepsilon_n)$ and let m be such that $s \in [t_m - \varepsilon_m, t_m + \varepsilon_m]$. This time we

obtain

$$q < t_m - \varepsilon_m < s < t_m + \varepsilon_m < t_n - \varepsilon_n < q + \delta.$$

In particular, $E[t_m - \varepsilon_m, t_m + \varepsilon_m] \subseteq E[q, q + \delta] \subseteq F_k$. Therefore, $F_k A F_k x = y_m$;

in particular, $\|F_k A F_k\| = 1$. Thus $\|A_p\| = 1$ and $A \notin R_N^{\infty}$.

Proposition 5 Let N be a nest. The following are equivalent:

(i) N is totally atomic

(ii) $R_N^{\infty} = S_N$.

Proof. The easy implication (i) \Rightarrow (ii) has already been given in the paragraph immediately preceding Proposition 2. So suppose that

N is not totally atomic; we must show $R_N^{\infty} \neq S_N$.

Let $\{E_i\}_{i \in I}$ be the (possibly empty) set of atoms from N .

Let $E = I - \sum_{i \in I} E_i$. By hypothesis, $E \neq 0$. Let K be the range of the projection E . Define a nest N_E on the Hilbert space K by $N_E = \{PE|_K \mid P \in N\}$. Observe that N_E is a continuous nest. Each operator A in $B(K)$ has a unique bounded linear extension to K^\perp which vanishes on the orthogonal complement, K^\perp , of K . We denote this extension by \tilde{A} . Note that $A \in \text{Alg}N_E$ if, and only if, $\tilde{A} \in \text{Alg}N$.

Since N_E is a continuous nest, $R_{N_E}^\infty$ is a proper subset of S_{N_E} . Fix an element A of S_{N_E} which is not in $R_{N_E}^\infty$. We shall show that $\tilde{A} \in S_N$ and $\tilde{A} \notin R_N^\infty$.

To prove that $\tilde{A} \in S_N$, let $x \in H$ and $\epsilon > 0$ be given. If Q is a projection in N_E , then one can, by adding appropriate atoms of N to Q , obtain a projection P in N so that $Q = PE|_K$. If $Q_0 = 0 < Q_1 < \dots < Q_n = I_K$ is a finite subnest of N_E , then we can obtain projections $P_0 = 0 < P_1 < \dots < P_n = I_H$ in N so that $Q_i = P_i E|_K$, $i = 1, \dots, n$. Since $\tilde{A}x = EAEx$, we have

$$\sum_{i=1}^n \|(P_i - P_{i-1})\tilde{A}(P_i - P_{i-1})x\|^2 = \sum_{i=1}^n \|(Q_i - Q_{i-1})A(Q_i - Q_{i-1})Ex\|^2.$$

From these remarks it is clear that $\tilde{A} \in S_N$.

Finally, we need to show that $\tilde{A} \notin R_N^\infty$. Assume the contrary; i.e. assume that $\tilde{A} \in R_N^\infty$. Let $\epsilon > 0$ be given. Then there is a partition $P = \{F_i\}_{i \in I}$ of N such that $\|F_i \tilde{A} F_i\| < \epsilon$, for all i . The set $P_E = \{F_i E \mid i \in I \text{ and } F_i E \neq 0\}$ is a partition of N_E and $\|F_i E A F_i E\| = \|\tilde{F}_i A \tilde{F}_i\| < \epsilon$, for all i . Thus $\tilde{A} \in R_{N_E}^\infty$, contrary to hypothesis. This completes the proof of the proposition.

Remark From Propositions 3 and 5 and the first sentence of the proof of Proposition 3, we see that the strongly strictly causal operators and the strongly causal operators on a nest N coincide if, and only if, N is order isomorphic to a subset of the extended integers $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$.

We conclude this note with a discussion of the effect of similarities on each of the classes of hypercausal operators considered above. The significance of similarities for system theory is indicated by the fact that Larson's theorem [4] that any two continuous nests are similar implies that there exist positive definite hermitian operators which do not admit spectral factorization. (See [3] for a discussion of factorization problems.)

If N is a nest and T is an invertible operator in $B(H)$ then, for each $P \in N$, TPT^{-1} is an idempotent (not necessarily self-adjoint). Let $\phi_T(P)$ be the orthogonal projection on the range of TPT^{-1} . Thus, T maps the range space of P onto the range space of $\phi_T(P)$. Let TN denote the nest $\{\phi_T(P) | P \in N\}$. We say that two nests M and N are similar if $M = TN$ for some invertible $T \in B(H)$. The map $\phi_T: N \rightarrow M$ induced by T is an order isomorphism of N onto M . If ϕ is any order isomorphism of N onto M then ϕ has a natural extension to a map from the set of intervals from N to the set of intervals from M : define $\phi(P-Q)$ to be $\phi(P) - \phi(Q)$. (We denote the extension by the same symbol.) In particular, atoms from N correspond to atoms from M . If corresponding atoms have the same dimension, we say that ϕ preserves dimension. It is evident that each order isomorphism of the form ϕ_T preserves dimension. Recently, Davidson [1] has proven the converse to this: if ϕ is an order isomorphism of N onto M which preserves dimension, then there is an invertible operator T such that $M = TN$ and $\phi = \phi_T$.

Fix a nest N and an invertible operator T and let $M = TN$. Then the two nest algebras $\text{Alg}N$ and $\text{Alg}M$ are similar: $\text{Alg}N = T^{-1}(\text{Alg}M)T$. Furthermore, $R_N = T^{-1}R_M T$ and $R_N^{\text{int}} = T^{-1}R_M^{\text{int}} T$, i.e. the strictly causal operators and the strongly strictly causal operators are preserved by similarities. The first of these two

facts is completely trivial - it follows immediately from the definition of the radical as the intersection of the kernels of all the irreducible representations of the algebra. It follows equally rapidly from the characterization of the radical as the largest ideal consisting entirely of quasi-nilpotent elements. Yet a third proof is available: both similarity results stated above follow from a lemma of Larson [4] which asserts that if E is any interval from N and $K = \|T\| \|T^{-1}\|$, then for any $A \in \text{Alg} N$, $\|EA E\| \leq K \| \phi_T(E) T A T^{-1} \phi_T(E) \|$ and $\| \phi_T(E) T A T^{-1} \phi_T(E) \| \leq K \|EA E\|$. To obtain the two similarity results one need merely observe that if $P = \{E_i\}_{i \in I}$ is a finite or integer ordered partition of N then $\{\phi_T(E_i)\}_{i \in I}$ is a finite or integer ordered partition of M .

If P is an arbitrary partition then it is not necessarily the case that $\{\phi_T(E_i)\}$ is a partition. As a consequence, R_N^∞ need not be preserved by similarities. A detailed discussion of this may be found in [4].

Finally, we turn to S_N . In light of Larson's results on R_N^∞ , it is not surprising that we find that S_N need not be preserved by similarities.

Example. We use the following standard construction to produce a pair of similar nests. If μ is a finite Borel measure on $[0,1]$, let $H_\mu = L^2([0,1], \mu)$. For each $t \in [0,1]$, we let P_t^μ (resp. P_{t-}^μ) denote the multiplication operator by the characteristic function of $[0,t]$ (resp. $[0,t)$). Let N^μ denote nest consisting of all the projections P_t^μ and P_{t-}^μ .

Let ν be a purely atomic measure on $[0,1]$ with support equal to $Q \cap (0,1)$. So, in the nest N^ν , we find that $P_t^\nu + P_{t-}^\nu$ if, and only if $t \in Q \cap (0,1)$. The nest is totally atomic and the atoms are the intervals $E_t^\nu = P_t^\nu - P_{t-}^\nu$, $t \in Q \cap (0,1)$. Let m be Lebesgue

measure on $[0,1]$ and let $\lambda = m + \nu$. In the nest N^λ the atoms are once again the intervals $E_t^\lambda = P_t^\lambda - P_{t-}^\lambda, t \in Q \cap (0,1)$, but this time the nest is not totally atomic. Indeed, $H_\lambda = H_m \oplus H_\nu$ and the sum of the atoms from N^λ is the projection on H_ν , not the identity on the whole Hilbert space H_λ . The map $\phi: N^\nu \rightarrow N^\lambda$ given by $\phi(P_t^\nu) = P_t^\lambda$ and $\phi(P_{t-}^\nu) = P_{t-}^\lambda$, for all t , is an order isomorphism which preserves dimension (all atoms are one dimensional). By Davidson's theorem [4], $\phi = \phi_T$ for some invertible operator T . So $N^\lambda = TN^\nu$ and $\text{Alg} N^\nu = T^{-1}(\text{Alg} N^\lambda)T$. We shall show that $S_{N^\nu} \neq T^{-1}S_{N^\lambda}T$.

Let A be a non-zero operator in $(\text{Alg} N^\lambda) \cap (\text{Alg} N^\lambda)^*$ with the property that $EAE = 0$ for every atom from N^λ . (A is simply a multiplication operator by a function $f \in L^\infty([0,1], \lambda)$ with the property that $f(r) = 0$, for all $r \in Q \cap (0,1)$.) Since A is memoryless, it commutes with each projection in N^λ ; therefore $A_p = A$ for any partition p . Thus $A \notin S_{N^\lambda}$.

Let $B = T^{-1}AT$. Then $B \in \text{Alg} N^\nu$. By Larson's lemma [4], $\|FBF\| \leq \|T\| \|T^{-1}\| \|\phi_T(F)TBT^{-1}\phi_T(F)\| = \|T\| \|T^{-1}\| \|\phi_T(F)A\phi_T(F)\| = 0$, for every atom F from N^ν . Since N^ν is totally atomic $B \in S_{N^\nu}$. But $B = T^{-1}AT \notin T^{-1}S_{N^\lambda}T$, so $S_{N^\nu} \neq T^{-1}S_{N^\lambda}T$ as desired.

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