1. INTRODUCTION

Group von Neumann algebras of discrete groups are an important source of examples of finite von Neumann algebras and several authors ([2], [6], [13], [15], [16]) have studied their natural $\star$-automorphisms, i.e. those induced by characters and by automorphisms of the group.

In this note we generalize some of the known results by elementary methods and complement the exposition given in [18, Sections 22.10-22.13]. We first describe the group generated by natural $\star$-automorphisms and give some criterions for properly outerness. Secondly, we obtain some relations between fixed point algebras and crossed products of group von Neumann algebras which may be of interest in view of the isomorphism problem for such algebras. The final example section is mainly devoted to non inner amenable groups, since the associated group factors are then known to be full ([1], [10]) and thus being far less understood.

We now fix some notation. When no reference or definition is given, the reader may consult [18] and/or some of the standard text books in the respective fields.

All groups will be considered as discrete groups and G will always denote such a non-trivial group, with identity element e. $\mathcal{L}(G)$ will denote the (group) von Neumann algebra generated by the left regular representation $g \to \lambda(g)$ of G on $\ell^2(G)$. $\mathcal{L}(G)$ is a (group) factor if and only if G is ICC, in which case it is a II_1-factor.

An element $A \in \mathcal{L}(G)$ is usually identified with $f_A = A\delta \in \ell^2(G)$, where δ denotes the characteristic function of $\{e\}$, and we set $\mathrm{supp}(A) = \{g \in G \,|\, f_A(g) \neq 0\}$. Then when H is a subgroup of G, we may identify $\mathcal{L}(H)$ with $\{A \in \mathcal{L}(G) \,|\, \mathrm{supp}(A) \subseteq H\}$.

Γ will always denote the character group of G, with identity element 1, and CG the commutator subgroup of G.

Let M be a von Neumann algebra. When ϕ is a group-homomorphism from a group K into $\mathrm{Aut}(G)$ (resp. $\mathrm{Aut}(M)$), the semi-direct product of G (resp. the crossed product of M) by the action ϕ of K is denoted by $G \times_\phi K$ (resp. $M \times_\phi K$).

At last, for $n \in \{1,2,\ldots,\infty\}$, $\mathbb{Z}_n$ will denote the cyclic group with n elements and $\mathbb{F}_n$ the free group on n generators.

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2. ON THE NATURAL $\star$-AUTOMORPHISMS OF $\mathcal{L}(G)$

2.1. For $\gamma \in \Gamma$ (resp. $\sigma \in \mathrm{Aut}(G)$), we let α_γ (resp. β_σ) denote the $\star$-automorphism of $\mathcal{L}(G)$ induced by γ (resp. σ), and α (resp. β) the associated action of Γ (resp. $\mathrm{Aut}(G)$) into $\mathrm{Aut}(\mathcal{L}(G))$. We recall that

$$\alpha_\gamma(\lambda(g)) = \gamma(g)\lambda(g), \quad \beta_\sigma(\lambda(g)) = \lambda(\sigma(g)) \quad (g \in G).$$

2.2. We define $N(\mathcal{L}(G))$ to be the subgroup of $\mathrm{Aut}(\mathcal{L}(G))$ generated by $\alpha(\Gamma) \cup \beta(\mathrm{Aut}(G))$. Let $\phi : \mathrm{Aut}(G) \to \mathrm{Aut}(\Gamma)$ be the action defined by

$$\phi_\sigma(\gamma) = \gamma \circ \sigma^{-1} \quad (\sigma \in \mathrm{Aut}(G),\ \gamma \in \Gamma),$$

and define $i : \Gamma \times_\phi \mathrm{Aut}(G) \to N(\mathcal{L}(G))$ by

$$i(\gamma, \sigma) = \alpha_\gamma \beta_\sigma.$$

Then we have:

<u>Proposition</u>: The mapping i is an isomorphism of $\Gamma \times_{\phi} \mathrm{Aut}(G)$ onto $N(\mathcal{L}(G))$.

<u>Proof</u>: One obtains immediately that

(1) $$\alpha_{\gamma}\beta_{\gamma} = \beta_{\sigma}\alpha_{\gamma\circ\sigma}\,,\quad \beta_{\sigma}\alpha_{\gamma} = \alpha_{\gamma\circ\sigma^{-1}}\beta_{\sigma} \quad (\gamma\in\Gamma,\ \sigma\in\mathrm{Aut}(G)).$$

Thus

$$\begin{aligned} i((\gamma_1,\sigma_1)(\gamma_2,\sigma_2)) &= i(\gamma_1\phi_{\sigma_1}(\gamma_2),\sigma_1\sigma_2)\\ &= i(\gamma_1(\gamma_2\circ\sigma_1^{-1}),\sigma_1\sigma_2)\\ &= \sigma_{\gamma_1(\gamma_2\circ\sigma_1^{-1})}\beta_{\sigma_1\sigma_2}\\ &= \alpha_{\gamma_1}\alpha_{(\gamma_2\circ\sigma_1^{-1})}\beta_{\sigma_1}\beta_{\sigma_2}\\ &= \alpha_{\gamma_1}\beta_{\sigma_1}\alpha_{\gamma_2}\beta_{\sigma_2}\\ &= i(\gamma_1,\sigma_1)i(\gamma_2,\sigma_2) \end{aligned}$$

for all $\gamma_j\in\Gamma$, $\sigma_j\in\mathrm{Aut}(G)$, $j = 1,2$.
Further, i is easily seen to be injective and it follows from (1) that i is onto. ■

2.3. Some of the results of [2; Sections 4, 5] may be viewed as criterions for the outerness of elements of $N(\mathcal{L}(G))$. On the other hand, recall that ([13], [18; prop. 22.12]), for $\sigma\in\mathrm{Aut}(G)$, we have: β_{σ} is <u>properly outer</u> (or freely acting) if and only if

(2) the set $\{\sigma(a)ga^{-1};a\in G\}$ is infinite for every $g\in G$.

These approaches may be unified as follows.

Lemma: Let $\theta = \alpha_\gamma \beta_\sigma$, $\gamma \in \Gamma$, $\sigma \in \mathrm{Aut}(G)$, and let $A \in \mathcal{L}(G)$ be a θ-dependent element, i.e. $AB = \theta(B)A$, for all $B \in \mathcal{L}(G)$.
Then we have:

i) $\gamma(\sigma(a))f_A(g) = f_A(\sigma(a)ga^{-1})$ $(a, g \in G)$,

ii) the set $\{\sigma(a)ga^{-1}; a \in G\}$ is finite for every $g \in \mathrm{supp}(A)$,

iii) $\mathrm{supp}(A)$ lies in a coset of G_0, where G_0 denotes the normal subgroup of G consisting of all elements in G with finite conjugacy classes.

Proof: By assumption we have: $A\lambda(a) = \theta(\lambda(a))A$ $(a \in G)$, which gives $\gamma(\sigma(a))A = \lambda(\sigma(a))^* A\lambda(a) = \lambda(\sigma(a)^{-1})A\lambda(a)$, from which i) follows. ii) follows from i) and the fact that $|f_A| \in \ell^2(G)$, while iii) is immediate from ii). ■

2.4. Proposition: Let $\theta \in N(\mathcal{L}(G))$ be given as $\theta = \alpha_\gamma \beta_\sigma$, $\gamma \in \Gamma$, $\sigma \in \mathrm{Aut}(G)$. Then θ is properly outer whenever:

i) σ satifies (2) or

ii) $\gamma \neq 1$, σ is inner and at least one of the following conditions is satisfied:

   a) G_0 agrees with the center Z of G.

   b) γ is of infinite order.

   c) the set $\{aga^{-1}; a \in G\}$ is infinite for every $g \in G^\gamma = \{g \in G \mid \gamma(g) = 1\}$, $g \neq e$.

Proof: Let A be a θ-dependent element in $\mathcal{L}(G)$. If i) holds then lemma 2.3 ii) implies that $\mathrm{supp}(A) = \emptyset$, i.e. $A = 0$ and thus θ is properly outer. Next, suppose $\gamma \neq 1$ and σ is inner. Then β_σ is inner, and θ will be properly outer if α_γ is. Thus we may suppose $\theta = \alpha_\gamma$. We will now apply lemma 2.3 i) and ii) (with σ = identity). Suppose $A \neq 0$ and let $b \in \mathrm{supp}(A)$. Here we obtain that $b \in G_0$ and that the centralizer of b in G is a subgroup of G^γ. Thus the index of G^γ in G must be finite, i.e. γ is of finite order, and further $G = G^\gamma$, i.e. $\gamma = 1$ if $G_0 = Z$. So by contraposition, θ is properly outer if a) or b) is satisfied. If now c) is satisfied, one easily obtains that $G^\gamma \cap \mathrm{supp}(A) = \emptyset$. Since $\alpha_\gamma(A) = A$ by [4], we have also $\mathrm{supp}(A) \subseteq G^\gamma$ (cf. proposition 3.1). Thus $\mathrm{supp}(A) = \emptyset$, i.e. θ is properly outer. ∎

The essence of [2; Cor. 1 and 2, p. 589] is that σ is outer if and only if σ satisfies (2) when G is an R-group or has no normal subgroups of finite index other than itself. Observe also that $G_0 = Z$ trivially when G is ICC or abelian and that c) is especially satisfied when G^γ is ICC.

2.5. Let $\gamma \in \Gamma$. Since $\gamma \circ \sigma = \gamma$ whenever $\sigma \in \mathrm{Aut}(G)$ is inner, i.e. $\sigma \in \mathrm{Int}(G)$, one may also consider $\Gamma \times_{\tilde{\phi}} \mathrm{Out}(G)$, where $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$ and $\tilde{\phi}:\mathrm{Out}(G) \to \mathrm{Aut}(\Gamma)$ is the action defined by:

$$\tilde{\phi}_{\dot{\sigma}}(\gamma) = \phi_\sigma(\gamma) \quad (\dot{\sigma} \text{ denoting the coset of } \sigma \in \mathrm{Aut}(G)).$$

Then we have:

<u>Proposition</u>: Suppose $G_0 = Z$. Then

i) The action σ of Γ in $\mathcal{L}(G)$ is properly outer.

ii) $\theta = \alpha_\gamma \beta_\sigma$ $(\gamma \in \Gamma, \sigma \in \mathrm{Aut}(G))$ is inner if and only if $\gamma = 1$ and σ is inner.

iii) $\mathrm{Out}(\mathcal{L}(G)) = \mathrm{Aut}(\mathcal{L}(G))/\mathrm{Int}(\mathcal{L}(G))$ contains a subgroup isomorphic to $\Gamma \times_{\tilde{\phi}} \mathrm{Out}(G)$.

Proof: i) This follows from proposition 2.4 ii) a).

ii) Suppose $\theta = Ad(U)$, U unitary in $\mathcal{L}(G)$. Let $b \in supp(U)$ and set $\sigma' = ad(b) \in Int(G)$. Then $\beta_{\sigma'}^{-1}\theta = Ad(\lambda(b)^{*}U) \in N(\mathcal{L}(G))$ and $\lambda(b)^{*}U$ is a $\beta_{\sigma'}^{-1}\theta$-dependent element in $\mathcal{L}(G)$ such that $e \in supp(\lambda(b)^{*}U)$ since $f_{\lambda(b)^{*}U}(e) = f_U(b) \neq 0$. By lemma 2.3 iii) we obtain: $supp(\lambda(b)^{*}U) \subseteq G_0 = Z$, i.e. $\beta_{\sigma'}^{-1}\theta$ is the identity automorphism and so $\gamma = 1$ and $\sigma = \sigma' \in Int(G)$ by proposition 2.1. The converse is trivial.

iii) Let $\pi: Aut(\mathcal{L}(G)) \to Out(\mathcal{L}(G))$ denote the canonical homomorphism and define $\pi' = \pi o i$ where $i: \Gamma \, x_{\phi} Aut(G) \to N(\mathcal{L}(G))$ is defined in 2.2. Then $\ker \pi' = \{(1,\sigma); \alpha \in Int(G)\}$ by ii), and one checks that $(\Gamma \, x_{\phi} Aut(G))/\ker \pi' \simeq \Gamma \, x_{\tilde{\phi}} Out(G)$ under the obvious isomorphism. Thus $\Gamma \, x_{\tilde{\phi}} Out(G) \simeq \pi'(\Gamma \, x_{\phi} Aut(G))$. ■

3. FIXED-POINT ALGEBRAS AND CROSSED PRODUCTS

3.1. Fixed-point algebras of $\mathcal{L}(G)$ under automorphisms induced by characters have a nice description:

Proposition: Let $\gamma \in \Gamma$, Γ' be a subgroup of Γ and set $G^{\gamma} = \{g \in G \,|\, \gamma(g) = 1\}$, $G^{\Gamma'} = \bigcap_{\gamma \in \Gamma'} G^{\gamma}$ and $\alpha' = \alpha_{|\Gamma'}$. Then we have:

i) $\mathcal{L}(G)^{\alpha_{\gamma}} \simeq \mathcal{L}(G^{\gamma})$, where $\mathcal{L}(G)^{\alpha_{\gamma}} = \{A \in \mathcal{L}(G) \,|\, \alpha_{\gamma}(A) = A\}$.

ii) $\mathcal{L}(G)^{\alpha'} \simeq \mathcal{L}(G^{\Gamma'})$, where $\mathcal{L}(G)^{\alpha'} = \bigcap_{\gamma \in \Gamma'} \mathcal{L}(G)^{\alpha_{\gamma}}$.

iii) $\mathcal{L}(G)^{\alpha} \simeq \mathcal{L}(CG)$. Especially, α is ergodic if and only if G is abelian.

iv) $N = G^{\gamma}$, $G^{\Gamma'}$ or CG is not inner amenable whenever G is neither.

Proof:

i) Let $A \in \mathcal{L}(G)$. Then $\alpha_\gamma(A) = A \iff$

$\gamma(g) f_A(g) = f_A(g) \ (g \in G) \iff f_A(g) = 0 \ (g \in G,\ g \notin G^\gamma)$.

Hence $\mathcal{L}(G)^{\alpha_\gamma} = \{A \in \mathcal{L}(G) \mid \mathrm{supp}(A) \subseteq G^\gamma\} \simeq \mathcal{L}(G^\gamma)$.

ii) $\mathcal{L}(G)^{\alpha'} = \bigcap_{\gamma \in \Gamma'} \mathcal{L}(G)^{\alpha_\gamma} = \{A \in \mathcal{L}(G) \mid \mathrm{supp}(A) \subseteq \bigcap_{\gamma \in \Gamma'} G^\gamma = G^{\Gamma'}\} \simeq \mathcal{L}(G^{\Gamma'})$

iii) We have that $CG = G^\Gamma$ and that $CG = \{e\}$ if and only if G is abelian (cf. [11; th. 23.8]), so iii) follows from ii).

iv) Since N contains CG, G/N is abelian and thus amenable. The result now follows from [1; Cor. 2 iv)]. ∎

Corollary: Suppose G is a countable ICC-group and let Γ' be a finite subgroup of Γ of order n. Set $\alpha' = \alpha_{|\Gamma'}$ and M_n = the $n \times n$-complex matrices. Then: $\mathcal{L}(G) \times_{\alpha'} \Gamma' \simeq \mathcal{L}(G^{\Gamma'}) \otimes M_n$.

Proof: Combine the proposition, proposition 2.5 i) and [6]. ∎

3.2. Consider now an exact sequence of groups:

$$1 \to H \to G \to K \to 1.$$

When the extension splits, one may write G as a semidirct product of H by K and so (cf. [18; 22.10]) there exists an action $\phi: K \to \mathrm{Aut}(\mathcal{L}(H)$ such that

$$\mathcal{L}(G) \simeq \mathcal{L}(H) \times_\phi K.$$

It follows from the deep [19; th. 6.1] that the same conclusion is true when $\mathcal{L}(H)$ is a II_1-algebra and K is finite. However, there exist extensions (cf. [9]) where such a conclusion is not possible and one is then forced to introduce a so-called regular extension of $\mathcal{L}(H)$ by K (see [7], [19] for definitions and other results).

We now show how cyclic extensions may be handled by elementary methods. From now on, we identify H with its image in G.

Proposition: Suppose the above extension is cyclic, i.e. K is cyclic. Then there exists an action $\phi: K \to \mathrm{Aut}(\mathcal{L}(H))$ such that:

(3) $$\mathcal{L}(G) \simeq \mathcal{L}(H) \times_{\phi} K.$$

Moreover, the action ϕ is properly outer whenever H satisfies:

(4) the set $\{hgh^{-1}; h \in H\}$ is infinite for every $g \in G$, $g \notin H$.

Proof: Suppose first $K = \mathbb{Z}_n$, $n < +\infty$. Then there exists an $a \in G$ such that G is generated by a and H, and such that:

$$a^m \notin H \quad (1 \leqslant m \leqslant n-1), \text{ while } a^n \in H.$$

Now, let V be an n-th root of $\lambda(a^n)$ in $\mathcal{L}(H)$ and set $U = V^* \lambda(a)$, which is an unitary in $\mathcal{L}(G)$. Then $\phi(A) = UAU^*$, $(A \in \mathcal{L}(H))$ defines an $*$-automorphism of $\mathcal{L}(H)$, since H is normal in G, which is such that

$$\begin{aligned}\phi^n(A) &= (V^*\lambda(a))^n A(\lambda(a)^* V)^n \\ &= ((V^*)^n \lambda(a^n))A(\lambda(a^n)^* V^n) = A \quad (A \in \mathcal{L}(H)),\end{aligned}$$

since V commutes with $\lambda(a)^*$. So we may define an action $\phi: K \to \mathrm{Aut}(\mathcal{L}(H))$ by

$$\phi_j(A) = \phi^j(A) = U^j A (U^j)^* \quad (A \in \mathcal{L}(H),\ j \in K).$$

Clearly, $\mathcal{L}(G)$ is generated by $\mathcal{L}(H)$ and U. Further, let $E: \mathcal{L}(G) \to \mathcal{L}(H)$ be the canonical conditional expectation, which is such that $E(\lambda(g)) = 0$ when $g \in G$, $g \notin H$. Then

$$E(U^j) = E((V^*)^j\lambda(a^j)) = (V^*)^j E(\lambda(a^j)) = 0 \qquad (j\in K,\ j\neq 0).$$

Thus the first part (3) follows from [18; 22.2] in this case. When $K = \mathbb{Z}$, the extention splits and (3) again follows. We may also clearly proceed along the same lines as above.

The second part may be verified by direct computation. It is also a consequence of [18; 22.3] since (4) is equivalent to $\mathcal{L}(H)'\cap\mathcal{L}(G) \subseteq \mathcal{L}(H)$. ■

<u>Corollary</u>: Let $\gamma\in\Gamma$ be of finite order n (resp. such that $G/G^{\gamma}\simeq \mathbb{Z}$) and let $K' = \mathbb{Z}_n$ (resp. $\mathbb{Z}$). Then there exists an action $\phi':K' \to \mathrm{Aut}(\mathcal{L}(G)^{\alpha_\gamma})$ such that: $\mathcal{L}(G) \simeq \mathcal{L}(G)^{\alpha_\gamma} \times_{\phi'} K'$, and which is properly outer if G^{γ} satisfies (4).

<u>Proof</u>: Combine the proposition and proposition 3.1. ■

3.3. If H satisfies:

(5) the set $\{hgh^{-1}:h\in H\}$ is infinite for every $g\in G$, $g\neq e$, (i.e. $(H)'\cap (G)$ reduces to the scalars)

then H satisfies (4) and both H and G are ICC.

We have:

<u>Proposition</u>: For $g\in G$, let $I_g = \{h\in H\,|\,hg=gh\}$. Suppose H is not amenable (resp. not inner amenable) while I_g is amenable (resp. inner amenable) for all $g\in G$, $g\notin H$. Then H satisfies (4). If the same is true for all $g\in G$, $g\neq e$, then H satisfies (5).

<u>Proof</u>: Suppose that $X = \{hg_0h^{-1}; h\in H\}$ is finite for some $g_0\in G$, $g_0 \neq e$. Then consider the action of H on X defined by conjugation. For $x\in X$, the isotropy subgroup is then I_x. Therefore, by [1; addendum], H is amenable (resp. inner amenable) if I_x is amenable (resp. inner amenable) for all $x\in X$. Since $X \subseteq G\setminus\{e\}$ and $X \subseteq G\setminus H$ if $g_0\in G\setminus H$, the result follows by contraposition. ■

3.4. Suppose that in 3.2, $K \simeq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Then we may apply proposition 3.2 twice and obtain that there exist an action $\phi_1 : \mathbb{Z}_{n_1} \to \mathrm{Aut}(\mathcal{L}(H))$ and an action $\phi_2 : \mathbb{Z}_{n_2} \to \mathrm{Aut}(\mathcal{L}(H) x_{\phi_1} \mathbb{Z}_{n_1})$ such that:

$$\mathcal{L}(G) \simeq \mathcal{L}(\mathcal{L}(H) x_{\phi_1} \mathbb{Z}_{n_1}) x_{\phi_2} \mathbb{Z}_{n_2}.$$

This generalizes clearly to the case when K is abelian and finitely generated.

4. SOME EXAMPLES AND OPEN PROBLEMS

In each case we only give some relevant details. For all assertions about non inner amenability, we refer to [1]. Non inner amenable groups are automatically ICC.

4.1. Let $G = \mathbb{F}_n = \langle a_1, \dots, a_n \rangle$, $2 \leq n \leq +\infty$. Then G is not inner amenable and we have that $\Gamma \simeq T^n$ (T denoting the circle group) under the isomorphism given by $\gamma \to (\gamma(a_1), \dots, \gamma(a_n))$.

Further G^γ (resp. $G^{\Gamma'}$) is a free subgroup of G whose rank depends on the order of G/G^γ (resp. $G/G^{\Gamma'}$) (cf. [14; th. 2.10]). Especially, $G^\gamma \simeq \mathbb{F}_{m(n-1)+1}$ if γ is of finite order m, while $G^\gamma \simeq \mathbb{F}_\infty$ otherwise. An easy application of proposition 3.3 shows that G^{γ} always satisfies condition (5). Corollary 3.2 may now be applied specifically to obtain analoguous statements of known results. For example, the case $n = 2$, $\gamma = (\lambda,\lambda)$, $\lambda \in T$, is the one studied in [6], while the case $\gamma = \exp(2\pi i/m, 1,\ldots,1)$ corresponds to [16; prop. 4.5]. At last, Aut(G) is described in [14; sect. 3.5]. The condition (2) is easily verified for many $\sigma \in \mathrm{Aut}(G)$.

4.2. Let G be the free product $\mathbb{Z}_p * \mathbb{Z}_q$, $2 \leq p,q \leq +\infty$. Then G is not inner amenable when p (or q) >2, while G is not ICC when $p = q = 2$. From [14; p. 193-197], we may obtain what follows. First, we have that

$$CG \simeq \mathbb{F}_{(p-1)(q-1)} \quad \text{and} \quad G/CG \simeq \mathbb{Z}_p \times \mathbb{Z}_q .$$

Hence $\mathcal{L}(G)^{\alpha} \simeq \mathcal{L}(\mathbb{F}_{(p-1)(q-1)})$ (by 3.1) and

$$\mathcal{L}(G) \simeq \mathcal{L}(\mathcal{L}(\mathbb{F}_{(p-1)(q-1)}) x_{\phi_1} \mathbb{Z}_p) x_{\phi_2} \mathbb{Z}_q \quad \text{(by 3.4).}$$

Let us be more specific for $G = \mathbb{Z}_2 * \mathbb{Z}_3 \simeq PSL(2,\mathbb{Z})$.
Here $CG \simeq \mathbb{F}_2$ and $\Gamma \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$.
Thus $\mathcal{L}(\mathbb{Z}_2 * \mathbb{Z}_3)^{\alpha} \simeq \mathcal{L}(\mathbb{F}_2)$, $\mathcal{L}(\mathbb{Z}_2 * \mathbb{Z}_3)\ x_{\alpha}\ \mathbb{Z}_6 \simeq \mathcal{L}(\mathbb{F}_2) \otimes M_6$ and $\mathcal{L}(\mathbb{Z}_2 * \mathbb{Z}_3) \simeq \mathcal{L}(\mathbb{F}_2)\ x_{\phi}\ \mathbb{Z}_6$, where $\phi : \mathbb{Z}_6 \to \mathrm{Aut}(\mathcal{L}(\mathbb{F}_2))$ is the action obtained in 3.2, this being outer by 3.3. At last, observe that $\mathrm{Out}(\mathbb{Z}_2 * \mathbb{Z}_3) \simeq \mathbb{Z}_2$.

4.3. As we have seen in 2.5, $\mathrm{Out}(\mathcal{L}(G))$ contains a copy of $\Gamma\ x_{\tilde{\phi}}\ \mathrm{Out}(G)$ when G is ICC. It would be interesting to know if these groups are more intimately related, at least in some cases. Connes has shown in [8] that $\mathrm{Out}(\mathcal{L}(G))$ is countable whenever G is ICC and has property T (G is then especially non inner amenable). We now mention two examples of such groups for which $\Gamma\ x_{\tilde{\phi}}\ \mathrm{Out}(G) \simeq \mathbb{Z}_2$. In both cases, it is an open problem whether $\mathrm{Out}(\mathcal{L}(G)) \simeq \mathbb{Z}_2$.

a) Let $G = SL(3,\mathbb{Z})$.

Here $\Gamma = \{1\}$ while $Out(G) \simeq \mathbb{Z}_2$ by [12]. A representative $\tau \in Aut(G)$ of the non trivial element in $Out(G)$ is defined by $\tau(a) = (a^t)^{-1}$ $(a \in G)$, where a^t denotes the transpose of a.

b) Let $G = SL(3,\mathbb{Z}) x_\tau \mathbb{Z}_2$.

Now one may check that $\Gamma \simeq \mathbb{Z}_2$ while $Out(G)$ is trivial.

Observe also that $\mathcal{L}(G)\ x_\alpha\ \mathbb{Z}_2 \simeq \mathcal{L}(SL(3,\mathbb{Z})) \otimes M_2$.

4.4. In the same spirit one may ask whether $Out(\mathcal{L}(G))$ and $\Gamma\ x_{\tilde{\phi}}\ Out(G)$ are isomorphic for some G. Another open problem is whether $\Gamma\ x_{\tilde{\phi}}\ Out(G)$ (or even more $Out(\mathcal{L}(G))$) may be trivial for some countable non inner amenable group G. Besides other ICC groups with property T, we mention as possible candidates:

- the Ol'shanskii group ([17])
- the amalgams of the type $F_2 \underset{F_\infty}{*} F_2$ constructed in [3].

These groups are simple and thus at least with trivial character group.

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