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WHEN IS A STOCHASTIC INTEGRAL A TIME CHANGE OF A DIFFUSION?

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Abstract

We give a necessary and sufficient condition (in terms of u, v, b, σ) that a time change of an n-dimensional Ito stochastic integral X_t on the form

$$dX_t = u(t,\omega)dt + v(t,\omega)dB_t$$

has the same law as a diffusion Y_t on the form

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t.$$

As an application we prove a change of time formula for n-dimensional Ito integrals.

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§1. The Main Result

In the following we will let $Y_t = Y_t^x$ denote an Ito diffusion, i.e. a (weak) solution in an open set $U \subset \mathbb{R}^n$ of the Ito stochastic differential equation

(1.1) $$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad Y_0 = x$$

where the functions $b: \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous and $(B_t, \Omega, \mathcal{F}_t, P^x)$ denotes m-dimensional Brownian motion. And we will let $X_t = X_t^x$ denote an Ito stochastic integral

(1.2) $$dX_t = u(t,\omega)dt + v(t,\omega)dB_t, \quad X_0 = x,$$

where $u(t,\omega) \in \mathbb{R}^n$, $v(t,\omega) \in \mathbb{R}^{n\times m}$ satisfy the usual conditions for existence of the stochastic integral: $u(t,\omega)$ and $v(t,\omega)$ are $\mathcal{F}_t$-adapted and

$$P^0\{\omega; \int_0^t |u(s,\omega)| + \sum_{ij} \int_0^t |v_{ij}(s,\omega)|^2 ds < \infty \quad \text{for all} \quad t\} = 1.$$

(See e.g. [4] or [7]). The time changes will consider are of the following form:

Let $c(t,\omega) \geq 0$ be an $\mathcal{F}_t$-adapted process. Define

(1.3) $$\beta_t = \beta(t,\omega) = \int_0^t c(s,\omega)ds$$

We will say that β_t is a time change with time change rate $c(t,\omega)$. Note that β_t is also $\mathcal{F}_t$-adapted and for each ω the map $t \to \beta_t$ is nondecreasing. Let $\alpha_t = \alpha(t,\omega)$ be the right continuous inverse of β_t:

(1.4) $$\alpha_t = \inf\{s;\ \beta_s > t\}$$

Then $\omega \to \alpha(t,\omega)$ is an $\{\mathcal{F}_s\}$-stopping time for each t, since

$$\{\omega;\ \alpha(t,\omega) < s\} = \{\omega;\ t < \beta(s,\omega)\} \in \mathcal{F}_s.$$

We now ask the question: When does there exist a time change β_t as above such that $X_{\alpha_t} \sim Y_t$, i.e. X_{α_t} is identical in law to Y_t? In §1 we give an answer to this question (Theorems 1-3) and in §2 we use this to prove a change of time formula for stochastic integrals.

Note that $\beta(\alpha_t) = t$ for all (t,ω), so that

(1.5) $$a_t'(\omega) = \frac{1}{c(\alpha_t,\omega)} \quad \text{for a.a } t \geq 0,\ \omega \in \Omega.$$

Moreover,

$$\int_0^t c(\alpha_r,\omega)\,d\hat{\alpha}_r = \int_0^{\alpha_t} c(s,\omega)\,ds = \int_0^t dr$$

or

(1.6) $$c(\alpha_t,\omega)\,d\hat{\alpha}_t = dt, \text{ for each } \omega \in \Omega,$$

where $d\hat{\alpha}_t$ denotes the measure $d\alpha_t$ with the point masses corresponding to the discontinuities of α_t taken out.

First we establish a useful measurability result. We let $\mathcal{M}_t$ and $\mathcal{N}_t$ denote the σ-algebras generated by $\{X_s;\ s \leq t\}$ and $\{Y_s;\ s \leq t\}$, respectively, and we define $\mathcal{M}_{\alpha_t}$ to be the σ-algebra in Ω generated by the functions $\omega \to X_{\alpha_s};\ s \leq t$.

We let $C_0^2(U)$ denote the twice continuously differentiable functions with compact support in U, and v^T denotes the transposed of the matrix v.

Lemma 1

Let $dX_t = u(t,\omega)dt + v(t,\omega)dB_t$, $c(t,\omega)$, α_t be as above. Then $(vv^T)(\alpha_t,\omega)a_t'$ is $\mathcal{M}_{\alpha_t}$-adapted

Proof.

By Ito's formula we have

$$X_t^{(i)}X_t^{(j)} = X_0^{(i)}X_0^{(j)} + \int_0^t X_s^{(i)}dX_s^{(j)} + \int_0^t X_s^{(j)}dX_s^{(i)} + \int_0^t (vv^T)_{ij}(s,\omega)ds$$

Hence, if we put

$$H_{ij}(t,\omega) = X_t^{(i)}X_t^{(j)} - X_0^{(i)}X_0^{(j)} - \int_0^t X_s^{(i)}dX_s^{(j)} - \int_0^t X_s^{(j)}dX_s^{(i)}$$

then $H(t,\omega)$ is $\mathcal{M}_t$-adapted and we have

$$\int_0^{\alpha_t} (vv^T)(s,\omega)ds = H(\alpha_t,\omega)$$

Therefore

$$(vv^T)(\alpha_t,\omega)\alpha_t' = \lim_{r\to 0} \frac{H(\alpha_t,\omega)-H(\alpha_{t-r},\omega)}{r},$$

which shows that $(vv^T)(\alpha_t,\omega)\alpha_t'$ is $\mathcal{M}_{\alpha_t}$-adapted.

Remarks

1) One may ask if it is also true that $u(\alpha_t,\omega)\alpha_t'$ is $\mathcal{M}_{\alpha_t}$-adapted. However, the following example, which was pointed out to me by the referee, shows that this fails even in the case when $\alpha_t = t$, $v= 1$, $m = n = 1$:

Put

$$u(t,\omega) = \begin{cases} \frac{B_1 - B_t}{1-t} & \text{if} \quad t<1 \\ 0 & \text{if} \quad t\geq 1 \end{cases}$$

and define

$$\tilde{B}_t = -\int_0^t u(s,\omega)ds + B_t$$

Then $\tilde{B}_t$ is a Brownian motion and

$$B_t = \int_0^t u(s,\omega)ds + \tilde{B}_t,$$

but $u(t,\omega)$ is not $\mathcal{F}_t$-adapted.

2) The next example shows that it need <u>not</u> be the case that $v(\alpha_t,\omega)\alpha_t'$ is $\mathcal{M}_{\alpha_t}$-adapted, even if $\alpha_t = t$: Choose $v(t,\omega)$ non-constant with the values ± 1 and independent of $\{B_t\}_{t\geq 0}$ $(m=n=1)$. Define

$$d\tilde{B}_t = v(t,\omega)dB_t$$

Then $\tilde{B}_t$ is a Brownian motion (see McKean [4], §2.9 and also Corollary 1 later in this article). Hence we have

$$dB_t = v(t,\omega)d\tilde{B}_t,$$

but $v(t,\omega)$ is not $\mathcal{F}_t$-adapted.

Let $\mathcal{B}$ denote the Borel σ-algebra of subsets of $[0,\infty)$. For $t\geq 0$ we define a measure Q_{α_t} on $\mathcal{B}\times\mathcal{F}$ by setting

$$Q_{\alpha_t}(f) = E^x[\int_0^{\alpha_t} f(s,\omega)ds]$$

if $f(s,\omega)$ is bounded and $\mathcal{B}\times\mathcal{F}$-measurable. Let $\mathcal{X}$ denote the σ-algebra in $[0,\infty)\times\Omega$ generated by the function $(s,\omega)\to X_s(\omega)$ and let $E_{\alpha_t}[g|\mathcal{X}] = E_{\alpha_t}[g|X]$ denote the conditional expectation of $g(s,\omega)$ wrt. $\mathcal{X}$ and wrt. the measure Q_{α_t}.

We can now state and prove the main result. First we consider the case when

$$(1.9) \qquad \beta_\infty = \infty \quad \text{a.s. (i.e. } \alpha_t<\infty \text{ for all } t<\infty \text{ a.s.)}$$

The general case will considered later in this section (Theorem 2).

Theorem 1.

Assume that (1.9) holds. Then the following 3 statements, (I), (II) and (III), are equivalent:

(I) (i) $E_{\alpha_t}[u|X] = b(X)E_{\alpha_t}[c|X]$ for all $t\geq 0$ and

(ii) $(vv^T)(t,\omega) = c(t,\omega)(\sigma\sigma^T)(X_t)$ for a.a. $t\in(0,\alpha_\infty)$, $\omega\in\Omega$.

(II) (i) $E_{\alpha_t}[u|X] = b(X)E_{\alpha_t}[c|X]$ for all $t\geq 0$ and

(iii) $E_{\alpha_t}[vv^T|X] = \sigma\sigma^T(X)E_{\alpha_t}[c|X]$ for all $t\geq 0$

(III) $X_{\alpha_t} \sim Y_t$

Proof.

(I) => (II): This follows by noting that (i) and (iii) state that

$$(1.10) \qquad E^x[\int_0^{\alpha_t} u(s,\omega)g(X_s)ds] = E^x[\int_0^{\alpha_t} b(X_s)g(X_s)c(s,\omega)ds] \quad \text{and}$$

$$(1.11) \qquad E^x[\int_0^{\alpha_t} (vv^T)(s,\omega)g(X_s)ds] = E^x[\int_0^{\alpha_t} (\sigma\sigma^T)(X_s)g(X_s)c(s,\omega)ds]$$

for all bounded functions g.

(II) => (III):

For $0 \leq t < \infty$ we define a bounded linear functional W_t on $C_b(U)$ (the bounded real continuous functions on U equipped with the sup norm) by

$$W_t f = E^x[f(X_{\alpha_t})]; \quad f \in C_b(U).$$

Since α_t is a stopping time we have by Ito's formula (see e.g. [7], Lemma 7.8) if $f \in C_0^2(U)$:

$$W_t f = E^x[f(X_0)] + E^x[\int_0^{\alpha_t} \{\sum_i u_i(s,\omega) \frac{\partial f}{\partial x_i}(X_s) +$$

$$\tfrac{1}{2}\sum_{i,j}(vv^T)_{ij}(s,\omega)\frac{d^2 f}{\partial x_i \partial x_j}(X_s)\}ds$$

So if (II) holds we obtain, using (1.10), (1.11) and (1.6)

$$W_t f = f(x) + E^x[\int_0^{\alpha_t}\{\sum_i b_i(X_s) \cdot \frac{\partial f}{\partial x_i}(X_s) + \tfrac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(X_s) \cdot$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\}c(s,\omega)ds] = f(x) + E^x[\int_0^{\alpha_t}\{\sum_i b_i(X_{\alpha_r}) \cdot \frac{\partial f}{\partial x_i}(X_{\alpha_r}) +$$

$$\tfrac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{ij}(X_{\alpha_r}) \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}(X_{\alpha_r})\}dr] = f(x) + E^x[\int_0^{\alpha_t} Af(X_{\alpha_r})dr]$$

where $A = \sum_i b_i(\partial/\partial x_i) + \tfrac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}(\partial^2/\partial x_i \partial x_j)$ is the generator of Y_t. Therefore

$$\frac{d}{dt}W_t f = W_t(Af) \quad ; \quad t \geq 0$$
$$W_0 f = f(x) \tag{1.12}$$

for all $f \in C_0^2(U)$. Similarly we obtain, if we put

$$V_t f = E^x[f(Y_t)] \quad , \; t \geq 0$$

that

$$(1.13) \qquad \begin{aligned} \frac{d}{dt} V_t f &= V_t(A_f) \quad , \ t \geq 0 \\ V_0 f &= f(x) \end{aligned}$$

for all $f \in C_0^2(U)$. Since the solution of the equations (1.12) and (1.13) is unique (see [6], Lemma 2.5) we conclude that

$$W_t f = V_t f \quad \text{for all} \quad t \geq 0, \ f \in C_0^2(U).$$

Similarly we prove by induction on k that

$$E^x[f(X_{\alpha_t}) g_1(X_{\alpha_t}) \ \ldots \ g_k(X_{\alpha_{t_k}})] = E^x[f(Y_t) g_1(Y_{t_1}) \ \ldots \ g_k(Y_{t_k})]$$

for all $t, t_1, \ldots, t_k \geq 0$ and $f, g_1, \ldots; g_k \in C_0^2(U)$ by applying the above argument to the $n(k+1)$ - dimensional processes

$$(X_{\alpha_t}, X_{\alpha_{t_1}}, \ldots, X_{\alpha_{t_k}}) \quad \text{and} \quad (Y_t, Y_{t_1}, \ldots, Y_{t_k}).$$

<u>(III) => (I)</u>. Suppose $X_{\alpha_t} \sim Y_t$. Since Y_t is a Markov process wrt. $\mathcal{N}_t$ it follows that X_{α_t} is a Markov process wrt. $\mathcal{M}_{\alpha_t}$ and with generator A. Therefore, using Dynkin's formula (see e.g. [7], Th. 7.10) and (1.6) we have, for $f \in C_0^2(U)$:

$$(1.14) \quad E^x[f(X_{\alpha_{t+h}}) | \ _{\alpha_t}] = E^{X_{\alpha_t}}[f(X_{\alpha_h})] = f(X_{\alpha_t}) +$$

$$E^{X_{\alpha_t}}[\int_0^h \{\sum_i b_i(X_{\alpha_t}) \cdot \frac{\partial f}{\partial x_i}(X_{\alpha_t}) + \tfrac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{ij}(X_{\alpha_t}) \cdot \frac{d^2 f}{\partial x_i \partial x_j}(X_{\alpha_t})\} dr]$$

$$= f(X_{\alpha_t}) + E^{X_{\alpha_t}} [\int_0^{\alpha_h} \{\sum_i b_i(X_s) \cdot \frac{\partial f}{\partial x_i} (X_s) +$$

$$\tfrac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{ij}(X_s) \cdot \frac{d^2 f}{\partial x_i \partial x_j}(X_s)\} c(s,\omega) ds]$$

On the other hand, from Ito's formula we get as before

$$(1.15)\quad E^x[f(X_{\alpha_{t+h}})|\mathcal{M}_{\alpha_t}] = f(X_{\alpha_t}) + E^x[f(X_{\alpha_{t+h}}) - f(X_{\alpha_t})|\mathcal{M}_{\alpha_t}]$$

$$= f(X_{\alpha_t}) + E^x[\int_{\alpha_t}^{\alpha_{t+h}} \{\sum_i u_i(s,\omega)\cdot\frac{\partial f}{\partial x_i}(X_s) +$$

$$\tfrac{1}{2}\sum_{i,j}(vv^T)_{ij}(s,\omega)\frac{\partial^2 f}{\partial x_i\partial x_j}(X_s)\}ds|\mathcal{M}_{\alpha_t}],$$

and a similar formula, denoted by (1.15)', if we replace α_t by 0. Comparing (1.14) and (1.15)' for $f(x_1,\ldots,x_n)=\exp(i(\lambda_1x_1+\ldots+\lambda_nx_n))$ (where $i=\sqrt{-1}$) we see that (1.10) and (1.11) holds by putting $t=0$. Thus it remains to prove property (ii).

From (1.14) and (1.15) we conclude that if we fix i,j and put

$$F_t(\omega) = \int_0^{\alpha_t}(vv^T)_{ij}(s,\omega)ds$$

then

$$(\sigma\sigma^T)_{ij}(X_{\alpha_t}) = \lim_{h\to 0}\frac{1}{h}E^{X_{\alpha_t}}[\int_0^h(\sigma\sigma^T)_{ij}(X_{\alpha_r})dr]$$

$$(1.16)\qquad = \lim_{h\to 0}\frac{1}{h}E^x[F_{t+h}-F_t|\mathcal{M}_{\alpha_t}]\quad\text{for all } t,\omega.$$

Choose a $t>0$ such that F_t' exists a.s. Let N be an integer. Define, for $h>0$,

$$G_h(\omega) = \frac{1}{h}(F_{t+h}(\omega) - F_t(\omega))$$

$$H_h(\omega) = \begin{cases} G_h(\omega) & \text{if } |G_h(\omega)| \leq N \\ -N & \text{if } G_h(\omega) < -N \\ N & \text{if } G_h(\omega) > N \end{cases}$$

and put

$$H_0(\omega) = \begin{cases} F_h'(\omega) & \text{if } |F_h'(\omega)| \leq N \\ -N & \text{if } F_h'(\omega) < -N \\ N & \text{if } F_h'(\omega) > N, \end{cases}$$

Then H_0 is measurable wrt. $\mathcal{M}_{\alpha_t}$ by Lemma 1. By bounded convergence we have

(1.17) $$\lim_{h\to 0} E^x[H_h|\mathcal{M}_{\alpha_t}] = E^x[\lim_{h\to 0} H_h|\mathcal{M}_{\alpha_t}] = H_0 \qquad \text{a.s.}$$

Put $$W = \{\omega;\ |F_t'(\omega)|\leq \tfrac{1}{2}N\} \in \mathcal{M}_{\alpha_t}.$$

Choose $\omega\in W$. Then there exists $h(\omega) > 0$ such that

$$h < h(\omega) \Rightarrow |G_h(\omega)|\leq N \quad \text{i.e.} \quad G_h(\omega) = H_h(\omega).$$

We want to conclude that

(1.18) $$\lim_{h\to 0} E^x[G_h|\mathcal{M}_{\alpha_t}] = \lim_{h\to 0} E^x[H_h|\mathcal{M}_{\alpha_t}]$$

for a.a $\omega\in W$.

To obtain this write

$$E^x[f|\mathcal{M}_{\alpha_t}](\omega) = \int f(\eta)dQ_\omega(\eta), \quad \text{for a.a. } \omega\in\Omega.$$

where Ω_ω is a conditional probability distribution of P given $\mathcal{M}_{\alpha_t}$. (See Stroock and Varadhan [8], Theorem 1.16)

Let

$$V(\omega) = \cap\{V\in\mathcal{M}_{\alpha_t};\ \omega\in V\} \in \mathcal{M}_{\alpha_t}$$

be the $\mathcal{M}_{\alpha_t}$-atom containing ω.

Since

$$Q_\omega(V(\omega)) = 1 \quad \text{for a.a. } \omega$$

([8], Theorem 1.18) and $V(\omega) \subset W$ for all $\omega \in W$ (since $W \in \mathcal{M}_{\alpha_t}$), we have for a.a. $\omega \in W$ and $h < h(\omega)$

$$E^x[G_h | \mathcal{M}_{\alpha_t}](\omega) = \int_W G_h dQ_\omega = \int_W H_h(\omega) dQ_\omega = E^x[H_h | \mathcal{M}_{\alpha_t}]$$

and (1.18) follows.

Combining (1.17) and (1.18) we obtain that

$$\lim_{h \to 0} E^x[G_h | \mathcal{M}_{\alpha_t}] = F_t' \qquad \text{a.s. in } W$$

And since N was arbitrary we conclude from (1.16)

(1.19) $$(\sigma\sigma^T)_{ij}(X_{\alpha_t}) = (vv^T)_{ij}(\alpha_t, \omega)\alpha_t' \qquad \text{for a.a. } t, \omega$$

or

(1.20) $$(vv^T)_{ij}(\alpha_t, \omega) = c(\alpha_t, \omega)(\sigma\sigma^T)_{ij}(X_{\alpha_t}) \quad \text{for a.a. } t, \omega.$$

Moreover, if we define

(1.21) $$F_t'(\omega) = \underline{\lim}_{h \to 0} \frac{1}{h}(F_{t+h} - F_t) \qquad \text{for all } t, \omega,$$

then using (1.15) and Fatou's lemma we get

$$F_t'(\omega) = E^x[F_t' | \mathcal{M}_{\alpha_t}] \leq \underline{\lim}_{h \to 0} \frac{1}{h} E^x[F_{t+h} - F_t | \mathcal{M}_{\alpha_t}]$$

(1.22) $$= (\sigma\sigma^T)_{ij}(X_{\alpha_t}) < \infty \qquad \text{for all } t, \omega$$

Thus $t \to F_t(\omega)$ is absolutely continuous for each ω. Therefore $(vv^T)_{ij}(s,\omega) = 0$ a.e. on each s-interval where $s \to \beta(s,\omega)$ is constant i.e. where $s \to c(s,\omega)$ is 0 a.e. and, by (1.6)

$$(vv^T)_{ij}(\alpha_r, \omega) d\alpha_r = (\sigma\sigma^T)_{ij}(X_{\alpha_r}) dr = (\sigma\sigma^T)_{ij}(X_{\alpha_r}) c(\alpha_r, \omega) d\alpha_r$$

This is equivalent to saying that.

$$\int_0^{\alpha_t} (vv^T)_{ij}(s,\omega)ds = \int_0^{\alpha_t} (\sigma\sigma^T)_{ij}(X_s)c(s,\omega)ds$$

for all t,ω. Thus (ii) holds and the proof of Theorem 1 is complete.

Remark. Consider the more general situation when Y_t is not ssumed to be a diffusion, but just a stochastic integral of the same type as X_t:

(1.1) $$dY_t = e(t,\omega)dt + f(t,\omega)dB_t, \quad Y_0 = x.$$

It is natural to ask if one can find conditions on the coefficients in order that $X_{\alpha_t} \sim Y_t$ in case.

We end this section by considering the case when we do not assume that (1.9) holds, i.e. we allow $\beta_\infty < \infty$. This case will be a special case of the following situation: Let

$$X_t = X_t^x(\omega) = x + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_t; \quad 0 \leq t \leq \tau$$

be a stochastic integral in an open set $W \subset U \subset R^n$, where τ is an $\mathfrak{F}_t$-stopping time such that $\tau \leq \tau_W$, the first exit time from W of X_t. The probability law of X_t starting at x, $\bar{P}^x$, is defined by

$$\bar{P}^x[X_{t_1\wedge\tau}\in F_1,\ldots,X_{t_k\wedge\tau}\in F_k] = P^0[X^x_{t_1\wedge\tau}\in F_1,\ldots,X^x_{t_k\wedge\tau}\in F_k],$$

and $\bar{E}^x$ denotes integration wrt. $\bar{P}^x$. Suppose Y_t is as before and let $\hat{P}^x$ denote the probability law of X_t starting at x. Then we say that X_t is a time change of Y_t (with time change rate $c(t,\omega)$) if the process Z_t defined by

$$(1.23) \qquad Z_t = \begin{cases} X_{\alpha_t} & ; \quad 0 \leq t < \beta_\tau \\ Y_{t-\beta_\tau} & ; \quad t \geq \beta_\tau \end{cases}$$

with probability law $\tilde{P}^x$ defined by

$$\tilde{E}^x[f_1(Z_{t_1})\dots f_k(Z_{t_k})\cdot\chi_{\{t_j<\beta_\tau\leq t_{j+1}\}}]=E^x[f_1(X^x_{\alpha_{t_1}})\dots f_j(X^x_{\alpha_{t_j}})\cdot$$

$$(1.24) \qquad f_{j+1}(Y^{X_\tau}_{t_{j+1}-\beta_\tau})\dots f_k(Y^{X_\tau}_{t_k-\beta_\tau})\cdot\chi_{\{t_j<\beta_\tau\leq t_{j+1}\}}]$$

coincide in law with Y_t for every $x\in W$.

(For simplicity we suppress the superscript x in what follows)

Then question when X_t is a time change of Y_t can now by given an answer similar to Theorem 1, except that in this case the measure Q_{α_t} must be modified to the measure $Q_{\alpha_t\wedge\tau}$ defined by

$$Q_{\alpha_t\wedge\tau}(f) = E^x[\int_0^{\alpha_t\wedge\tau} f(s,\omega)ds]$$

if $f\geq 0$ is $\mathcal{B}\times\mathcal{F}$-measurable. The corresponding conditonal expectation is denoted by $E_{\alpha_t\wedge\tau}[\ |\ ]$.

Theorem 2. The following are equivalent:

(A) $E_{\alpha_t\wedge\tau}[u|X] = b(X)E_{\alpha_t\wedge\tau}[c|X]$ for all $t\geq 0$ and $(vv^T)(t,\omega)=c(t,\omega)(\sigma\sigma^T)(X_t)$ for a.a. t,ω $_\tau$such that $t<\beta_\tau$.

(B) X_t is a time change of Y_t, with time change rate $c(t,\omega)$.

Proof. (A) => (B): We proceed as in the proof of (II) => (III) in Theorem 1, except that now we put

$$W_t f = \tilde{E}[f(Z_t)]; \quad f\in C_0^2(U),\ t\geq 0.$$

Then by Ito's formula we get

$$\tilde{E}[f(Z_t)\cdot\chi_{\{t<\beta_\tau\}}]=\tilde{E}[f(x)\cdot\chi_{\{t<\beta_\tau\}}]+\tilde{E}[\int_0^{\alpha_t}(\nabla f)^T(X_s)v(s,\omega)dB_s\cdot\chi_{\{t<\beta_\tau\}}]+$$

$$\tilde{E}[\int_0^{\alpha_t}\{\sum_i u_i(s,\omega)\frac{\partial f}{\partial x_i}(X_s)+\sum_{i,j}(vv^T)_{ij}(s,\omega)\frac{\partial^2 f}{\partial x_i\partial x_j}(X_s)ds\cdot\chi_{\{t<\beta_\tau\}}]$$

Similarly

$$\tilde{E}[f(Z_t)\cdot\chi_{\{t\geq\beta_\tau\}}] = E[f(Y^{X_\tau}_{t-\beta_\tau})\cdot\chi_{\{t\geq\beta_\tau\}}]$$

$$= E[f(X_\tau)\cdot\chi_{\{t\geq\beta_\tau\}}] + E[\int_0^{t-\beta_\tau}(Af)(Y_u^{X_\tau})du\cdot\chi_{\{t\geq\beta_\tau\}}]$$

$$(1.26)\qquad = E[f(X_\tau)\cdot\chi_{\{t\geq\beta_\tau\}}] + E[\int_{\beta_\tau}^t(Af)(Y^{X_\tau}_{v-\beta_\tau})dv\cdot\chi_{\{t\geq\beta_\tau\}}]$$

By Ito's formula we get

$$E[f(X_\tau)\cdot\chi_{\{t\geq\beta_\tau\}}]=E[f(x)\cdot\chi_{\{t\geq\beta_\tau\}}]+E[\int_0^\tau(\nabla f)^T(X_s)v(s,\omega)dB_s\cdot\chi_{\{t\geq\beta_t\}}]$$

(1.27)

$$+ E[\int_0^\tau\{\sum_i u_i(s,\omega)\cdot\frac{\partial f}{\partial x_i}(X_s)+\sum_{i,j}(vv^T)_{ij}(s,\omega)\cdot\frac{\partial^2 f}{\partial x_i\partial x_j}(X_s)\}ds\cdot\chi_{\{t\geq\beta_\tau\}}]$$

so by adding (1.26) and (1.27) we obtain

$$\tilde{E}[f(Z_t)] = f(x) + E[\int_0^{\alpha_t\wedge\tau}(\nabla f)^T(X_s)v(s,\omega)dB_s]$$

$$+ E[\int_0^{\alpha_t\wedge\tau}\{\sum_i u_i(s,\omega)\cdot\frac{\partial f}{\partial x_i}(X_s)+\sum_{i,j}(vv^T)_{ij}(s,\omega)\cdot\frac{\partial^2 f}{\partial x_i\partial x_j}(X_s)\}ds$$

$$(1.28)\qquad + E[\int_{\beta_\tau}^t(Af)(Y^{X_\tau}_{v-\beta_\tau})dv\cdot\chi_{\{t\geq\beta_\tau\}}].$$

Since $\alpha_t\wedge\tau$ is a stopping time the second term on the right of (1.28) is 0 and by (A) the third term is the same as

$$E\Big[\int_0^{\alpha_\tau\wedge\tau}(Af)(X_s)c(s,\omega)ds\Big]=E\Big[\int_0^{\alpha_t}(Af)(X_s)c(s,\omega)ds\cdot\chi_{\{t<\beta_\tau\}}\Big]$$

$$+\ E\Big[\int_0^{\tau}(Af)(X_s)c(s,\omega)ds\cdot\chi_{\{t\geq\beta_\tau\}}\Big]$$

$$(1.29)\qquad = E\Big[\int_0^{t}(Af)(X_{\alpha_r})dr\cdot\chi_{\{t<\beta_\tau\}}\Big]+E\Big[\int_0^{\beta_\tau}(Af)(X_{\alpha_r})dr\cdot\chi_{\{t\geq\beta_\tau\}}\Big]$$

(Note that

$$(1.30)\qquad \int_0^{\tau}(Af)(X_s)c(s,\omega)ds=\int_0^{\alpha_{\beta_\tau}}(Af)(X_s)c(s,\omega)ds,$$

since $c(s,\omega) = 0$ for a.a. $s \in (\tau, \alpha_{\beta_\tau})$).

Substituting (1.29) in (1.28) and comparing with (1.24) we conclude that

$$\tilde{E}[f(Z_t)] = f(x) + \tilde{E}\Big[\int_0^t (Af)(Z_s)ds\Big].$$

Thus we have obtained (1.11) and the rest of the proof of (i) => (ii) follows the proof of (II) => (III) in Theorem 1.

<u>(B) => (A)</u>: We reverse the argument just given. If Z_t is a Markov process with generator A we get by the Dynkin formula

$$\tilde{E}(f(Z_t)] = f(x) + \tilde{E}\Big[\int_0^t (Af)(Z_s)ds\Big]$$

$$=f(x)+\tilde{E}\Big[\int_0^{t\wedge\beta_\tau}(Af)(Z_s)ds\Big]+\tilde{E}\Big[\int_{t\wedge\beta_\tau}^{t}(Af)(Z_v)dv\Big]$$

$$=f(x)+E\Big[\int_0^{t\wedge\beta_\tau}(Af)(X_{\alpha_r})dr\Big]+\tilde{E}\Big[\Big(\int_{\beta_\tau}^{t}(Af)(Z_v)dv\Big)\chi_{\{t\geq\beta_\tau\}}\Big]$$

$$(1.31)\qquad =f(x)+E\Big[\int_0^{\alpha_t\wedge\tau}(Af)(X_s)c(s,\omega)ds\Big]+E\Big[\Big(\int_{\beta_\tau}^{t}(Af)(Y^{X_\tau}_{v-\beta_\tau})dv\Big)\cdot\chi_{\{t\geq\beta_\tau\}}\Big]$$

Comparing (1.28) and (1.31) we conclude that

$$E[\int_0^{\alpha_t\wedge\tau} u(s,\omega)g(X_s)ds] = E[\int_0^{\alpha_t\wedge\tau} b(X_s)c(s,\omega)g(X_s)ds]$$

and

$$E[\int_0^{\alpha_t\wedge\tau} (vv^T)(s,\omega)g(X_s)ds] = E[\int_0^{\alpha_t\wedge\tau} (\sigma\sigma^T)(X_s)c(s,\omega)g(X_s)ds]$$

for all bounded functions g.

This proves the first identity in (A). To obtain the second identity we proceed as in the proof of (III) => (I) in Theorem 1: Let $\tilde{\mathcal{M}}_t$ denote the σ-algebra generated by $\{Z_s;\ s\leq t\}$. Then by the strong Markov property we have for all t,ω

$$(1.32)\quad \lim_{h\to 0}\frac{1}{h}\tilde{E}[f(Z_{t+h})-f(Z_t)|\tilde{\mathcal{M}}_t]=\lim_{h\to 0}\frac{1}{h}\tilde{E}^{Z_t}[f(Z_h)-f(Z_0)]=(Af)(Z_t)$$

On the other hand, from the general calculation in (1.28) we get

$$\lim_{h\to 0}\frac{1}{h}\tilde{E}[f(Z_{t+h}) - f(Z_t)|\tilde{\mathcal{M}}_t] =$$

$$\lim_{h\to 0}\frac{1}{h}\tilde{E}[\int_{\alpha_t\wedge\tau}^{\alpha_{t+h}\wedge\tau}\{\sum_i u_i(s,w)\cdot\frac{\partial f}{\partial x_i}(X_s)+\sum_{ij}(vv^T)_{ij}(s,\omega)\cdot\frac{\partial^2 f}{\partial x_i\partial x_j}(X_s)\}ds|\tilde{\mathcal{M}}_t]$$

$$\lim_{h\to 0}\frac{1}{h}\tilde{E}[\int_t^{t+h}(Af)(Y^{X_\tau}_{v-\beta_\tau})dv\cdot\chi_{\{t\geq\beta_\tau\}}|\tilde{\mathcal{M}}_t]\qquad(1.33)$$

Applying this to the function $f(x_1,\ldots,x_n) = x_i x_j$ we get by combining (1.32) and (1.33):

$$(\sigma\sigma^T)_{ij}(Z_t) = \lim_{h\to 0}\frac{1}{h}E[\int_{\alpha_t}^{\alpha_{t+h}}(vv^T)_{ij}(s,\omega)ds\cdot\chi_{\{t<\beta_\tau\}}|\tilde{\mathcal{M}}_t]$$

$$+\lim_{h\to 0}\frac{1}{h}\tilde{E}[\int_t^{t+h}(\sigma\sigma^T_{ij})(Z_v)dv\cdot\chi_{\{t\geq\beta_\tau\}}|\tilde{\mathcal{M}}_t]$$

$$= (vv^T)_{ij}(\alpha_t,\omega)\alpha_t' E[\chi_{\{t\geq\beta_\tau\}}|\tilde{\mathcal{M}}_t] + (\sigma\sigma^T)_{ij}(Z_t)\cdot E[\chi_{\{t\geq\beta_\tau\}}|\tilde{\mathcal{M}}_t],$$

by the same argument as in the proof of (1.19).

Hence

$$(1.34)\quad (\sigma\sigma^T)_{ij}(Z_t)E[\chi_{\{t<\beta_\tau\}}|\tilde{\mathcal{M}}_t] = (vv^T)_{ij}\alpha_t' E[\chi_{\{t\geq\beta_\tau\}}|\tilde{\mathcal{M}}_t]$$

Put $B = \{\omega;\ t<\beta_\tau\}$ and let

$$A_0 = \{\omega;\ E[\chi_B|\tilde{\mathcal{M}}_t] = 0\} \in \tilde{\mathcal{M}}_t .$$

Then

$$P(B\cap A_0) = \int_{A_0} \chi_B \cdot dP = \int_{A_0} E[\chi_B|\tilde{\mathcal{M}}_t]dP = 0,$$

so

$$E[\chi_B|\tilde{\mathcal{M}}_t] > 0 \qquad \text{a.s. on } B$$

Therefore we can conclude from (1.34) that for all $t\geq 0$

$$(vv^T)_{ij}(\alpha_t,\omega) = c(\alpha_t,\omega)(\sigma\sigma^T)_{ij}(X_{\alpha_t})$$

for a.a. ω s.t. $t < \beta_\tau(\omega)$.

Thus we obtain the same conclusion (I) as in Theorem 1, except that it is only valid for a.a. t,ω such that $t < \beta_\tau(\omega)$. That completes the proof of Theorem 2.

Corollary 1. Suppose

$$u(t,\omega) = c(t,\omega)b(X_t) \quad\text{and}\quad (vv^T)(t,\omega) = c(t,\omega)(\sigma\sigma^T)(X_t)$$

for a.a. t,ω sucht that $t < \beta_\tau$.

Then X_t is a time change of Y_t, with time change rate $c(t,\omega)$.

Theorem 2 allows us to extend the characterization of Markovian path-preserving functions given in Csink and Øksendal [1] to the case when the time change β_t is not necessarily strictly increasing:

Theorem 3. Let $dS_t = a(S_t)dt + \gamma(S_t)dB_t$ and $dY_t = b(Y_t)dt + \sigma(Y_t)dB_t$ be Ito diffusions on open sets $G \subset \mathbb{R}^p$ and $U \subset \mathbb{R}^n$, respectively. Denote the generators of S_t and Y_t by $\bar{A}$ and A, respectively. Let $\phi: G \to U$ be a C^2 function. Then the following are equivalent:

(1) There exists a continuous function $\lambda \geq 0$ on G such that

$$(1.35) \qquad \bar{A}[f \circ \phi] = \lambda A[f] \circ \phi \quad \text{for all} \quad f \in C^2(U)$$

(2) For each open set D with $\bar{D} \subset G$ the stochastic integral $\phi(S_t)$, $t \leq \tau_{\bar{D}}$ is a time change of X_t, with time change rate $\lambda(S_t)$ (in the sense of (1.23)-(1.24)).

Proof. By the Ito formula we have that $X_t = \phi(S_t)$, $t \leq \tau_W$, satisfies

$$dX_t^{(k)} = (A\phi_k)(S_t)dt + \nabla\phi_k^T(S_t)\gamma(S_t)dB_t, \quad k=1,\ldots,m \quad ,$$

where $X_t^{(k)}$ is component no. k of X_t. Therefore by Theorem 2 (2) holds if and only if

$$(1.36) \qquad E_{\alpha_t \wedge \tau}[A\phi_k(S_t)|X] = b_k(X)E_{\alpha_t \wedge \tau}[\lambda(S_t)|X]$$

and

$$(1.37) \qquad (\nabla\phi_k^T \gamma\gamma^T \nabla\phi_\ell)(S_t) = \lambda(S_t)(\sigma\sigma^T)_{k\ell}(X_t) \quad ; \quad 1 \leq k, \ \ell \leq m,$$

for a.a. t, ω such that $t < \beta_\tau$. Letting $t \to 0$ we se that equation (1.37) is equivalent to

$$(1.38) \qquad \nabla\phi_k^T \gamma\gamma^T \nabla\phi_\ell(x) = \lambda(x)(\sigma\sigma^T)_{k\ell}(\phi(x)), \quad 1 \leq k, \ \ell \leq m$$

for all $x \in G$.

Similarly we claim that (1.36) is equivalent to

(1.39) $\quad A\phi_k(x) = \lambda(x) b_k(\phi(x)) \quad ; \; 1 \le k \le m, \; x \in G.$

It is clear that (1.39) implies (1.36). Conversely, if (1.36) holds we consider two cases:

Case 1: x belongs to the S-fine interior D of $N = \{z; \lambda(z)=0\}$; i.e. $\tau_N = \inf\{t>0; S_t \notin N\} > 0$ a.s. Since $\alpha_{0^+} = \lim_{t \downarrow 0} \alpha_t = \tau_N$ we then get from (1.36) that

$$K(x) = E^x\Big[\int_0^{\tau_N \wedge \tau} (A\phi_k)(S_t)\,dt\Big] = 0 \qquad \text{for all } x \in D.$$

Applying the characteristic operator $\mathfrak{A}$ of S_t to the function K we get (se [7], p.138)

$$0 = \mathfrak{A}K(x) = (A\phi_k)(x) \qquad \text{for all } x \in D,$$

so (1.39) holds in this case.

Case 2: $\tau_N = 0$ a.s. Then we have $\alpha_0 = 0$ a.s. and therefore from (1.36)

$$A\phi_k(x) = \lim_{t \downarrow 0} \frac{1}{E^x[\alpha_t \wedge \tau]} E^x\Big[\int_0^{\alpha_t \wedge \tau} (A\phi_k)(S_r)\,dr\Big]$$

$$= \lim_{t \downarrow 0} \frac{1}{E^x[\alpha_t \wedge \tau]} \cdot E^x\Big[\int_0^{\alpha_t \wedge \tau} \lambda(S_r) b_k(\phi(S_r))\,dr\Big] = \lambda(x) b_k(\phi(x)),$$

as claimed.

We now note that (1.38) and (1.39) are equivalent to requiring that

$$\bar{A}[f \circ \phi] = \lambda A[f] \circ \phi$$

for all polynomials

$$f(x_1, \ldots, x_n) = \sum_i c_i x_i + \sum_{i,j} d_{ij} x_i x_j$$

of degree ≤ 2, and hence that (1.35) holds for all $f \in C^2(U)$.

Remark. It is natural to ask what happens if we allow a more general time change rate $c(t,\omega)$ (not necessarily of the form $\lambda(S_t)$) which makes $\phi(S_t)$ a time change of X_t. However, the argument above gives that if such a $c(t,\omega)$ exists, then as in (1.37)

$$(\nabla\phi_k^T\gamma\gamma^T\nabla\phi_\ell)(S_t) = c(t,\omega)(\sigma\sigma^T)_{k\ell}(X_t) \quad \text{for} \quad 1\leq k,\ \ell\leq m,$$

and so

$$c(t,\omega) = \lambda(S_t)$$

with

$$\lambda(x) = \frac{(\nabla\phi_k^T\gamma\gamma^T\nabla\phi_\ell)(x)}{(\sigma\sigma^T_{k\ell})(\phi(x))},$$

i.e. we have a time change of the type discussed in Theorem 3.

§2. A TIME CHANGE FORMULA FOR ITO INTEGRALS

As an illustration we first use Theorem 1 to characterize the stochastic integrals which are time changes of Brownian motion. If $u=0$ the corresponding result without time change (and with time change if $n=1$) was first proved by McKean ([4], §2.9). The sufficiency of condition (2.1) has been proved by F. Knight [3] (in a martingale setting).

Corollary 2. Let X_t be the n-dimensional stochastic integral in (1.2). Then there exists a time change α_t as above with time change rate $c(t,\omega) \geq 0$ such that

$$X_{\alpha_t} \sim B_t \text{ (n-dimensional Brownian motion)}$$

if and only if

(2.1) $E_{\alpha_t}[u|X] = 0$ for all t and $(vv^T)(t,\omega) = c(t,\omega)I_n$

for all a.a $t \geq 0$, a.a. $\omega \in \Omega$

where I_n is the $n \times n$ identity matrix.

Example 1. If X_t is a 2-dimensional process the form

$$dX_t = v(t,\omega)dB_t$$

where $v \in R^{2\times 2}$ and B_t is 2-dimensional Brownian motion, then X_t is a conformal martingale if and only if

$$(vv^T)(t,\omega) = \eta(t,\omega)I_2 \quad \text{for some} \quad \eta(t,\omega) \geq 0.$$

(See [2]). Thus it follows from Corollary 2 that a conformal martingale is a change of time of Brownian motion (in R^2). This was proved by Getoor and Sharpe ([2]), p. 292-293) and it follows from the result by Knight in [3].

A special case of Corollary 2 is the following:

Corollary 3. Let $c(t,\omega) \geq 0$ be given and let α_t correspond to c as before. Put

$$X_t = \int_0^t \sqrt{c(s,\omega)}\,dB_s$$

where B_s is n-dimensional Brownian motion. Then X_{α_t} is also an n-dimensional Brownian motion.

We now use this to prove that a time change of a stochastic integral is again a stochastic integral, but driven by a different Brownian motion $\tilde{B}_t$. First we construct $\tilde{B}_t$:

Lemma 2. Suppose $t \to \alpha(t,\omega)$ is continuous, $\alpha(0,\omega) = 0$ for a.a

ω. Fix $t>0$. For $k = 1, 2, \ldots$ put

$$t_j = \begin{cases} j \cdot 2^{-k} & \text{if} \quad j \cdot 2^{-k} \leq t \\ t & \text{if} \quad j \cdot 2^{-k} > t \end{cases}$$

and choose r_j such that $\alpha_{r_j} = t_j$.

Suppose $f(s,\omega) \geq 0$ is $\mathcal{F}_s$-adapted and satisfies

$$P^0[\int_0^t f(s,\omega)^2 ds < \infty] = 1$$

Then

$$(2.2) \qquad \lim_{k\to\infty} \sum_j f(\alpha_j,\omega)\Delta B_{\alpha_j} = \int_0^{\alpha_t} f(s,\omega)dB_s \qquad \text{a.s.,}$$

where $\alpha_j = \alpha_{r_j}$, $\Delta B_{\alpha_j} = B_{\alpha_{j+1}} - B_{\alpha_j}$ and the limit is in $L^2(\Omega,P^0)$.

Proof. For all k we hve

$$E[(\sum_j f(\alpha_j,\omega)\Delta B_{\alpha_j} - \int f(s,\omega)dB_s)^2]$$

$$= \sum_j E[(\int_{\alpha_j}^{\alpha_{j+1}} (f(\alpha_j,\omega) - f(s,\omega))dB_s)^2]$$

$$= \sum_j E[\int_{\alpha_j}^{\alpha_{j+1}} (f(\alpha_j,\omega) - f(s,\omega))^2 ds] = E[\int_0^{\alpha_t} (f-f_k)^2 ds],$$

where $f_k(s,\omega) = \sum_j f(t_j,\omega)\chi_{[t_j,t_{j+1})}(s)$ is the elementary approximation to f. (See [7], Ch. III). This implies (2.2) in the case when f is bounded and $t \to f(t,\omega)$ is continuous, for a.a. ω. The proof in the general case follows by approximation in the usual way. (See Ch. III, Steps 1-3 in [7]).

The following result extends a 1-dimensional time change formula proved by Mckean ([4], §2.8).

Theorem 4. (Time change formula for Ito integrals)

Let $(B_s, \mathcal{F}_s)$ be m-dimensional Brownian motion and $v(t,\omega) \in \mathbb{R}^{n\times m}$ as before. Suppose α_t satisfies the conditions in Lemma 2. Define

$$(2.3) \qquad \tilde{B}_t = \lim_{k\to\infty} \sum_j \sqrt{c(a_j,\omega)}\Delta B_{\alpha_j} = \int_0^{\alpha_t} \sqrt{c(s,\omega)}dB_s$$

Then $\tilde{B}_t$ is an (m-dimensional) $\mathcal{F}_{\alpha_t}$-Brownian motion (i.e $\tilde{B}_t$ is a Brownian motion and $\tilde{B}_t$ is a martingale wrt. $\mathcal{F}_{\alpha_t}$) and

$$(2.4) \qquad \int_0^{\alpha_t} v(s,\omega)dB_s = \int_0^t v(\alpha_r,\omega)\cdot\sqrt{\alpha_r'}\, d\tilde{B}_r, \quad \text{a.s. } P^0.$$

where $\alpha_r'(\omega)$ is the derivative of α_r wrt. r, so that

$$(2.5) \qquad \alpha_r'(\omega) = \frac{1}{c(\alpha_r,\omega)} \quad \text{for a.a } r\geq 0,\ \omega\in\Omega.$$

Proof. The existence of the limit in (2.3) and the second identity in (2.3) follows by applying Lemma 2 to the function

$$f(s,\omega) = \sqrt{c(s,\omega)}.$$

Then by Corollary 2 we have that $\tilde{B}_t$ is an $\mathcal{F}_{\alpha_t}$-Brownian motion. It remains to prove (2.4):

$$\begin{aligned}\int_0^{\alpha_t} v(s,\omega)dB_s &= \lim_{k\to\infty}\sum_j v(\alpha_j,\omega)\Delta B_{\alpha_j}\\ &= \lim_{k\to\infty}\sum_j v(\alpha_j,\omega)\sqrt{\frac{1}{c(\alpha_j,\omega)}}\sqrt{c(\alpha_j,\omega)}\Delta B_{\alpha_j}\\ &= \lim_{k\to\infty}\sum_j v(\alpha_j,\omega)\sqrt{\frac{1}{c(\alpha_j,\omega)}}\Delta\tilde{B}_j\end{aligned}$$

$$= \int_0^t v(\alpha_r,\omega)\sqrt{\frac{1}{c(\alpha_r,\omega)}}\,\Delta\tilde{B}_r,$$

and the proof is complete.

We now apply Theorem 4 to the case when the stochastic integral X_t is an Ito diffusion

(2.6) $$dX_t = a(X_t)dt + \gamma(X_t)dB_t$$

where $a: \mathbb{R}^n \to \mathbb{R}^n$, $\gamma: \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous.

Corollary 4. Let X_t be the Ito diffusion given by (2.6) and let $t \to \alpha(t,\omega)$ be absolutely continuous, $\alpha(0,\omega) = 0$ for a.a. ω. Then X_{α_t} is a Markov process wrt. $\mathcal{M}_{\alpha_t}$ if and only if there exists a function $q: R^n \to [0,\infty)$ such that

(2.7) $$c(t,\omega) = q(X_t(\omega))$$

for a.a. $t < \alpha_\infty$, $\omega\in\Omega$, and in that case

(2.8) $$d(X_{\alpha_t}) = \frac{a(X_{\alpha_t})}{q(X_{\alpha_t})}dt + \frac{\gamma(X_{\alpha_t})}{q(X_{\alpha_t})}d\tilde{B}_t$$

where $\tilde{B}_t$ is the $\mathcal{F}_{\alpha_t}$-Brownian motion from Theorem 4.

Proof. If (2.7) holds then (2.8) follows from Theorem 4. Hence X_{α_t} is a weak solution of the stochastic differential equation (2.8) and therefore X_{α_t} is a Markov process. Conversely, if X_{α_t} is a Markov process wrt. $\mathcal{M}_{\alpha_t}$ then by the proof of (III)=>(I)(ii) in Theorem 1 we obtain

(2.9) $$(\gamma\gamma^T)(X_t) = c(t,\omega)(\sigma\sigma^T)(X_t) \qquad \text{for a.a. } t<\alpha_\infty,\ \omega\in\Omega$$

i.e.

$$c(t,\omega) = q(X_t)$$

with

$$q(x) = \frac{(\gamma\gamma^T)(x)}{(\sigma\sigma^T)(x)} .$$

Remark. The last part of this proof does not require that α_t is absolutely continuous.

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