A STOCHASTIC APPROACH TO QUASI-EVERYWHERE BOUNDARY CONVERGENCE OF HARMONIC FUNCTIONS.

Rodrigo Bañuelos and Bernt Øksendal

ABSTRACT

Given a Dirichlet form $\mathcal{E}(\cdot,\cdot)$ on the unit sphere S in $\mathbf{R}^n$ $(n \geq 2)$ associated to a continuous, symmetric convolution semigroup of measures on a group G of isometries on S and given a (G - invariant) Markov process X_t on the open unit ball $\mathbf{B}$, it is shown that for any real function $u \in L^2(S)$ with $\mathcal{E}(u,u) < \infty$ the X_t-harmonic extension $\tilde{u}$ has limit $\check{u}(\theta)$ along a.a. paths X_t conditioned to exit from $\mathbf{B}$ at θ, for quasi-all $\theta \in S$, where $\check{u}$ is a quasi-continuous version of u.

This extends in several ways classical results due to Beurling and Broman about the existence of radial limits quasi-everywhere for a harmonic function in the open unit disc in the plane with a finite Dirichlet integral.

1. Introduction. In 1940 A. Beurling [1] proved the following:

(1.1) If h is a harmonic function in the open unit disc D in the plane $\mathbf{R}^2$ such that h has a finite Dirichlet integral, i.e.

$$\int_D |\nabla h(x)|^2 \, dx < \infty$$

(where dx denotes Lebesgue measure), then

$$\lim_{r \to 1} h(re^{i\theta})$$

exists for quasi-all θ, i.e. for all $\theta \in \partial D \setminus F$ where F is some set in ∂D with $\operatorname{cap} F = 0$ (cap denotes logarithmic capacity).

The following extension of Beurling's theorem was obtained by Carleson in 1967 ([3], Theorem V. 3):

(1.2) Let f be continuous in D with partial derivatives a.e. in D and such that $\theta \to f(re^{i\theta})$ is absolutely continuous for a.a. r and $r \to f(re^{i\theta})$ is absolutely continuous for a.a. θ. Suppose

$$\int_D |\nabla f|^2 (1 - |z|)^\alpha \, dxdy < \infty \qquad (z = x + iy)$$

for some α, $0 \leq \alpha < 1$. Then

$$\lim_{r \to 1} f(re^{i\theta})$$

exists C_α quasi-everywhere, where C_α is the capacity defined by using the kernel $|x|^{-\alpha}$ if $\alpha > 0$ and the kernel $\log 1/|x|$ if $\alpha = 0$ (thus C_0 has the same null sets as logarithmic capacity). In the special case when f is harmonic in D

this result was obtained by A. Broman in 1947 [2].

In this paper, which was inspired by an approach used by Fukushima [11] to quasi-everywhere convergence of Fourier series on ∂D, we prove a stochastic result of this type. The convergence along radial lines / non-tangential convergence is replaced by convergence along the paths of certain Markov processes X_t (e.g. Brownian motion) in the unit ball $\mathbf{B}$ in $\mathbf{R}^n$ for $n \geq 2$ conditioned to exit at specified boundary points and the functions we consider are X_t-harmonic extensions of boundary functions with finite norm wrt a Dirichlet form on the boundary. In the special case when $n = 2$ and X_t is Brownian motion we get Broman's result by choosing the Dirichlet form on ∂D appropriately.

More precisely, let m denote the normalized Lebesgue measure on the unit sphere S of $\mathbf{R}^n$ and let $\mathcal{E}(\cdot,\cdot)$ be the Dirichlet form on $L^2(m)$ associated to a continuous, symmetric convolution semigroup of probability measures on a group G of isometries on S. (See Fukushima [10].) Let Cap denote the capacity associated to $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)$ where $(\cdot,\cdot)$ is the usual inner product in $L^2(m)$. By "Cap quasi-everywhere" we mean "except on a set F with Cap F $= 0$". Let X_t be a Markov process in $\mathbf{B}$ satisfying a certain 'G-invariance' requirement. For $f \in L^1(m)$ let $\tilde{f}$ denote the X_t-harmonic extension of f to $\mathbf{B}$. (If X_t is Brownian motion B_t then $\tilde{f}$ coincide with the classical harmonic extension of f.) Then our main result is the following (Theorem 1):

(1.3) If u is a real function on S such that $\mathcal{E}(u,u) < \infty$ then

$$\lim_{t \to T} \tilde{u}(X_t^\theta) = \check{u}(\theta) \qquad \text{a.s.}$$

for quasi-all $\theta \in S$, where $\check{u}$ is a quasi-continuous version of u and X_t^θ is the process X_t conditioned to exit from $\mathbf{B}$ at θ.

2. Quasi-everywhere boundary convergence. Let $\{\nu_t\}_{t\geq 0}$ be a continuous symmetric convolution semigroup of probability measures ν_t on a group G of isometries on S, i.e.

(i) $\nu_t * \nu_s = \nu_{t+s}$, $\quad t,s > 0$

(ii) $\int_G f(\gamma)\, d\nu_t(\gamma) = \int_G f(\gamma^{-1})\, d\nu_t(\gamma)$ for all bounded Borel functions f on G

(iii) $\lim_{t\to 0} \nu_t = \delta$

where $*$ denotes convolution and δ is the Dirac measure at $1 \in G$.

Let (X_t,Ω,P^x) be a strong Markov process in $\mathbf{B}$ with continuous paths and a (possibly infinite) lifetime τ. We assume that no killing of X_t occurs inside $\mathbf{B}$ and that X_t satisfies the following conditions (2.1), (a) - (c), (2.2): (Note that these conditions are satisfied for Brownian motion B_t)

(2.1) $$X_\tau = \lim_{t\to\tau} X_t \in S \quad \text{exists a.s.} \quad P^x$$

for all $x \in \mathbf{B}$. Moreover, if we define the X_t-harmonic measure λ_x by

$$\lambda_x(F) = E^x[X_\tau \in F], \qquad F \subset S \text{ Borel set}$$

then λ_x is absolutely continuous wrt m and

$$\frac{d\lambda_x}{dm} = K(x,\theta)$$

where

(a) $K(x,\theta) > 0$ for all $x \in \mathbf{B}, \quad \theta \in S$

(b) $K(x,\theta) \to 0$ as $x \to \xi \in S\setminus\{\theta\}$

(c) $\theta \to K(x,\theta)$ is uniformly continuous for $x \in H$, if $H \subset \mathbf{B}$ is compact.

(2.2) (G-invariance.) For any isometry $\gamma \in G$ we have that X_t with probability law P^x has the same finite-dimensional distributions as γX_t with probability law $P^{\gamma x}$.

In particular, this implies that

(i) $K(x,\theta) = K(\gamma x, \gamma\theta)$

and

(ii) $K(0,\theta) = 1$.

For $f \in L^1(m)$ we define its X_t-harmonic extension $\tilde{f}$ by

$$\tilde{f}(x) = E^x[f(X_T)] = \int f(\phi)K(x,\phi)\, dm(\phi)\,. \tag{2.3}$$

Now define

$$p_t(\xi,f) = \int_G f(\gamma\xi)\, d\nu_t(\gamma), \qquad \xi \in S,\ f \in C(S). \tag{2.4}$$

Then $p_t(\cdot,\cdot)$ is a strongly continuous Markovian transition function. Moreover, p_t is m-symmetric in the sense that

$$\int_S u(\xi)p_t(\xi,v)\, dm(\xi) = \int_S v(\eta)p_t(\eta,u)\, dm(\eta)$$

for all $u,v \in C(S)$.

This is because

$$\int u(\xi)p_t(\xi,v)\,dm(\xi) = \int\int u(\xi)v(\gamma\xi)\,d\nu_t(\gamma)\,dm(\xi)$$

$$= \int\int u(\xi)v(\gamma^{-1}\xi)\,d\nu_t(\gamma)\,dm(\xi) = \int\int u(\gamma\xi)v(\xi)\,d\nu_t(\gamma)\,dm(\xi)\,,$$

using (ii) and that m is isometry invariant on S.

Let $\mathcal{E}$ be the regular Dirichlet form on $L^2(m)$ associated with $\mathfrak{p}_t$ (see [10], p. 29-30). Put $\mathcal{F} = \mathcal{D}(\mathcal{E})$ (the domain of definition of $\mathcal{E}$) and let A be the non-positive definite self-adjoint operator given by

$$\mathcal{E}(u,v) = -(u,Av), \quad \mathcal{D}(A) = \mathcal{F} \tag{2.5}$$

where $(\cdot,\cdot)$ denotes the usual inner product in $L^2(m)$. As in Fukushima [11] we now define

$$V = (I - A)^{-1/2}\,. \tag{2.6}$$

Then we have:

LEMMA 1. *(See Fukushima [11], p. 131-132.)*

(a) $(Vf)(\xi) = \int_0^\infty \frac{1}{\sqrt{\pi s}}\, e^{-s}\,(p_s f)(\xi)\,ds; \quad f \in L^2(m).$

(b) Vf *is quasi-continuous for each* $f \in L^2(m)$.

(c) $\mathcal{F} = \{Vf; f \in L^2(m)\}$.

(d) $\mathcal{E}_1(Vf,Vg) = (f,g)$ for $f,g \in L^2(m)$, where $\mathcal{E}_1(\cdot,\cdot) = \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)$.

(e) If $u \in \mathcal{F}$ and $\check{u}$ denotes a quasi-continuous version of u then

$$\mathrm{Cap}\{\theta; |\check{u}(\theta)| > \lambda\} \leq \frac{1}{\lambda^2}\, \mathcal{E}_1(u,u) .$$

Here Cap denotes the capacity wrt $\mathcal{E}$, i.e.

$$\mathrm{Cap}(U) = \inf\{\mathcal{E}_1(u,u);\, u \in \mathcal{F},\ u \geq 1 \text{ a.e. on } U\} \tag{2.7}$$

if U is open and

$$\mathrm{Cap}(H) = \inf\{\mathrm{Cap}(U);\, U \text{ open},\ U \supset H\}$$

for general H.

We say that g is quasi-continuous (wrt Cap) if for all $\epsilon > 0$ we can find a set H with $\mathrm{Cap}(H) < \epsilon$ such that $g \mid S \backslash H$ is continuous.

Combining (a) from Lemma 1 with the definition (2.5) of p_t we obtain:

LEMMA 2. *If* $f \in L^2(m)$ *then*

$$(Vf)\,(\xi) = \int_G f(\gamma\xi)\, d\mu(\gamma) \tag{2.8}$$

where μ *is the measure on* G *defined by*

$$\mu(F) = \int_0^\infty \frac{1}{\sqrt{\pi s}}\, e^{-s} \nu_s(F)\, ds, \quad F \subset G .$$

If g is a function on $\mathbf{B}$ we can for each $r \in (0,1)$ associate a function g_r on S by

$$(g_r)\,(\xi) = g(r\xi)\,, \quad \xi \in S . \tag{2.9}$$

With this notation we have:

LEMMA 3. *Let* $f \in L^2(m)$ *and let* $\tilde{f}$ *denote the* X_t*-harmonic extension of* f*, given by (2.2).*

Then

$$(\widetilde{Vf})_r(\xi) = V(\tilde{f})_r(\xi) \quad \text{for all } \xi \in S .$$

Proof.

$$\begin{aligned}
(\widetilde{Vf})_r(\xi) &= \int (Vf)\,(\phi) K(r\xi,\phi)\, dm(\phi) \\
&= \int\int f(\gamma\phi) K(r\xi,\phi)\, d\mu(\gamma)\, dm(\phi) && \text{(by Lemma 2)} \\
&= \int\int f(\phi) K(r\xi,\gamma^{-1}\phi)\, d\mu(\gamma)\, dm(\phi) && (m \text{ isometry invariant}) \\
&= \int\int f(\phi) K(\gamma(r\xi),\phi)\, d\mu(\gamma)\, dm(\phi) && \text{(by (2.1),d))} \\
&= \int (\tilde{f})_r(\gamma\xi)\, d\mu(\gamma) = V(\tilde{f})_r(\xi) .
\end{aligned}$$

If g is a bounded Borel function on $\mathbf{B}$ we define

$$(Vg)\,(x) = (Vg_r)\,(\theta) \quad \text{where} \quad \theta = \frac{x}{|x|}\,, \quad r = |x| .$$

Note that by (2.8) and (2.9) we still have

$$(2.10) \qquad (Vg)\,(x) = \int_G g(\gamma x)\, d\mu(\gamma) \quad \text{for} \quad x \in \mathbf{B} .$$

Next we explain Doob's concept of a conditioned X_t-process (see Doob [7] and [8] for details):

Let $h > 0$ be an X_t-*harmonic* function, i.e.

$$h(x) = E^x[h(X_\beta)] \quad \text{for all stopping times} \quad \beta < \tau .$$

Then we put

$$(T_t^h f)(x) = \frac{T_t(fh)(x)}{h(x)} \quad ; \quad f \in C_0(\mathbf{B})$$

$(C_0(\mathbf{B}) = \{f \in C(\mathbf{B});\ f$ has compact support$\})$ where $(T_t f)(x) = E^x[f(X_t)]$ is the transition function of X_t.

The semigroup $\{T_t^h\}_{t\geq 0}$ will be the transition function of a strong Markov process denoted by X_t^h with probability law P_h^x, i.e. we have

$$E_h^x[f(X_t^h)] = \int_\Omega f(X_t^h)\, dP_h^x = \frac{E^x[f(X_t)h(X_t)]}{h(x)} .$$

In particular, for a fixed $\theta \in S$ we have that

(2.11) $\qquad h(x) = K(x,\theta)$ is X_t- harmonic in $\mathbf{B}$.

This can be seen as follows:

By the strong Markov property we know that for all $f \in L^2(m)$ the function $\bar{f}(x) = E^x[f(X_\tau)]$ is X_t-harmonic in $\mathbf{B}$ (see e.g. [12], (7.17) in Ch. VII). Therefore

$$\bar{f}(x) = E^x[\bar{f}(X_\beta)] = E^x[\int f(\phi)K(X_\beta,\phi)\, dm(\phi)]$$

$$= \int f(\phi)E^x[K(X_\beta,\phi)]\, dm(\phi), \quad \text{for all stopping times} \quad \beta < \tau .$$

Since this holds for all $f \in L^2(m)$ we conclude that

$$E^x[K(X_\beta,\phi)] = K(x,\phi) \text{ for a.a. } \phi \text{ with respect to } m .$$

So by condition (2.1) (c) we obtain (2.11).

From now on we will let X_t^θ denote the $K(\cdot,\theta)$-conditioned X_t-process and we abbreviate $P^x_{K(\cdot,\theta)}$ to P^x_θ.

By condition (2.1) (b) we know that

$$(2.12) \qquad X_t^\theta \to \theta \text{ a.s. } P^x_\theta \text{ as } t \to \tau^\theta ,$$

where τ^θ is the life time (i.e. the first exit time from $\mathbf{B}$) of X_t^θ.

The next result gives a crucial connection between the expectation involving the conditioned process and the conditional expectation of the original process: (We interpret X_t as X_τ if $t \geq \tau$ and similarly with X_t^θ in order to simplify the notation.)

LEMMA 4. *Let g be a bounded Borel function on $\overline{\mathbf{B}}$. Then*

$$E^x[g(X_t) \mid X_\tau = \theta] = E^x_\theta[g(X_t^\theta)] .$$

Proof. We must show that

$$E^x[g(X_t) \mid X_\tau] = (E^x_\theta[g(X_t^\theta)])_{\theta=X_\tau}$$

i.e. that

$$(2.13) \qquad E^x[f(X_\tau)g(X_t)] = E^x[f(X_\tau)E^x_\theta[g(X_t^\theta)]_{\theta=X_\tau}]$$

for all bounded Borel functions f.

The right hand side of (2.13) is

$$E^x\left[f(X_\tau)\left(\frac{E^x[g(X_t)K(X_t,\theta)]}{K(x,\theta)}\right)_{\theta=X_\tau}\right] = \int f(\theta)\,\frac{E^x[g(X_t)K(X_t,\theta)]}{K(x,\theta)}\,K(x,\theta)\,dm(\theta)$$

$$= E^x[g(X_t)\int f(\theta)K(X_t,\theta)\,dm(\theta)] = E^x[g(X_t)\,E^{X_t}[f(X_\tau)]]$$

$$= E^x[E^x[g(X_t)f(X_\tau)|\mathcal{M}_\tau]] = E^x[g(X_t)f(X_\tau)],$$

where $\mathcal{M}_\tau$ is the σ-algebra generated by $\{X_{t\wedge\tau}\,; t \geq 0\}$ and we have used the strong Markov property for X_t (see e.g. (7.15), Ch. VII in [12]).

LEMMA 5. *Let* g *be a bounded Borel function on* $\overline{\mathbf{B}}$. *Then, with* $E_\theta = E^0_\theta,\ E = E^0$

$$E_\theta[(Vg)\,(X^\theta_t)] = V(E_\theta[g(X^\theta_t)]), \quad \theta \in S\,.$$

Proof.

$$E_\theta[(Vg)\,(X^\theta_t)] = E_\theta[\int g(\gamma X^\theta_t)\,d\mu(\gamma)]$$

$$= E[(\int g(\gamma X_t)\,d\mu(\gamma))K(X_t,\theta)] \qquad \text{(by (2.2) (ii))}$$

$$= \int E[g(\gamma X_t)K(\gamma X_t,\gamma\theta)]\,d\mu(\gamma) \qquad \text{(by (2.2) (i))}$$

$$= \int E[g(X_t)K(X_t,\gamma\theta)]\,d\mu(\gamma) \qquad \text{(by (2.2))}$$

$$= V(E_\theta[g(X_t^\theta)]) \ .$$

LEMMA 6. *Let* $f \in L^2(m)$. *Then*

$$- f(X_\tau)$$

$$E^x[|\bar{f}(X_{t\wedge\tau})|^2] \to 0 \quad \text{as} \quad t \to \infty$$

Proof. Let $M_t = \bar{f}(X_{t\wedge\tau})$.

Then we see that M_t is a martingale wrt the σ- algebras $\mathcal{M}_t$ generated by $\{X_{s\wedge\tau}\,;\, s \leq t\}$:

$$E^x[M_s|\mathcal{M}_t] = E^x[\bar{f}(X_{s\wedge\tau})|\mathcal{M}_t] = E^{X_{t\wedge\tau}}[\bar{f}(X_{(s-t)\wedge\tau})] = \bar{f}(X_{t\wedge\tau}) \ ,$$

for all $s > t$, since $\bar{f}$ is X_t-harmonic.

Moreover,

$$E[|M_t|^2] \leq E[|M_\tau|^2] = E[|f(X_\tau)|^2] = \|f\|^2 \ .$$

So Lemma 6 follows from the martingale convergence theorem. (See e.g. [13].)

We are now ready to prove the main result of this paper:

THEOREM 1. *Let* $\{\nu_t\}$ *be a continuous symmetric convolution semigroup of measures on a group* G *of isometries on* S *and let* X_t *be any Markov process in* **B** *satisfying the conditions (2.1), (2.2) above.*

Let $u \in \mathcal{F}$. Then for quasi-all $\theta \in S$ we have that

$$\lim_{t\to\tau} \bar{u}(X_t^\theta) = \check{u}(\theta) \qquad \text{a.s.} \quad P_\theta^0 \ .$$

Proof. Write $E_\theta = E_\theta^0$ and $E = E^0$.

For $0 \le r < 1$ put

$$\rho = \rho_r = \inf\{t > 0 \,;\, |X_t^\theta| = r\}$$

$$\tau_r = \inf\{t > 0 \,;\, |X_t| = r\}\,.$$

Choose $f \in L^2(m)$ such that $u = Vf$ (Lemma 1 c)).

Then for $\lambda > 0$ we have

$$\mathrm{Cap}\{\theta\,;\, E_\theta[\sup_{t>\rho} |\tilde{u}(X_t^\theta) - \check{u}(\theta)|] > \lambda\}$$

$$= \mathrm{Cap}\{\theta\,;\, E_\theta[\sup_{t>\rho} |(Vf)\,(X_t^\theta) - (Vf)\,(\theta)|] > \lambda\}$$

$$= \mathrm{Cap}\{\theta\,;\, E_\theta[\sup_{t>\rho} |V\{\tilde{f}(X_t^\theta) - f(\theta)\}|] > \lambda\} \quad \text{(by Lemma 3)}$$

$$\le \mathrm{Cap}\{\theta\,;\, V(E_\theta[\sup_{t>\rho} |\tilde{f}(X_t^\theta) - f(\theta)|]) > \lambda\} \quad \text{(because } g \le h \Rightarrow Vg \le Vh \text{ and by Lemma 5)}$$

$$\le \frac{1}{\lambda^2}\, \mathcal{E}_1(V(E_\theta[\sup_{t>\rho} \cdots]),\, V(E_\theta[\sup_{t>\rho} \cdots])) \qquad \text{(by Lemma 1 b), e))}$$

$$= \frac{1}{\lambda^2} \| E_\theta[\sup_{t>\rho} |\tilde{f}(X_t^\theta) - f(X_\tau^\theta)|] \|_{L^2(m)}^2 \qquad \text{(by Lemma 1 d))}$$

$$= \frac{1}{\lambda^2} \| E[\sup_{t>\tau_r} |\tilde{f}(X_t) - f(X_\tau)| \,|\, X_\tau = \theta] \|_{L^2(m)}^2 \quad \text{(by Lemma 4)}$$

$$= \frac{1}{\lambda^2}\, E[(E[\sup_{t>\tau_r} |\tilde{f}(X_t) - f(X_\tau)| \,|\, X_\tau])^2] \qquad \text{(by (2.1) c))}$$

$$\le \frac{1}{\lambda^2}\, E[\sup_{t>\tau_r} |\tilde{f}(X_t) - f(X_\tau)|^2] \qquad \text{(cond. exp. reduces } L^2 \text{ norm)}$$

$$\leq \frac{2}{\lambda^2} E[|\bar{f}(X_\tau) - \bar{f}(X_{\tau_r})|^2] \qquad \text{(martingale inequality)}$$

$\to 0$ as $r \to 1$ by Lemma 6.

So we have proved that for all $\lambda > 0$

$$\text{Cap}\{\theta \,;\, \liminf_{r\to 1} (E_\theta[\sup_{t>\rho} |\bar{u}(X_t^\theta) - \check{u}(\theta)|]) > \lambda\} = 0 \,.$$

Hence

$$\text{Cap}\{\theta \,;\, \liminf_{r\to 1} (E_\theta[\sup_{t>\rho} |\bar{u}(X_t^\theta) - \check{u}(\theta)|] > 0\} = 0 \,.$$

So for quasi-all θ we have by monotone convergence

$$E_\theta[\lim_{r\to 1} (\sup_{t>\rho} |\bar{u}(X_t^\theta) - \check{u}(\theta)|)] = 0 \,.$$

Hence

$$\lim_{r\to 1} (\sup_{t>\rho} |\bar{u}(X_t^\theta) - \check{u}(\theta)|) = 0 \quad \text{a.s. } P_\theta^0$$

i.e.

$$\lim_{t\to\tau} \bar{u}(X_t^\theta) = \check{u}(\theta) \qquad \text{a.s. } P_\theta^0$$

for quasi-all $\theta \in S$. That completes the proof.

3. Examples. We now look at the special case when $n = 2$, i.e. **B** is the unit disc D in the plane. Then it is known (see [10] p. 31) that there is a 1-1 correspondence between the continuous symmetric convolution semigroups $\{\nu_t\}_{t\geq 0}$ and the set of all real sequences $\lambda = \{\lambda_n\}_{-\infty}^{\infty}$ satisfying

(3.1) $$\lambda_0 = 0, \quad \lambda_n = \lambda_{-n}$$

and

(3.2) $$\sum(\lambda_n + \lambda_m - \lambda_{n-m})\rho_n\rho_m \geq 0$$

for any real sequence $\{\rho_n\}$ with finite support. This correspondence is given by

(3.3) $$\hat{\nu}_t(n) = \int e^{in\theta}\, d\nu_t(\theta) = e^{-t\lambda_n} \quad \text{for all } n ,$$

and the Dirichlet form corresponding to λ_n is

$$\mathcal{E}(u,u) = \sum_n |\hat{u}(n)|^2 \lambda_n .$$

Examples of sequences $\{\lambda_n\}$ satisfying (3.1) and (3.2) are

(3.4) $$\lambda_n = |n|^{1-\alpha} \quad \text{where} \quad -1 < \alpha < 1$$

and

(3.5) $$\lambda_n = \log(1 + |n|) .$$

In particular, if we choose

$$\lambda_n = |n|$$

then the corresponding Dirichlet form $\mathcal{C}$ on $L^2(\partial D)$ is given by

(3.6) $$\mathcal{C}(u,u) = \pi \sum_{-\infty}^{\infty} |\hat{u}(n)|^2 \, |n| ,$$

where $\hat{u}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta}\, u(\theta)\, d\theta$ is the n'th Fourier coefficient of

$u(\theta) \in L^2(\partial D)$.

Moreover, we have

$$C(u,u) = D(\tilde{u},\tilde{u}) , \tag{3.7}$$

where $\tilde{u}$ is the classical harmonice extension of u to D and D denotes the classical Dirichlet form

$$D(f,g) = \frac{1}{2} \int_D \nabla f \cdot \nabla g \, dx . \tag{3.8}$$

(See [10], p. 12.) Therefore the (classical) harmonic functions h in D with bounded Dirichlet integral are exactly the harmonic extensions $\tilde{u}$ of functions $u \in L^2(\partial D)$ with $C(u,u) < \infty$.

In fact, we have the following more general connection between Dirichlet forms on ∂D and in D:

LEMMA 7. *Let $\mathcal{E}_\alpha(\cdot,\cdot)$ be the Dirichlet form on ∂D corresponding to*

$$\lambda_n = |n|^{1-\alpha} \quad \text{where} \quad -1 < \alpha < 1 .$$

Then we have

$$\mathcal{E}_\alpha(u,u) < \infty \Leftrightarrow \int_D |\nabla \tilde{u}|^2 (1 - |z|)^\alpha \, dxdy < \infty \tag{3.9}$$

where $z = x + iy$.

Proof. We may assume $u(\theta) = \sum_{n=0}^\infty a_n e^{in\theta}$. Then $\tilde{u}(re^{i\theta}) = \sum a_n r^n e^{in\theta} = \sum a_n z^n$, $|\nabla \tilde{u}|^2 = |\tilde{u}'(z)|^2 = |\sum n a_n z^{n-1}|^2$.

Hence

$$\int_D |\nabla \bar{u}|^2 (1-|z|)^\alpha \, dxdy = \int_0^1 \int_0^{2\pi} \sum_{n,m} nm a_n \bar{a}_m r^{n+m-1} e^{i\theta(n-m)} (1-r)^\alpha \, d\theta dr$$

$$= \sum_n |a_n|^2 n^2 \int_0^1 r^{2n-1} (1-r)^\alpha \, dr \, .$$

Now $\int_0^1 r^{2n-1}(1-r)^\alpha \, dr = B(2n, 1+\alpha) = \dfrac{\Gamma(2n)\Gamma(1+\alpha)}{\Gamma(2n+1+\alpha)}$

$$\sim \frac{(2n)^{2n-1/2} e^{-2n}}{(2n+1+\alpha)^{2n+1+\alpha-1/2} \cdot e^{-2n-1-\alpha}}$$

$$\sim \left(\frac{2n}{2n+1+\alpha}\right)^{2n-1/2} \cdot (2n+1+\alpha)^{-1-\alpha} \sim n^{-1-\alpha} \, ,$$

where $a \sim b$ means that $1/c \, b \leq a \leq cb$ for some constant c.

Therefore

$$\int_D |\nabla \bar{u}|^2 (1-|z|)^\alpha \, dxdy \sim \sum_n |a_n|^2 |n|^{1-\alpha} \, ,$$

which proves Lemma 6.

It remains to relate the capacity wrt $\mathcal{E}_\alpha$, Cap_α, to the classical capacities C_α. The following result (as the preceding) is well known to experts, but it seems to be hard to find it in the literature.

LEMMA 8. Let Cap_α denote the capacity associated to the Dirichlet form $\mathcal{E}_\alpha$ corresponding to

$$\lambda_n = |n|^{1-\alpha} \, ; \quad 0 \leq \alpha < 1 \, .$$

Then

(3.10) $$\mathrm{Cap}_\alpha(F) \sim C_\alpha(F), \quad F \subset \partial D .$$

For completeness we sketch a proof:

Put $\gamma_n = |n|^{1-\alpha} + 1$ and define

$$K(x) = \sum_{-\infty}^{\infty} \frac{\cos nx}{\gamma_n} .$$

Then

$$K(x) \sim |x|^{-\alpha} ,$$

because $\int_0^{2\pi} x^{-\alpha} \cos nx \, dx = |n|^{\alpha-1} \int_0^{2\pi n} u^{-\alpha} \cos u \, du \sim 1/\gamma_n$.

The energy $E[\mu]$ of a measure μ wrt K is

$$E[\mu] = \int\int K(x-y) \, d\mu(x) \, d\mu(y) = \sum_n \frac{|\hat{\mu}(n)|^2}{\gamma_n} .$$

If $u \geq 1$ on an open set $U \subset \partial D$ and μ is a positive measure on U we have

$$(\mu(U))^2 \leq (\int u d\mu)^2 = |\sum_n \hat{u}(n)\hat{\mu}(n)|^2 \leq (\sum_n |\hat{u}(n)|^2 \gamma_n) \left(\sum_n \frac{|\hat{\mu}(n)|^2}{\gamma_n}\right)$$

$$\leq A_1 \mathcal{E}_1(u,u) E[\mu], \quad \text{where } A_1 \text{ is a constant} .$$

Hence

$$\mathcal{E}_1(u,u) \geq \frac{1}{A_1} \frac{\mu(U)^2}{E[\mu]} .$$

By taking the supremum of the right hand side over all μ with $\mu(U) = 1$ we obtain

$$\mathcal{E}_1(u,u) \geq \frac{1}{A_1} C_\alpha(U) .$$

Hence

$$\mathrm{Cap}_\alpha(U) \geq \frac{1}{A_1} C_\alpha(U) .$$

To get the opposite inequality we use that if β is the positive measure on U with $\beta(U) = 1$ which minimizes $E[\beta]$, then

$$v(x) = \int K(x-y)\, d\beta(y)$$

satisfies $v(x) = E[\beta]$ a.e. on U (see Carleson [5], p. 17).

Hence

$$\mathrm{Cap}_\alpha(U) \leq \mathcal{E}_1\left(\frac{u}{E[\beta]}, \frac{u}{E[\beta]}\right) \leq \frac{A_2}{E[\beta]^2} \sum_n \frac{|\hat{\beta}(n)|^2}{\gamma_n} = \frac{A_2}{E[\beta]} = A_2 C_\alpha(U) ,$$

where A_2 is a constant.

That completes the proof of Lemma 7.

Combining Theorem 1 with Lemma 6 and 7 we obtain the following stochastic analogue of Broman's theorem:

COROLLARY 1. Let h be a harmonic function in D such that

$$\int_D |\nabla h|^2 (1 - |z|)^\alpha \, dxdy < \infty$$

for some α, $0 \le \alpha < 1$. Then

$$\lim_{t \to \tau} h(B_t^\theta)$$

exists a.s. P_θ^0 for quasi-all $\theta \in \partial D$ wrt the capacity C_α.

A natural question is: Does convergence of a given harmonic function along a.a. conditional paths B_t^θ for a fixed θ imply non-tangential convergence at θ? For $n = 2$ the answer is yes. This is a result essentially due to Davis [6] and Burkholder, Gundy and Silverstein [3]. For a complete proof see Durrett [9]. Therefore, Corollary 1 implies the result by Broman stated earlier.

In order to obtain similar results for the unit ball $\mathbf{B}$ in $\mathbf{R}^n$ for $n > 2$ one would have to investigate the continuous symmetric convolution semigroups of probability measures on the given group G of isometries on S and then try to relate the capacity corresponding to the associated Dirchlet forms to the classical capacities. This topic will not be discussed here.

Finally we mention that the technique used above also applies to continuous symmetric convolution semigroups of probability measures on $\mathbf{R}^n$. Using the description of such semigroups given by the Lévy-Khinchin formula (see [10], p. 29) one can proceed as above and obtain results about quasi-everywhere boundary convergence of harmonic functions in the half-space $\mathbf{R}^n \times [0,\infty)$ along conditional Brownian paths. This raises the question whether convergence of a given harmonic function in $\mathbf{R}^n \times [0,\infty)$ along (a.a.) conditional paths B_t^x for *quasi-all* $x \in \mathbf{R}^n$ (for example wrt Newtonian capacity in $\mathbf{R}^{n+1}$) implies non-tangential convergence for quasi-all x. If one replaces quasi-all with almost all (Lebesgue measure) then the answer is known to be *no*, by an example due to Burkholder and Gundy [4].

Acknowledgments. We are grateful to L. Carleson, J. L. Doob and R. Durrett for useful conversations. B. Øksendal is partially supported by Norges Almenvitenskapelige Forskningsrad (NAVF), Norway.

REFERENCES

[1] A. Beurling: Ensembles exceptionelles. Acta Math. 72 (1940), 1-13.

[2] A. Broman: On two classes of trigonometrical series. Thesis, Uppsala 1947.

[3] D. L. Burkholder, R. F. Gundy and M. L. Silverstein. A maximal function characterization of the class H^p. Trans. Amer. Math. Soc. 157 (1971), 137-153.

[4] D. L. Burkholder and R. F. Gundy: Boundary behaviour of harmonic functions in a half-space and Brownian motion. Ann. Inst. Fourier 23 (1973), 195-212.

[5] L. Carleson: Selected Problems on Exceptional Sets. Van Nostrand 1967.

[6] B. Davis: Brownian motion and analytic functions. The Annals of Probability 7(1979), 913-932.

[7] J. L. Doob: Conditional Brownian motion and the boundary limits of harmonic functions. Bull. Soc. Math. France 85 (1957), 431-458.

[8] J. L. Doob: Classical Potential Theory and Its Probabilistic Counterpart. Springer-Verlag 1984.

[9] R. Durrett: Brownian Motion and Martingales in Analysis. Wadsworth 1984.

[10] M. Fukushima: Dirichlet Forms and Markov Processes. North-Holland/Kodansha 1980.

[11] M. Fukushima: Capacitary maximal inequalities and an ergodic theorem. In Probability Theory and Mathematical Statistics (Tbilisi 1982), Springer LNM 1021 (1983), 130-136.

[12] B. Øksendal: An Introduction to Stochastic Differential Equations with Applications. Springer-Verlag 1985

[13] D. W. Stroock and S. R. S. Varadhan: Multidimensional Diffusion Processes. Springer-Verlag 1979.

R.Bañuelos
Department of Mathematics
Caltech
Pasadena, CA 91125

B. Øksendal
Department of Mathematics
UCLA
Los Angeles, CA 90024

Current address of B. Øksendal:
Dept. of mathematics
University of Oslo, Box 1053
N-0316 Blindern, Oslo 3
NORWAY