A NXN CLASS OF SYSTEMS OF HYPERBOLIC CONSERVATION LAWS

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Abstract

The Riemann problem for a nonlinear class of systems of first order hyperbolic conservation laws is studied. In the system the matrix which is the derivative of the flux function, is lower triangular. In the class there is both strictly and non-strictly hyperbolicy. There is no assumptions on genuine nonlinearity. Existence and uniqueness are proved except in an area with measure zero where there is multiple solution. An example show that the solution does not depend continuously on the data. Numerical methods are discussed.

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1 INTRODUCTION

In this paper we study the Riemann problem for the system of differential equation

(1.1) $$u_{i_t} + f_i(u_1,\dots,u_i)_x = 0, \qquad i=1,2,\dots,n$$

where f is continuous and f and $\frac{\partial f_i}{\partial u_i}$ where defined, is piecewise monotone with a finite number of intervals where the functions are monotone. In the Riemann problem the initial condition is

(1.2) $$u_i(x,0) = \begin{cases} u_{i,-} & \text{for } x < 0 \\ u_{i,+} & \text{for } x > 0 \end{cases} \qquad i=1,2,\dots,n.$$

In order to classify the problem we study the matrix

$$\{ \frac{\partial f_i}{\partial u_j} \}_{i,j}.$$

In problem (1.1) this matrix is lower triangular. The eigenvalues to the problem are the diagonal elements. The problem is therefore hyperbolic. We will name the problem a lower triangular hyperbolic system. There is no assumption that the eigenvalues are distinct. Therefore the class contains both strictly and non-strictly hyperbolicy. Genuine nonlinearity in this case reduces to that

$$\frac{\partial^2 f_i}{\partial^2 u_j}$$

does not vanish for any values of u. This assumption is not done in this paper.

For n=1, i.e. the scalar problem, existence and uniqueness are well-known. See e.g. Oleinik [9] and [10] and Smoller [11]. For systems most of the published papers are either for n=2 see e.g. Smoller [12], Keyfitz and Kranzer [5] and [6] or for the strictly hyperbolic case see e.g. Lax [7].

The Riemann problem is a particular physical problem where it is possible to find an analytic solution. In addition it is used as building blocks in the Cauchy problem with general initial data. In fact, the Riemann problem is used both for existence and uniqueness theorems and as a numerical method. It is used in both ways in the celebrated paper by Glimm [1] and in a paper by Holden, Holden and Høegh-Krohn [3]. Godunov [2] uses the Riemann problem in a numerical method.

It is two main reasons to study a lower triangular hyperbolic system.

By restricting us to lower triangular systems we are able to solve a general nxn system, and therefore find some caracteristics for the general problem.

It is possible to approximate the solution of some physical problems with the solution of (1.1).

There exist no smooth solution of (1.1) and (1.2), no matter how smooth the flux function is. Therefore we are interested in weak solutions. There are several smooth solutions to the problem. We use an entropy criteria in order to find the relevant solution. The origin for the entropy criteria is that the solution is the limiting solution when a second order term vanish. A unique solution must be on the form

$$u(x,t) = v(x-st).$$

Substituted into the equation

$$u_t + f(u)_x = \varepsilon\, u_{xx}$$

we get

$$-s\, v' + (f(v))' = \varepsilon v''.$$

Scaling the equation and integrating gives

$$v'(t) = f(v(t)) - s\, v(t) - C.$$

Thuse the following entropy criteria for a shock with speed s between v_- and v_+ is used in the paper. There exist an integral curve

$$v'(t) = f(v(t)) - s\, v(t) - (\, f(v_\pm) - s\, v_\pm) .$$

and $\quad v(t) \to v_\pm \quad$ when $\quad t \to \pm\infty$.

We name this integral curve an entropy curve in order to separate it from other integral curves.

The solution of the Riemann problem consist of several shocks satisfying the entropy condition and smooth solution satisfying the equation following each other with increasing speed s. Smooth solutions are in this connection called rarefaction waves.

In order to allways get a solution we have to accept shocks following each other and have the same speed. This is necessary also in the scalar equation.

In the following chapter we prove existence of a solution of (1.1) and (1.2) for all initial values and uniqueness almost everywhere. Some characteristics of the solution are discussed in chapter 3. We show that the Lax shock inequalities are not valid for non-strictly hyperbolic systems. An example show that the solution does not depend continuously on the initial function. Finally some numerical methods are discussed. The general solution depends on the entropy curves. This slow down the numerical method. When the eigenvalues are in distinct intervals and f is piecewise linear, it is possible to find the solution only with convex/concave envelopes.

2 EXISTENCE AND UNIQUENESS

(1.1) and (1.2) is solved by one equation at a time. The first is a scalar equation and existence and uniqueness is well-known. This is stated as the first theorem

Theorem 2.1

The scalar Riemann problem

$$u_t + f(u)_x = 0$$

where f is locally Lipchitz continuous with inital value

$$u(x,0) = \begin{cases} u_+ & \text{for } x > 0 \\ u_- & \text{for } x < 0 \end{cases}$$

has a unique solution which may be described uniquely by a function $u(s)$ where $s = \frac{x}{t}$. $u(s)$ is piecewise continuous and there is a s_- and a s_+ such that $u(s)$ is constant for $s<s_-$ and $s>s_+$. There exist a unique integral curve $w(\xi)$ except for a shift in ξ, such that

$$w'(\xi) = f(u(\xi)) - s\,u(\xi) - (f(u(s_\pm)) - s\,u(s_\pm)),$$

$w(\xi)$ is monotone

and $w(\xi) \to u(s_\pm)$ when $\xi \to \pm\infty$

In a discontinuity of $u(s)$ the left and rigth value on each side of the discontinuity is denoted u_- and u_+ respectively.

It is not proved earlier that ths solution may be described as above. The proof is however straigthforward.

The general problem is solved by induction on the number of equations. Assume that the problem is solved for n equations. We will then prove it for n+1 equations. The n+1 equation problem may be written as

(2.1) $$v_t + g(u,v)_x = 0$$

and

(2.2) $$v(x,0) = \begin{cases} v_+ & \text{for } x > 0 \\ v_- & \text{for } x < 0 \end{cases}$$

$u(s)$, $s=\frac{x}{t}$, is a known piecewise continuous n dimensional function which is constant for $s>s_-$ and $s<s_+$ for som s_- and s_+. Where $u(s)$ is discontinuous, there exist a piecewise monotone entropy curve

(2.3) $$w'(\xi) = f(u(\xi)) - s\,u(\xi) - (f(u(s_\pm)) - s\,u(s_\pm))$$

and $w(\xi) \to u(s_\pm)$ when $\xi \to \pm\infty$.

Similarly the solution v may be described by a function $v(s)$ and for each discontinuity in $v(s)$ there is an entropy curve $y(t)$.

Assume g is continuous and g and g_v where defined, is piecewise monotone.

In the argument below we assume there is a fixed left value v_L for v.

We use v_L instead of v_- in order to separate it from left values in single shocks. Similarly v_R is used. Then the values for v_R which is possible to connect to v_L only using speed less than a speed s is found. When this maximum speed is large enough, it is possible to connect the fixed v_L to all possible v_R values. In describing the v_R values which may be connected to v_L, the function $h_s(v)$ is used in addition to the function $g(u(s),v)$. The values which is possible to connect to v_L with speed less than or equal s, is the v values where $h_s(v) = g(u(s),v)$. The $h_s(v)$ function has the following properties:

- $h_s(v) = g(u(s),v)$ in a finite number of intervals. An interval may consist of one point. There is at least one interval.
- Between these intervals $h_s(v)$ is linear with slope s.
- There exist a v_0 such that for $v > v_0$
  either $h_s(v) = g(u(s),v)$,
  or $h_s(v) < g(u(s),v)$.
- There exist a v_0 such that for $v < v_0$
  either $h_s(v) = g(u(s),v)$,
  or $h_s(v) > g(u(s),v)$.

See figure 2.1 for a typical $h_s(v)$ and $g(u(s),v)$.

The argument is made a litle more complicated because there is not always a unique entropy curve. When the entropy curve is not unique the solution v(s) is stil unique, but we get problems in the induction. When there are several entropy curves w(t) in the u variable between the same u_- and u_+, there is not a unique solution in general. Luckily this does not happen often. There is only for (v_L,v_R) in an area with measure zero, where there is a shock with not unique entropy curves. Therefore using induction on the number of equations, the solution is unique except for $(u_{1_-},\dots,u_{n_-},u_{1_+},\dots,u_{n_+})$ in an area of measure zero in R^{2n}.

We may then start with the proofs.

<u>Proposition 2.2</u>

The v_R values that are possible to connect to a fixed v_L value with speed less than or equal s, may be described as stated above by a function $h_s(v)$ with the properties listed above, and the function $g(u(s),v)$. Where there is a shock the entropy curve is unique except that for each v_L there is a finite number of v_R values for which the entropy curve is not unique.

<u>Comment to Propsition 2.2</u>

We will prove that the entropy curve is unique for convergence to points where $h_s(v)=g(u_+,v)$ and $g_v > s$. These points are important because the solution $v(s)$ pass these points for v_R in an interval. See figure 2.2 where $s=0$ and $u(s)$ is constant for $s>0$. We see that $v(0)=c$ for all $v_R \varepsilon (a,c)$. When $g_v(u_0,v_0)<s$, (u_0,v_0) is only passed for at most one single v_R value. Therefore the proposition is correct when there only is non-unique entropy curve for a finite number of points where $g_v<s$.

Before this proposition is proved, some lemmas must be proved.

Lemma 2.3

Proposition 2.2 is correct for $s<s_0$ if $u(s) = u_-$ for $s<s_0$.

Proof of Lemma 2.3

When $u(s)$ is constant (2.1) and (2.2) is equivalent with the scalar problem. The solution is then well-known. If v_- is smaler than v_+, the solution is described by the convex envelope from v_- to v_+, and if v_- is larger than v_+, the solution is described by the concave envelope from v_- to v_+. It is easily seen that Proposition 2.2 is satisfied. See figure 2.3 for a typcal $h_s(v)$ when $u(s)$ is constant. The entropy curve is always unique.●

Lemma 2.4 u continuous

Assume that Proposition 2.2 is satisfied for $s=s_0$ and that $u(s)$ is continuous for $s\varepsilon[s_0,s_1]$. Then Proposition 2.2 is satisfied for $s=s_1$.

Proof of Lemma 2.4

When $u(s)$ is continuous, we will prove that the solution of (2.1) - (2.3) is a combination of smooth rarefaction waves in all the v variables combined with shocks only in the v variable. Since the shocks are only in the v variable, they appear in the same manner as shocks in the scalar equation.

When $u(s)$ is continuous, the equation is

$$v_t + g(u(s),v)_x = 0.$$

Since the solution is on the form $v(s)$, $s=\frac{x}{t}$, the equation may be rewritten as

$$- s\, v_s + g_u(u,v)\, u_s + g_v(u,v)\, v_s = 0$$

or equivalent

$$v_s = \frac{g_u\, u_s}{s - g_v} .$$

When $g_v(u,v) = s$, the equation may be treated as the scalar equation with rarefactions waves where $s=g_v$ and shocks from (u_-,v_-) to (u_+,v_+) with speed s,

$$s = \frac{g(u_-,v_-) - g(u_+,v_+)}{v_0 - v_+} .$$

There is a rarefaction wave starting in every point v_0 where $h_{s_0}(v_0) = g(u(s_0),v_0)$ and $g_v \neq s_0$. The rarefaction wave is defined by the integral curve

(2.4) $$v(s_0) = v_0 ,$$

(2.5) $$v_s(s) = \frac{g_u u_s}{s - g_v} .$$

These curves are well-defined as far as $g_v \neq s$. Two curves can not pass each other, i.e. if $v_1(s_1) < v_2(s_1)$, then $v_1(s) < v_2(s)$ for all s.

In (v,g) plane the curves (v(s),g(u(s),v(s))) are parallel with slope s.

$$\frac{g_s}{v_s} = \frac{g_u u_s + g_v v_s}{v_s} = \frac{v_s(s-g_v) + g_v v_s}{v_s} = s.$$

Even when $g_v = s$, the curves (v(s),g(u(s),v(s)) are parallel with slope s in the (v,g) plane.

The entropy curves are unique exactly as in the scalar case. Points with multiple entropy curves and $g_v<s$ is transformed to other points where $g_v<s$. It is then trivial to see that Proposition 2.2 is satisfied.●

Then we are left with the most difficult case where there is a shock in u. Assume u(s) is discontinuous in s_0 with the left and right values u_- and u_+ respectively. We assume that Proposition 2.2 is satisfied for s_-, where s_- is speed s_0 but before the shock. s_+ is defined correspondingly. Assume also that there exist a piecewise monotone integral curve w(t) such that

$$w'(t) \to u_\pm \qquad \text{when} \quad t \to \pm \infty.$$

We use the notation $h_-(v)$ and $h_+(v)$ instead of $h_{s_-}(v)$ and $h_{s_+}(v)$.

First we will prove that there starts and ends integral curves from almost all points on $h_-(v)$.

<u>Lemma 2.5</u>

Assume that g is continuous and $g(w(t),v) - s\,v$ is strictly monotone in the v variable and monotone in t for t smal and for the v variable in a neighbourhood to a v_- and that w(t) is piecewise monotone and converges to u_- when $t \to -\infty$.

Then there exist a piecewise monotone integral curve such that

(2.6) $v'(t) = g(w(t),v(t)) - s\,v(t) - (g(u_-,v_-) - s\,v_-)$

and $v(t) \to v_-$ when $t \to -\infty$.

If $g(u,v) - s\,v$ is strictly decreasing in a neighbourhood to (u_-,v_-), then the curve is unique.

Correspondingly when $t \to \infty$, there exist an integral curve where $v(t)$ converges to v_+. Then there is uniqueness when $g(u,v) - s\,v$ is strictly increasing in v.

<u>Proof of Lemma 2.5</u>

We will only pove the lemma when $t \to -\infty$. We may assume s=0. By the assumption there exist a N and a unique monotone curve a(t) such that

$$g(w(t),a(t)) = g(u_-,v_-) \text{ for } t < -N.$$

A curve v(t) is uniquely defined by (2.6) and v(a) = b for a fixed a and b. We divide into to cases depending on g is increasing or decreasing in v.

<u>g increasing in v</u>

We may assume g increasing in u and $u_- > w(t_2) > w(t_1)$. Then a(t) is increasing. See figure 2.4. v(t) is defined by

$$v(t_0) = \frac{1}{2}\,(a(t_0) + a(t_0+1)) \quad \text{for an arbitrary } t_0 < -N-1.$$

Then $v(t) \geq a(t) > v_-$, $v'(t) > 0$ and

$$v'(t) \to g(u_-,v(t)) - g(u_-,v_-) \text{ when } t \to -\infty.$$

The convergence is monotonically.

<u>g decreasing in v</u>

We may assume g increasing in u and $u_- < w(t_2) < w(t_1)$. Then a(t) is increasing. See figure 2.5. We will prove existence of v(t) by defining a sequence $\{v_i(t)\}$ for $i > N$ which converge towards v(t). Define $v_i(t)$ by

$$v_i(-i) = \frac{1}{2}\,(v_- - a(-i)).$$

Let us first prove that $v_- < v_i(t) < a(t)$ for $-i < t < -N$. From the definition $v_i(-i)$ is between v_- and a(t). The interval $(v_-,a(t))$ increases with t. While $v_- < v_i(t) < a(t)$, $v_i'(t) > 0$. But $v_i(t)$ does not pass a(t) since if $v_i(t)=a(t)$ then $v_i'(t)=g(w(t),a(t))=0$.

Secondly we prove that for $i,j > N$, $|v_i(t) - v_j(t)|$ decreases with t for $t < -N$. If $v_i(t) < v_j(t)$, then $v_i'(t) > v_j'(t)$ since g is decreasing in v.

Then it is easy to see that $v_i(-N)$ converge when $i \to \infty$. Assume j>i.

Then $|v_i(-N) - v_j(-N)| \leq |v_i(-i) - v_j(-i)| \leq |v_- - a(-i)| \to 0$, when $i \to \infty$ and $v(t)$ may be defined by (2.6) and $v(-N) = \lim_{i \to \infty} v_i(-N)$.
Since $v_- < v_i(t) < a(t)$ for $-i < t < -N$, we have $v_- < v(t) < a(t)$ for $t < -N$. Then $v(t) \to v_-$.

Since also $v'(t) > 0$ for $t < -N$, the convergence is monotone. It is easy to see that it is piecewise monotone. Finally uniqueness must be proved. Assume that there is two integral curves $v_1(t)$ and $v_2(t)$ which both satisfies the conditions. In order to converge towards v_-, both must be between v_- and $a(t)$. But since $|v_1(t) - v_2(t)|$ increases, when $t \to -\infty$, it is not possible that both converges towards v_-, except when $v_1(t) = v_2(t)$ for all t.●

<u>Comment to Lemma 2.5</u>

For smooth g, $g(u,v) - s\,v$ not strictly monotone means $g_v = s$.

When $g_v(u_-,v_-) = s$, there may exist integral curves which converges to v_-, and there may not. In the first of the following examples there is continuum of integral curves, in the next example there is no such curves.

<u>Example 2.1</u>

$$g(u,v) = u + v^2 \qquad (u_-,v_-) = (0,0) \quad \text{and } w(t) = -\frac{1}{t} \text{ for } t<0.$$

All curves defined by
$v'(t) = g(w(t),v(t))$ and $v(0) = a$ for $a>0$
converge towards v_-. See figure 2.6.

<u>Example 2.2</u>

$$g(u,v) = u + v^2 \qquad (u_-,v_-) \quad \text{and } w(t) = \frac{1}{t} \text{ for } t<0.$$

All curves is at the form
$v_a'(t) = g(w(t),v(t))$ and $v_a(-1)=a$
Then

$$|v_a(-\infty)-v_a(-1)| \geq \int_{\infty}^{-1} |v_a'|\,dx \geq \int_{\infty}^{-1} \left|\frac{1}{t}\right|\,dx = \infty,$$

thus all integral curves diverges. See figure 2.7.

<u>Lemma 2.6</u>

Assume that there is two entropy curves $v_1(t)$ and $v_2(t)$, which converges towards respectively $v_{1,-}$ and $v_{2,-}$ with $v_{1,-} < v_{2,-}$. Assume further that $h_-(v_{1,-}) = g(u,v_{1,-})$ and $h_-(v_{2,-}) = g(u,v_{2,-})$. Then $v_1(t) < v_2(t)$ for all t.

Proof of Lemma 2.6

We may assume s=0. Assume there is a t_0 such that $v_1(t_0) > v_2(t)$. Then there must be a t_1 such that $v_1(t_1) = v_2(t_1)$ and $v_1^{\cdot}(t_1) > v_2^{\cdot}(t_1)$. But since $h_-(v)$ is nonincreasing, $g(u_-,v_{1,-}) > g(u_-,v_{2,-})$. Then

$$\begin{aligned} v_1^{\cdot}(t_1) &= g(w(t),v_1(t_1)) - g(u_-,v_{1,-}) \\ &< g(w(t),v_2(t_1)) - g(u_-,v_{2,-}) \\ &= v_2^{\cdot}(t_1). \bullet \end{aligned}$$

Then we are ready for the lemma which handles u(s) discontinuous, i.e. there is a shock in one of the equation higher up in the equation system.

Lemma 2.7 u(s) discontinuous

Assume u(s) is discontinuous for $s=s_0$ and that Proposition 2.2 is correct for s_-. Then Proposition 2.2 is correct for s_+.

Proof of Lemma 2.7

Let us start the argument with $h_-(v)$. Using this function we find out where it is possible to end using a speed less than or equal s_0 but not passed the shock in the u variable. Everywhere where $g(u_-,v)=h_-(v)$ and $g(w(t),v) - s\ v$ is strictly monotone in v and monotone in t, there is an integral curve which converges this point. By the assumption on g and w(t) the function is always monotone in t for t smal. Where $g(u,v) - s_0\ v$ is not strictly monotone in v it is possible to make a shock with speed s_0 only in the v variable before the shock with speed s_0 in the u variable.

Let us follow an integral curve which starts in $(v_-,h_-(v_-))$. The integral curve $(v(t),h_-(v_-) + s_0(v(t)-v_-))$ describes a straight line. When $t \to \infty$, the curve either diverges to ∞ or $-\infty$, or it converges to a point where the straight line crosses $g(u_-,v)$. Lemma 2.6 tells that two integral curves does not pass each other. Following the end points of all the integral curves starting at $h_-(v)$ having slope s_0, we find some parts of $g(u_+,v)$ which may be connected to each other by straight lines with slope s_0. See figure 2.8. If Proposition 2.2 is satisfied after the shock, this must be the new $h_+(v)$. It remains to prove that it satisfies the conditions on $h_s(v)$. It is easy to prove that this $h_s(v)$ satisfies the conditions when $|v|$ is large. It left to further that $h_-(v_{1,-}) = g(u,v_{1,-})$ and $h_-(v_{2,-}) = g(u,v_{2,-})$. Then $v_1(t) < v_2(t)$ for all t.

Proof of Lemma 2.6

We may assume s=0. Assume there is a t_0 such that $v_1(t_0) > v_2(t)$. Then there must be a t_1 such that $v_1(t_1) = v_2(t_1)$ and $v_1'(t_1) > v_2'(t_1)$. But since $h_-(v)$ is nonincreasing, $g(u_-,v_{1,-}) > g(u_-,v_{2,-})$. Then

$$\begin{aligned} v_1'(t_1) &= g(w(t),v_1(t_1)) - g(u_-,v_{1,-}) \\ &< g(w(t),v_2(t_1)) - g(u_-,v_{2,-}) \\ &= v_2'(t_1). \bullet \end{aligned}$$

Then we are ready for the lemma which handles u(s) discontinuous, i.e. there is a shock in one of the equation higher up in the equation system.

Lemma 2.7 u(s) discontinuous

Assume u(s) is discontinuous for $s=s_0$ and that Proposition 2.2 is correct for s_-. Then Proposition 2.2 is correct for s_+.

Proof of Lemma 2.7

Let us start the argument with $h_-(v)$. Using this function we find out where it is possible to end using a speed less than or equal s_0 but not passed the shock in the u variable. Everywhere where $g(u_-,v)=h_-(v)$ and $g(w(t),v) - s\,v$ is strictly monotone in v and monotone in t, there is an integral curve which converges this point. By the assumption on g and w(t) the function is always monotone in t for t smal. Where $g(u,v) - s_0\,v$ is not strictly monotone in v it is possible to make a shock with speed s_0 only in the v variable before the shock with speed s_0 in the u variable.

Let us follow an integral curve which starts in $(v_-,h_-(v_-))$. The integral curve $(v(t),h_-(v_-) + s_0(v(t)-v_-))$ describes a straight line. When $t \to \infty$, the curve either diverges to ∞ or $-\infty$, or it converges to a point where the straight line crosses $g(u_-,v)$. Lemma 2.6 tells that two integral curves does not pass each other. Following the end points of all the integral curves starting at $h_-(v)$ having slope s_0, we find some parts of $g(u_+,v)$ which may be connected to each other by straight lines with slope s_0. See figure 2.8. If Proposition 2.2 is satisfied after the shock, this must be the new $h_+(v)$. It remains to prove that it satisfies the conditions on $h_s(v)$. It is easy to prove that this $h_s(v)$ satisfies the conditions when $|v|$ is large. It left to prove that all points where $h_+(v) = g(u_+,v)$ is possible to reach with speed s_+. The difficulty is to prove that where $h_+(v)$ is a straight line and crosses $g(u_+,v)$ there is an integral curve which converges to this point.

In order to make the notation easy assume g_v is defined everywhere. Let us consider all integral curves

$$v_c'(t) = g(w(t),v_c(t)) - c$$

for a constant c. One of these integral curves is defined uniquely by $v_c(0) = a$. When $t \to -\infty$, $v_c(t)$ diverges to $-\infty$ or ∞ or converges to a v_- value where $g(u_-,v_-) = c$. Correspondingly when $t \to \infty$, $v_c(t)$ diverges to $-\infty$ or ∞ or converges to a v_+ value where $g(u_+,v_+) = c$. There is a unique curve which converges to a point where $g_v(u_-,v_-) < s_0$ and a unique curve which converges to a point where $g_v(u_+,v_+) > s_0$. Let us consider which values of a and c where the curve $v_c(0) = a$ converges to $h_-(v)$. According to Lemma 2.6 two integral curves which both converges to $h_-(v)$ does not pass each other. Using this fact it is easy to prove that there is a continuous function $c = \gamma(a)$ in the (a,c) plane such that for $c = \gamma(a)$ the curve defined by $v_c(0) = a$ converges to $h_-(v)$ where $h_-(v) = c$. See figure 2.9. $\gamma(a)$ is decreasing where the corresponding $h_-(v)$ is decreasing and $\gamma(a)$ is constant, where the corresponding $h_-(v)$ is constant. Where $\gamma(a)$ is constant there are several integral curves which converges to $h_-(v)$ and where $\gamma(a)$ is decreasing there is only one such curve.

Correspondingly we may study which (a,c) values which the curve $v_c(0) = a$ converges to $h_+(v)$. For convergence to $h_+(v)$ the properties are changed; there is a single (a,c) value for which $v_c(0) = a$ converges to a point where $g_v(u_+,v) > s$, and an interval with a values for each c value for which $v_c(0) = a$ converges to a point where $g_v(u_+,v) > s$. See figure 2.10 where the different curves for convergence to points where $g_v(u_+,v) > s$ is showed.

Since $c = \gamma(a)$ is continuous it crosses the curves in the (a,c) plane for convergence to $g(u_+,v)$ where $g_v > s$. Therfore there is always an integral curve from $h_-(v)$ to $h_+(v)$ where $h_+(v)$ is as indicated in the first part of this proof.

Where $c = \gamma(a)$ is constant, i.e. where h_- is constant, there usually are several curves converging from h_- to h_+. If there are several curves h_- has in this point slope s_0 and h_+ has slope less than or equal s_0. See example in the following chapter for a typical example with multiple integral curves. There is a finite number of intervals where h_- is linear with slope s_0. Then for each v_L it is only a finite number of v_R values for which the entropy curve is not unique.

Proof of Proposition 2.2

$u(s)$ is piecewise continuous and constant for s small and s large. In Lemma 2.3 we have proved that the proposition is correct for s small. Lemma 2.4 gives that if $u(s)$ is continuous in an interval and Proposition 2.2 is correct in the beginning of the interval then it is correct in the end of the interval. Lemma 2.7 states that if $u(s)$ is discontinuous for the s value to the left of the discontinuity, then it is correct to the right of the discontinuity. Then the proposition is correct for any s.●

In order to prove existence for every initial values the following lemma is needed.

Lemma 2.8

There is a s_+ such that for $s > s_+$, $h_s(v) = g(u_+,v)$.

The poof is trivial.

We may then state the theorem of existence and uniqueness of the system (2.1) - (2.3). This theorem is used in the induction step for proving existence and uniqueness for the general system (1.1) and (1.2).

Theorem 2.9

Given v_L, v_R, $g(u,v)$, $u(s)$ and $w(t)$ where $u(s)$ is discontinuous with the properties listed in the beginning of this chapter. Then there exist a unique solution to the Riemann problem

$$v_t + g(u,v)_x = 0$$

$$v(x,0) = \begin{cases} v_L & \text{for } x < 0 \\ v_R & \text{for } x > 0. \end{cases}$$

There exist also integral curves in the shocks in v. These integral curves are unique except for a (v_-,v_+) in an area with measure 0 in R^2.

The theorem follow easily from Proposition 2.2 and Lemma 2.8.

Then finally, we may state the existence and uniqueness proof for the system (1.1) and (1.2).

Theorem 2.10

There exist a solution to the Riemann problem (1.1) and (1.2) and the solution is unique except for some inital values u_- and u_+. The area where there exist several solutions has measure 0 in R^{2n}. For n=1 and n=2 there is always uniqueness.

Proof of Theorem 2.10

The theorem is proved by using induction. For $n=1$ the result is well-known. This is used as the induction hypothesis. Theorem 2.9 is used as the induction step. For $n=2$ there may be an initial value in an area with measure 0 where there are several entropy curves, but the solution is stil unique. For $n>2$ this may lead to several solutions.●

3 SOME CHARACTERISTICS OF THE SOLUTION

In this chapter we study some of the characteristics of the solution of lower triangular hyperbolic systems. First we show that the Lax entropy inequalities are not satisfied. Afterwards we prove that the solution does not depend continuously on the data.

For genuinly nonlinear and strictly hyperbolic systems the following inequalities

$$\lambda_k(u_+) < s < \lambda_{k+1}(u_+)$$

and

$$\lambda_{k-1}(u_-) < s < \lambda_k(u_-)$$

where λ_k are the ordered eigenvalues to the system, was proved by Lax [7] for local solutions. In lower triangular hyperbolic systems the eigenvalues are the derivatives $\frac{\partial f_i}{\partial u_i}$. Let us name this eigenvalue λ^i. Assume there is a simple rarefaction solution in equation $1,\dots,k-1$. Then a shock with speed s in equation k. This shock influences the solution in equation $k+1,\dots,n$. Then $\lambda^i = s$ on both side of the shock for $i=1,\dots,k-1$. For $i=k$ the eigenvalues appear as in the scalar equation i.e. $\lambda^i(u_+) \leqslant s \leqslant \lambda^i(u_-)$. For $i > k$ usually $\lambda^i(u_+)$, $\lambda^i(u_-)$ are both less than s according to the proof of lemma 2.7. But we may also have situations where $\lambda^i(u_+) \geqslant s$ and where $\lambda^i(u_-) \geqslant s$. When $\lambda^i(u_-) \geqslant s$, the solution is not unique. In one or two of the possible solutions $\lambda^i(u_-) = s$. $\lambda^i(u_+) \geqslant s$ is an ordinary sitation which is not possible to exclude. We see that the Lax shock inequality is not correct for general non-strict hyperbolic systems. In the argument above it is not assumed that the system is not genuinely nonlinear. Johansen and Winther [4] have come to the same conclusion by a study of a particular non-strictly hyperbolic system.

The solution in the scalar equation depend continuously on the data, see Lucier [8] and Holden, Holden and Høegh-Krohn [3]. For the scalar equation the following theorem is valid.

Theorem 3.1

If f and g are Lipschitz continuous functions, u_0 and $v_0 \varepsilon BV(R)$ and u and v are the solutions of

$$u_t + f(u)_x = 0 \qquad \text{for } x \varepsilon R \text{ and } t>0$$
$$u(x,0) = u_0(x) \qquad \text{for } x \varepsilon R$$

and

$$v_t + g(v)_x = 0 \qquad \text{for } x \varepsilon R \text{ and } t>0$$
$$v(x,0) = v_0(x) \qquad \text{for } x \varepsilon R$$

then for any t>0

$$||u(.,t)-v(.,t)||_{L_1} \leq ||u_0(x)-v_0(x)||_{L_1} + ||f-g||_{Lip} \min(|u_0|_{BV(R)}, |v_0|_{BV(R)})$$

where we have defined

$$||g||_{Lip} = \sup_{x \neq y} \left| \frac{g(x)-g(y)}{x-y} \right| .$$

In lower triangular hyperbolic systems the solution does not depend continuously on the data. The problem arises in connection to where the solution is not unique. Using the second part of the proof of Lemma 2.7 we see that the solution when it is not unique is a member in a one parameter family where the parameter is in an interval. This interval is either bounded in both ends or bounded below or above. In the following example we approach a point where the solution is not unique along different curves where the solution is unique. Then we find the endpoints in the parameter interval and see that the solution does not depend continuously on the inital data.

In the example n=3. We take one equation at a time.

$$f_1(u_1) = - u_1^2$$

and

$$u_{0,1}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

The solution is easily found

$$u_1(x,t) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

See figure 3.1. f_2 is defined a litle more complicated

$$f_2(u_1,u_2) = \begin{cases} g_1(u_2) & \text{for } u_1 < -1 \\ \frac{1}{2}(1-u_1)\, g_1(u_2) + \frac{1}{2}(1+u_1)\, g_2(u_2) & \text{for } -1 < u_1 < 1 \\ g_2(u_2) & \text{for } u_1 > 1 \end{cases}$$

where

$$g_1(u) = \begin{cases} |u| & \text{for } u < 1 \\ 2 - u & \text{for } u > 1 \end{cases}$$

and

$$g_2(u) = -2 - u.$$

See figure 3.2 for the definition of f_2. We use two different initial values in the Riemann problem which is arbitrary near each other. The initial values are

$$u_0^+(x) = \begin{cases} -1 & \text{for } x < 0 \\ \sout{1+}\ -2+ & \text{for } x < 0 \end{cases}$$

respectively

$$u_0^-(x) = \begin{cases} -1 & \text{for } x < 0 \\ \sout{1-}\ -2- & \text{for } x < 0. \end{cases}$$

The + and - sign just after a number indicate a litle larger respectively a litle lower value. The exact solution are

$$u_3^+(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ 0 & \text{for } -1 < \frac{x}{t} < 0- \\ 2+ & \text{for } 0- < \frac{x}{t} < 0 \\ \sout{1+}\ -2+ & \text{for } 0 < \frac{x}{t} \end{cases}$$

and

$$u_3^-(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ 0 & \text{for } -1 < \frac{x}{t} < 0 \\ \sout{1+}\ -2- & \text{for } 0 < \frac{x}{t} \end{cases}$$

See figure 3.3 and 3.4. We see that when the right value approaches 4 then these two solution becomes equal. But the entropy curves with speed 0 does not approach each other. This becomes evident when we add the third equation with

$$f_3(u_2,u_3) = \begin{cases} g_3(u_3) & \text{for } u_2 < 0 \\ \frac{1}{2}(2-u_2)\, g_3(u_3) + u_2 g_4(u_3) & \text{for } 0 < u_2 < 4 \\ g_4(u_3) & \text{for } u_2 > 2 \end{cases}$$

where

$$g_3(u) = |u| \sout{+2}$$

and

$$g_4(u) = |u|. + 2$$

See figure 3.5. The initial value is

$$u_{0,3}(x) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x < 0. \end{cases}$$

The solution depend on the initial value for u_2.

$$u_3^-(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ -2- & \text{for } -1 < \frac{x}{t} < 0- \\ 0 & \text{for } 0- < \frac{x}{t} < 0 \\ 2 & \text{for } 0 < \frac{x}{t} < 1 \\ 1 & \text{for } 1 < \frac{x}{t} \end{cases}$$

and

$$u_3^+(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ 0 & \text{for } -1 < \frac{x}{t} < 1 \\ 1 & \text{for } 1 < \frac{x}{t} \end{cases}$$

See figure 3.6 and figure 3.7. When the right hand initial value for u_2 is 4 there is a continuum with entropy curves between the two entropy curves we get when the inital value appraches 4 from both sides. The corresponding solution for u_3 is changing from u_3^+ to u_3^-. The sector with value 0 is increasing and finally ends up as in u_3^-.

4 NUMERICAL METHODS FOR LOWER TRIANGULAR HYPERBOLIC SYSTEMS

There is a lot of different numerical methods for the scalar equation. It is possible to generalize most of these to lower triangular hyperbolic systems. Here we will use a method which follow the proofs in chapter 2, except that the entropy curves are found by a numerical method for the integral curve.

If one want to solve one Riemann problem or several problems but with different v_L and v_R then it is to cumbersome to handle the whole $h_s(v)$ function. Instead a shooting method is valueable. The system is solved by one equation at a time. Then a shooting method runs as follows:

Try to connect the v_- value to any v_+ value. This is done by following the curve u(s). When u(s) is constant, convex or concave envelops are used. The integral curve (2.4) and (2.5) is used when u(s) is constinuous but not constant. Use an ordinary numerical method for (2.4) and (2.5). It is a litle more difficult when u(s) is discontinuous since there is no initial value for the integral curve. Numerically, this is solved by setting $v(t_0) = g(w(t_0),v_-)$ for t_0 smal. Following the u(s) curve we finally reaches a v_R value which probably is to low or to high. This scheme is monotone, i.e. when following the u(s) curve if we move a litle shorter in v variable for a specific s value, the v_R value that we end up with is smaler than the original v_R value independently of what is done for larger s values. Then it is easy to approximate any v_R value.

If we assume that the eigenvalues of (1.1) are in distinct intervals, it is easy to find the solutions for shocks in u(s). In this case it is not necessary to use the entropy curves since the shocks are uniquely defined by the equation

$$s = \frac{g(u_+,v_+) - g(u_-,v_-)}{v_+ - v_-}. \tag{4.1}$$

If f_i, i=1,2,...,n are approximated by piecewise linear functions the solution only consist of shocks and therefore is piecwise constant.

Thuse if f_i, i=1,2,...,n are piecewise linear and the eigenvalues are in distinct intervals it is no need to use any integral curves. Then it is possible to solve the problem exactly only using convex and concave envelopes and shock with speed defined by (4.1).

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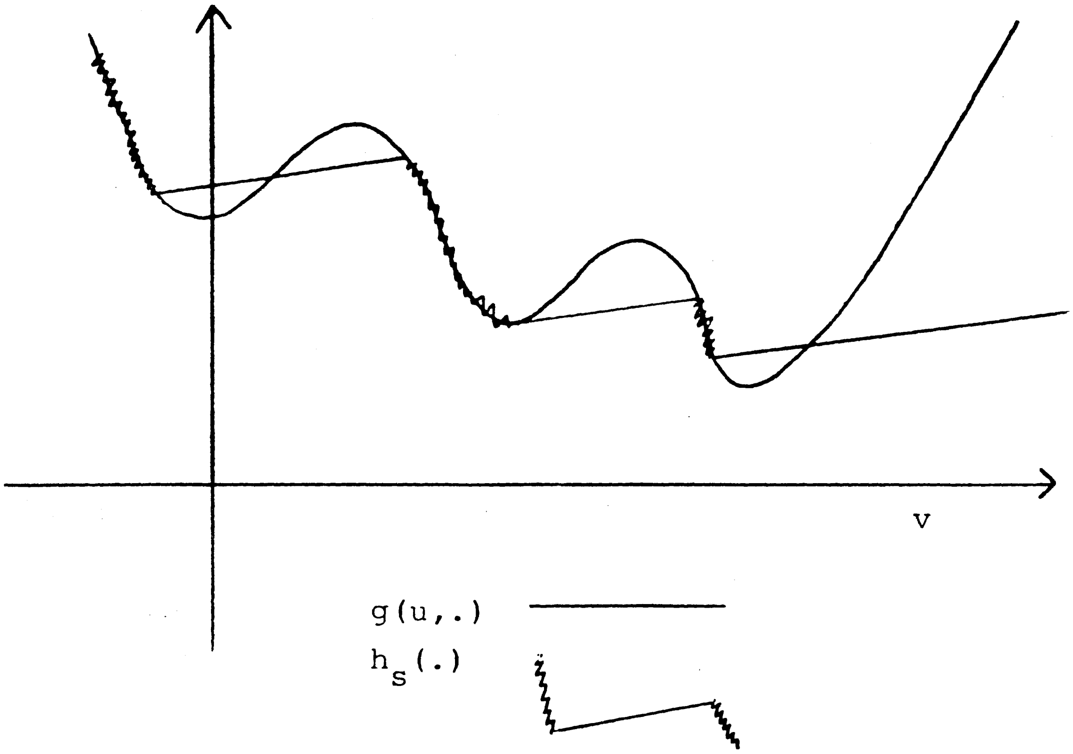

Figure 2.1. A typical $g(u,.)$ and $h_s(.)$

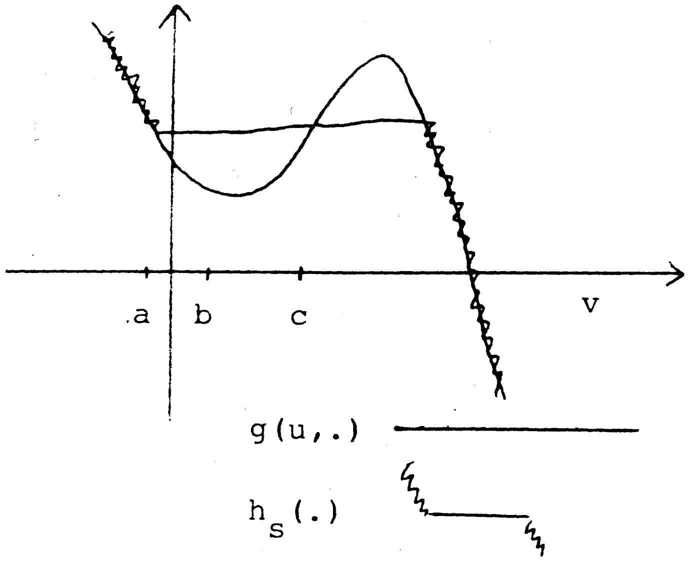

Figure 2.2a.

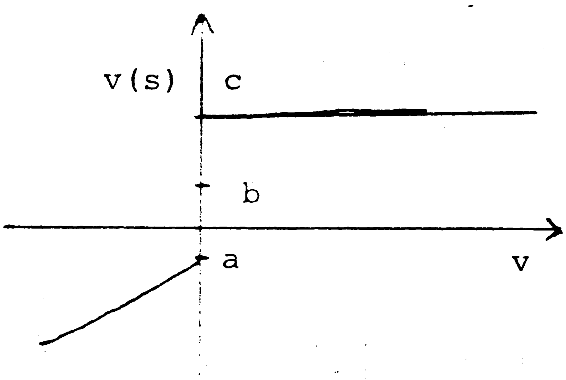

Figure 2.2b. $v(s)$ for $v_R = c$.

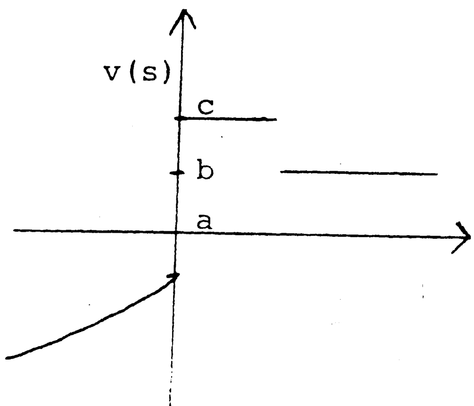

Figure 2.2c. $v(s)$ for $v_R = b$

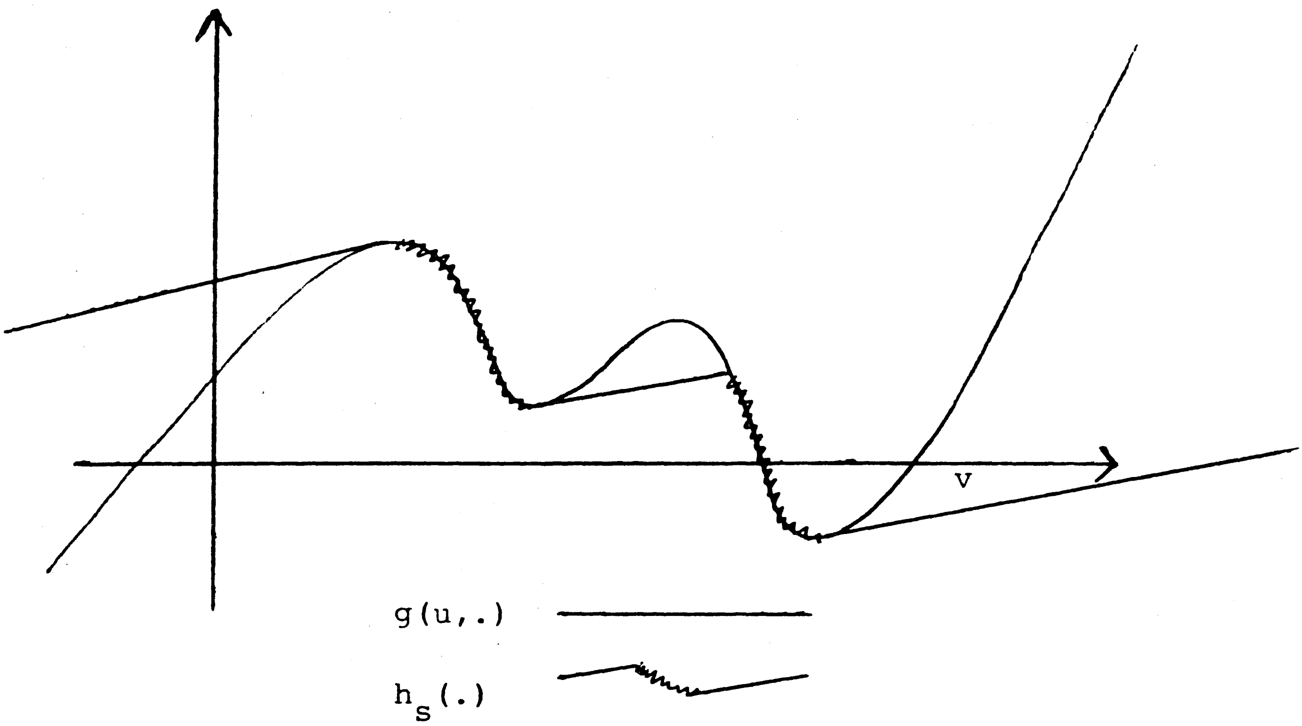

Figure 2.3. $g(u,.)$ and $h_s(.)$ for $u(s)$ constant, $s<0$.

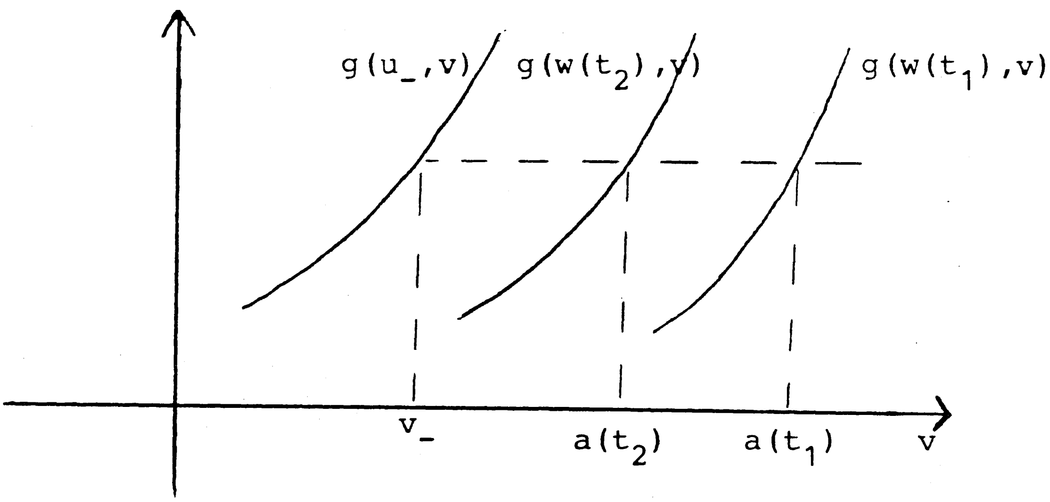

Figure 2.4. $g(u,v)$ increasing in v.

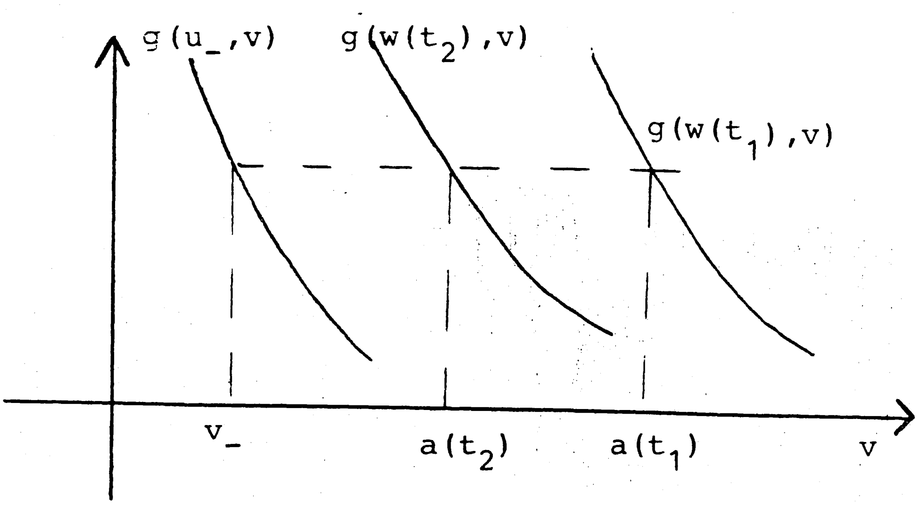

Figure 2.5. $g(u,v)$ decreasing in v.

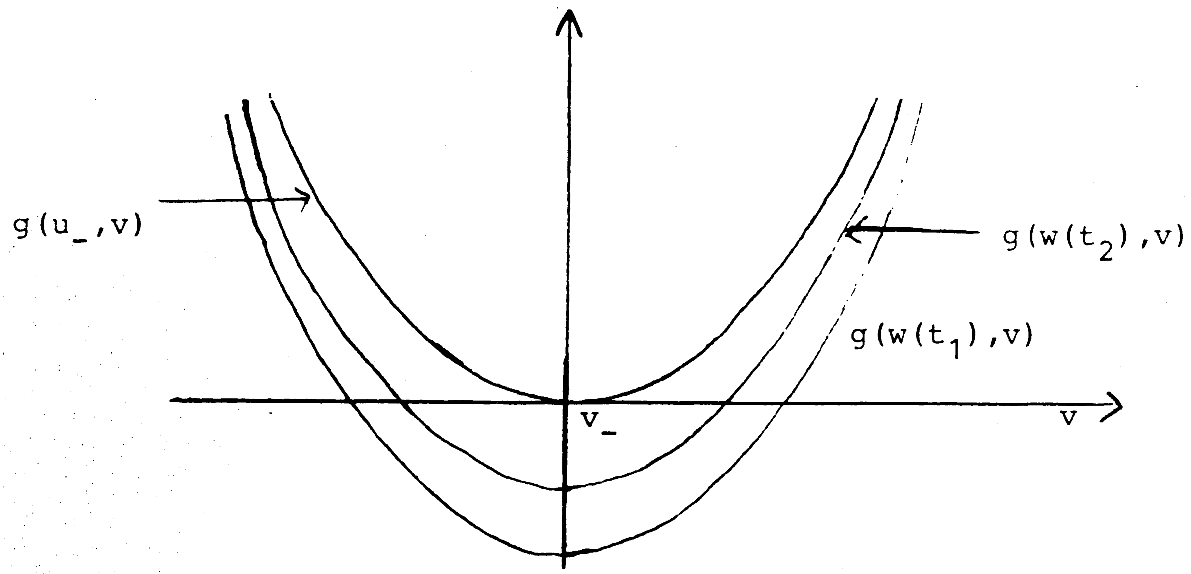

Figure 2.6. All integral curves converge to v_-.

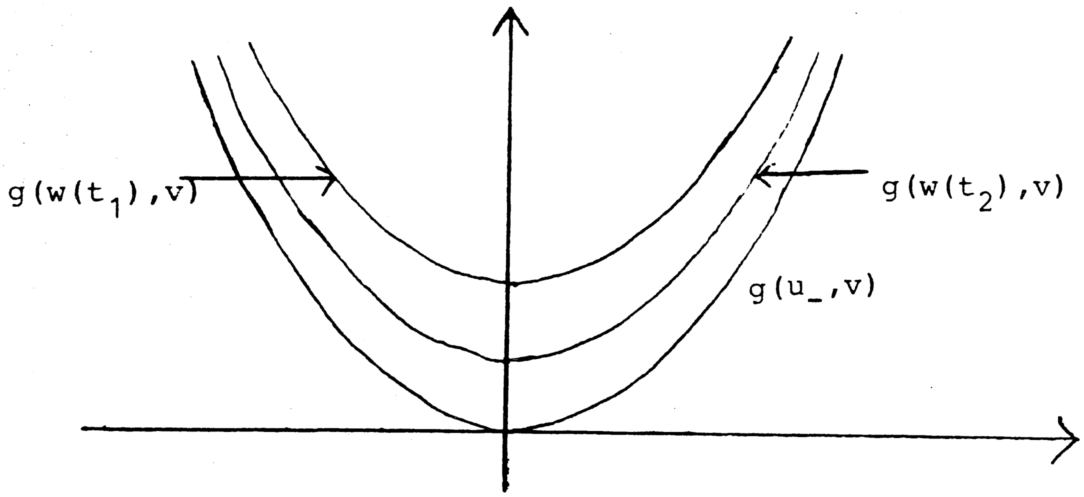

Figure 2.7. All integral curves diverges.

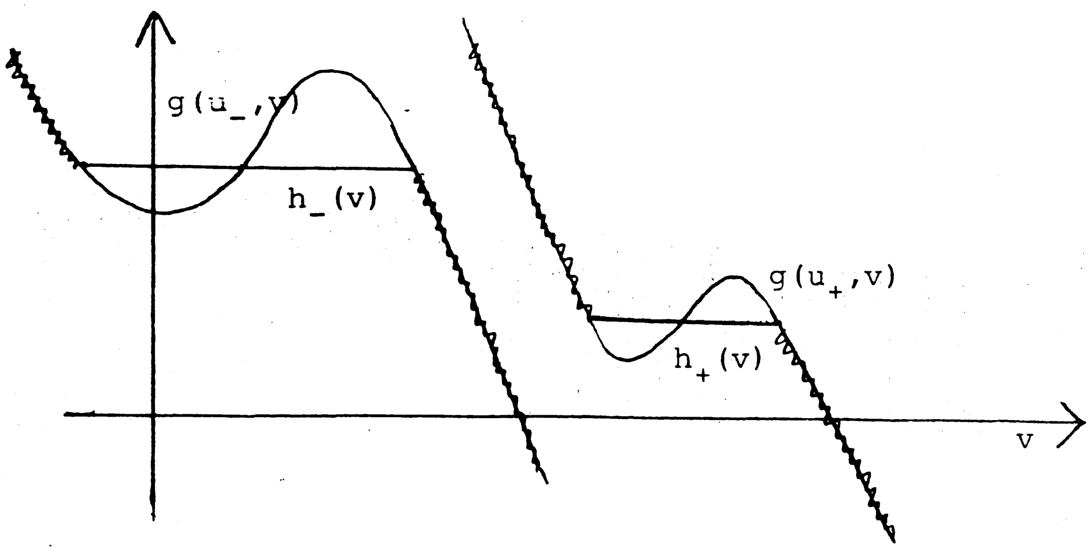

Figure 2.8. h_- and h_+.

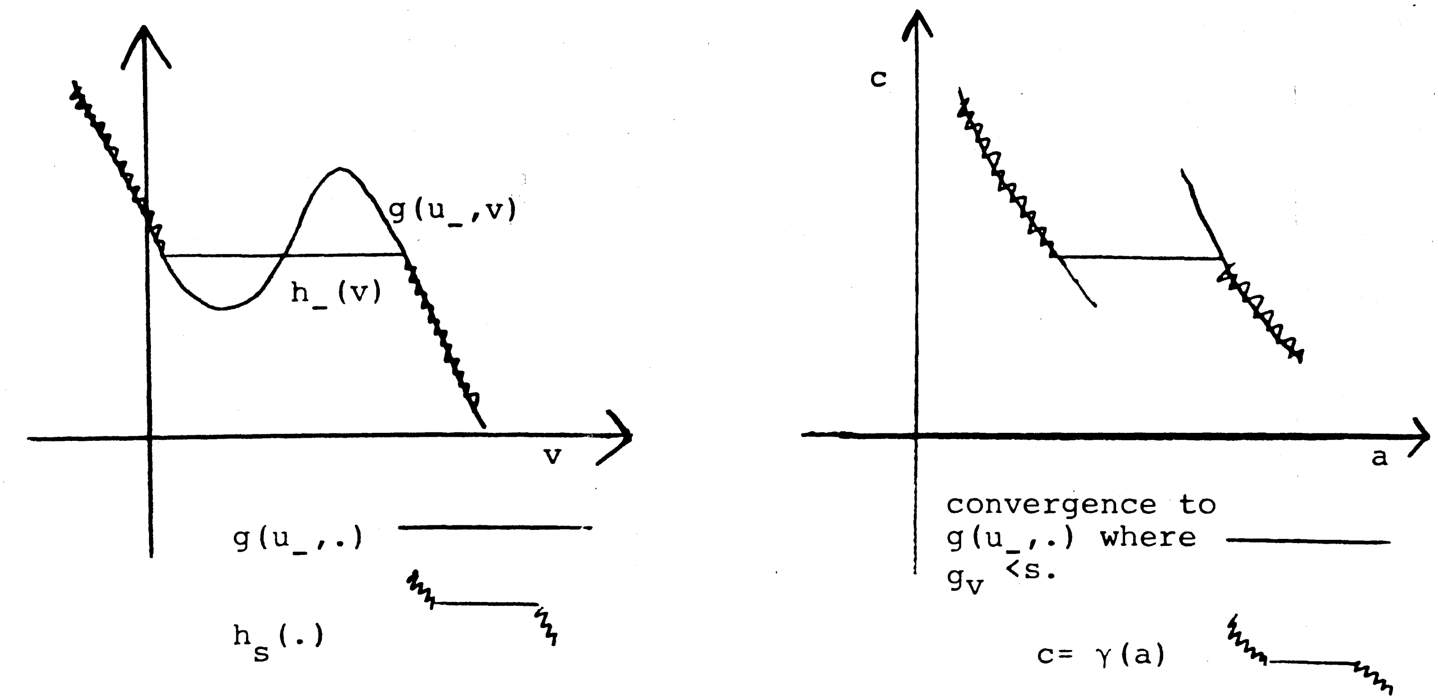

Figure 2.9. $h_s(v)$ and $\gamma(a)$. s=0 in figure.

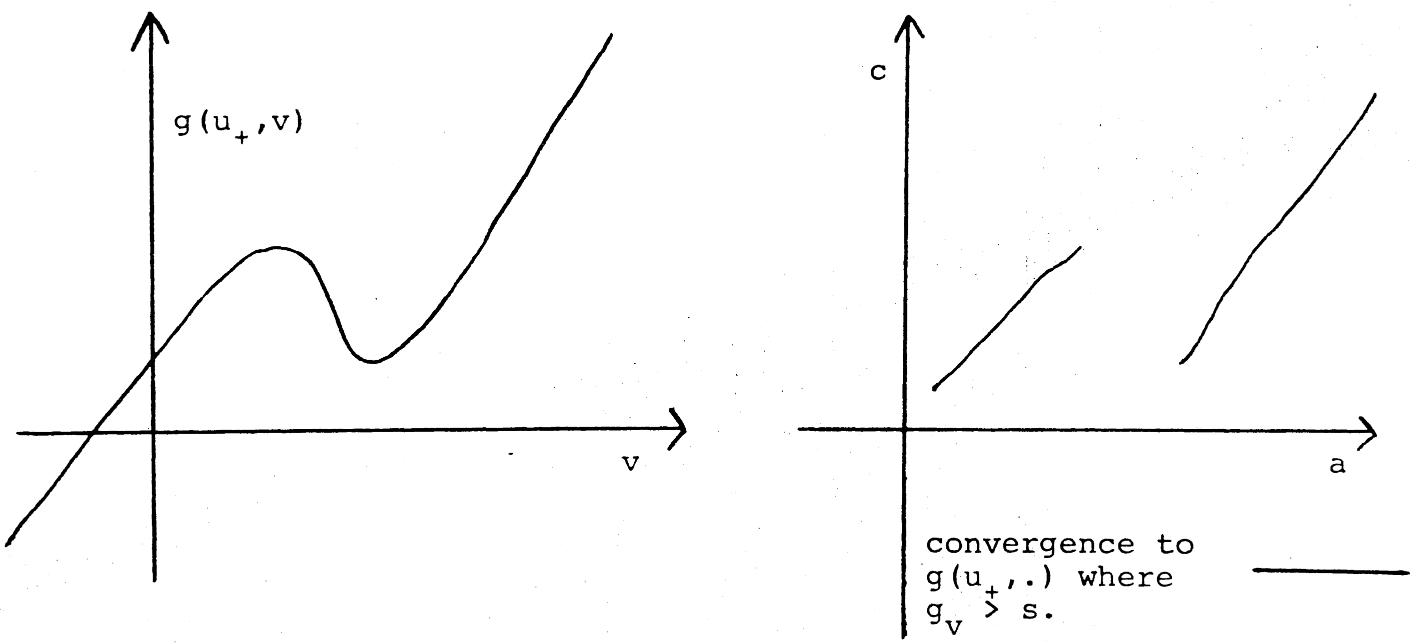

Figure 2.10. Convergence to $g(u_+,v)$

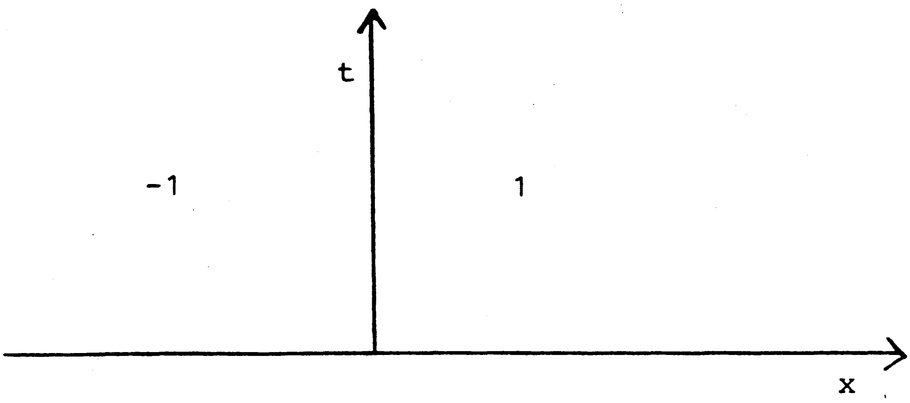

Figure 3.1. $u_1(x,t)$

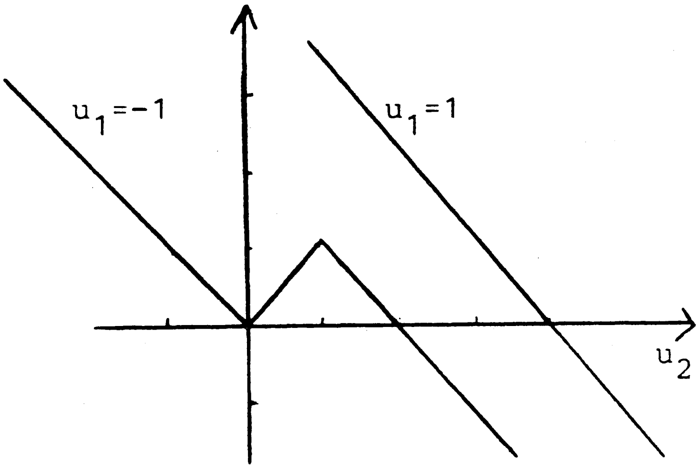

Figure 3.2. $f_2(u_1,u_2)$

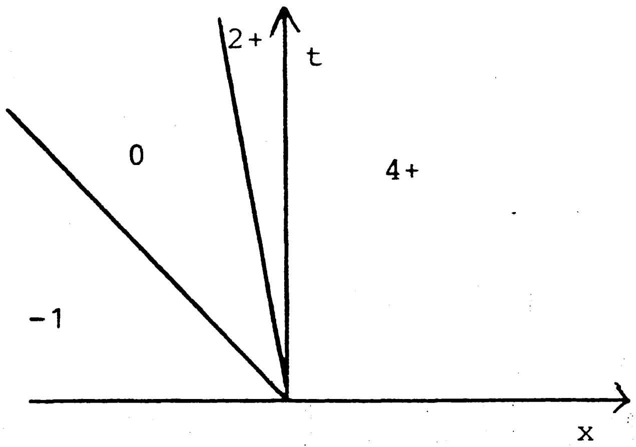

Figure 3.3 $u_2^+(x,t)$

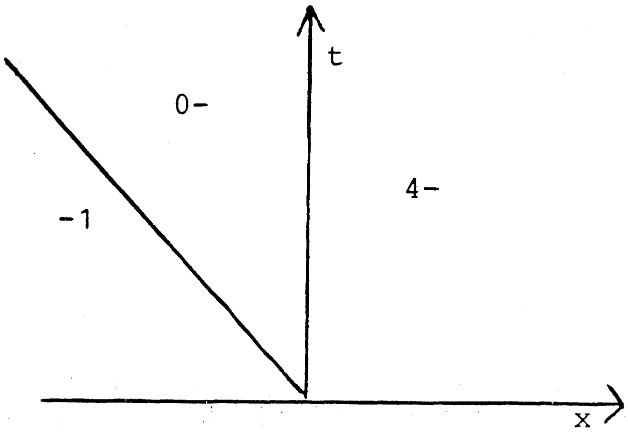

Figure 3.4. $u_2^-(x,t)$

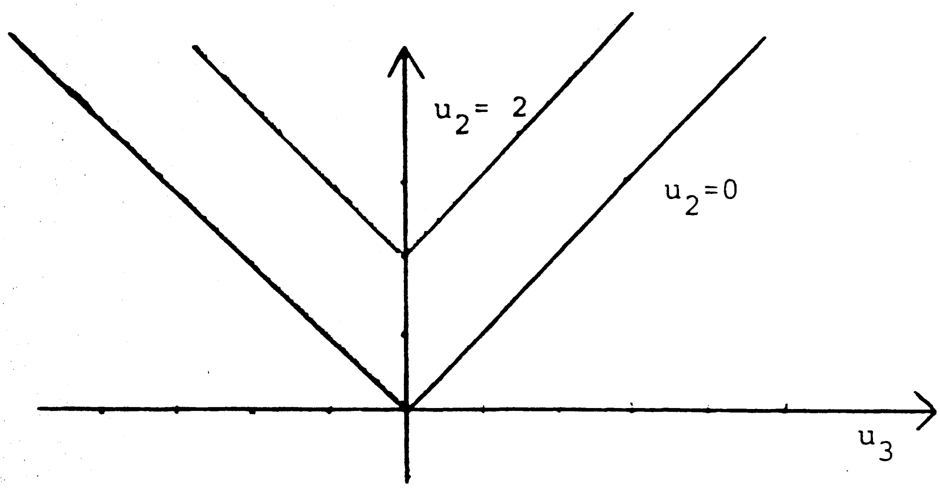

Figure 3.5. $f_3(u_2, u_3)$

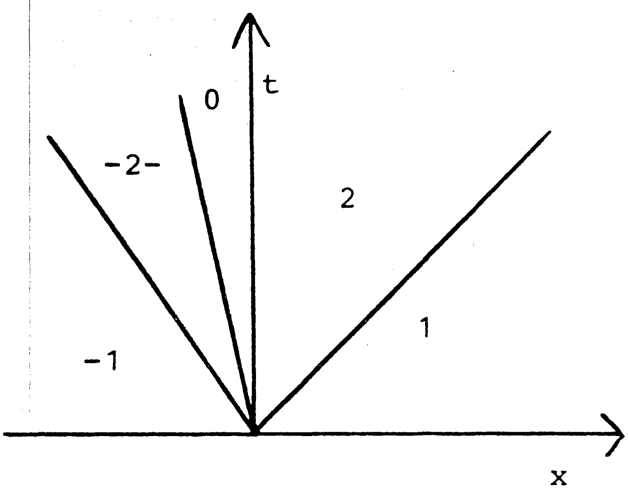

Figure 3.6. $u_3^+(x,t)$

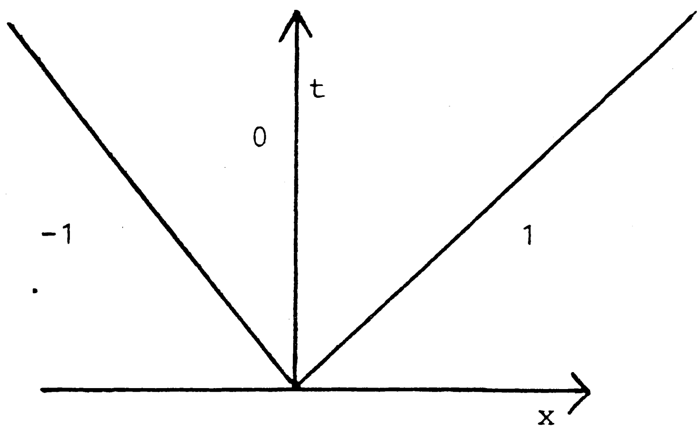

Figure 3.7. $u_3^-(x,t)$