Abstract

Let V be a linear system on a curve C. In Part 1 we constructed a method for studying the secant varieties V_d^r locally. The varieties V_d^r are contained in the d-fold symmetric product $C^{(d)}$.

In this paper (Part 2) we apply the methods from Part 1. We give a formula for local tangent space dimensions of the varieties V_d^1 valid in all characteristics. (Theorem 2.4.)

Assume rank V = n+1, and char K = 0. In §3 and §4 we describe in detail the projectivized tangent cones of the varieties V_n^1 for a large class of points. The description is a generalization of earlier work on trisecants for a space curve.

In §5 we study the curve in $C^{(2)}$ consisting of divisors D such that $2D \in V_4^1$. We give multiplicity formulas for all points on this curve in $C^{(2)}$ in terms of local geometrical invariants of C. We assume char K = 0.

At last we use our set-up to reproduce two well-known formulas; one for the δ-invariant of a plane cusp, and one for the weights of Weierstraß points of a linear system.

§1. Introduction

Let C be a non-singular curve over a field K, and let $V \subset H^0(C,L)$ be a linear system on C, where L is a line bundle. Denote by $C^{(d)}$ the d'th symmetric product of C. The subschemes V_d^r of $C^{(d)}$ consist of those divisors that impose at most d-r independent conditions on V. The V_d^r are secant varieties.

As an example consider the case where rank V = 4 and V is very ample. Then V defines an embedding of C into P^3. The variety V_3^1 parametrizes those divisors of degree 3 that consist of

3 collinear poins on C in P^3. Roughly speaking: V_3^1 parametrizes the 3-secant lines of the embedded curve.

It is a well-known fact that the V_d^r can be defined scheme-theoretically as the zero schemes

$$Z(\overset{d-r+1}{\Lambda}\sigma), \qquad \text{for } r=1,\dots,d$$

where σ is a canonical $C^{(d)}$-bundle map

$$\sigma\colon V\otimes C^{(d)} \to E_L,$$

and E_L is a vector bundle of rank d on $C^{(d)}$ obtained from L by a socalled symmetrization process.

In Part 1 we constructed a computational device for studying the map σ and the varieties V_d^r locally. Our main results were given in Theorem 4.2 and Proposition 4.4 of Part 1. We constructed a local matrix description of σ and described the formal completion $\hat{O}_{V_d^r,D}$ of the local ring of V_d^r at a point (divisor) D. Such a local descripton is often trivial when D consists of d distinct points. The main purpose with our results is to study the V_d^r at points on the diagonal in $C^{(d)}$.

Part 1 is inspired by the papers [Ma] and [Ma-Ma]. In Part 2 we will use the results of Part 1 to give some geometrical results.

In §2 we give a formula for the tangent space dimension of the variety V_d^1 at a point D. The formula is valid in any characteristic.

In §3 we study a large class of points on the variety V_n^1, where rank $V = n+1$. We describe the tangent cones of V_n^1 at such points, and in particular we give a formula for the multiplicity of V_n^1 at these points.

In §4 we find further properties of the tangent cones described in §3. We will indicate when the projectivized tangent cones are singular. This is a generalization of a result in [J] concerning 3-secant lines for a space curve.

In §5 we study stationary bisecants for a non-singular space curve. A stationary bisecant is a bisecant line, where the curve tangents at the points of secancy meet, or a tangent line at a point where the osculating plane of the curve is hyperosculating. We define a curve in $C^{(2)}$ which parametrizes these situations, and we describe the local structure of this curve. We find out how the tangent cone of the curve in $C^{(2)}$ at a secant divisor is determined by the local geometry of C a the points of secancy.

In the two last sections we give some further applications of our local methods. These sections contain no essentially new results.

In §6 we study singularities of plane curves. We reproduce a well-known formula for the δ-invariant of a cusp.

In §7 we reproduce a well-known formula for the weights of Weierstraß points on C with respect to an arbitrary linear system.

First we recall the main results from Part 1.

<u>The main results from Part 1.</u>

Let $X_0,\dots,X_n$ be independent sections spanning a linear system V and set

$$D = \sum_{i=1}^{k} d_i P_i, \text{ where } \sum_{i=1}^{k} d_i = d,$$

and the P_i are distinct points on the curve C. Choose t_i as a local parameter for C at P_i, for $i=1,\dots,k$, and let

$$\sum_{j=0}^{\infty} a_{r,i,j} t_i^j$$

be a local parametrization of X_r at P_i, for $i=1,\ldots,k$, and $r=0,\cdots,n$.

Regard $\{s_{1,1},\ldots,s_{1,d_1},\ldots,s_{k,1},\ldots,s_{k,d_k}\}$ as a set of (formal) algebraically independent variables. Let $s_{i,\ell} = 0$, when $\ell > d_i$, and set

$$(1.1) \qquad W_j(\underline{s}^{(i)}) = \begin{vmatrix} s_{i,1} & s_{i,2} & \cdots & & s_{i,j} \\ 1 & s_{i,1} & \cdots & & s_{i,j-1} \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & s_{i,1} \end{vmatrix}$$

when $j \in N$, set $W_0(\underline{s}^{(i)})=1$, and $W_j(\underline{s}^{(i)})=0$, when $j<0$ for $i=1,\cdots,k$.

Denote by M the following matrix:

$$(1.2) \qquad \begin{bmatrix} \sum a_{0,1,j} W_j(\underline{s}^{(1)}) & \cdots & \sum a_{n,1,j} W_j(\underline{s}^{(1)}) \\ \vdots & & \vdots \\ \sum a_{0,1,j} W_{j-d_1+1}(\underline{s}^{(1)}) & \cdots & \sum a_{n,1,j} W_{j-d_1+1}(\underline{s}^{(1)}) \\ \vdots & & \vdots \\ \sum a_{0,k,j} W_j(\underline{s}^{(k)}) & \cdots & \sum a_{n,k,j} W_j(\underline{s}^{(k)}) \\ \vdots & & \vdots \\ \sum a_{0,k,j} W_{j-d_k+1}(\underline{s}^{(k)}) & \cdots & \sum a_{n,k,j} W_{j-d_k+1}(\underline{s}^{(k)}) \end{bmatrix}$$

Theorem 1.1.

$$\hat{O}_{V_d^r, D} = K[[s_{1,1},\ldots,s_{k,d_k}]]/J$$

where J is generated by the $d-r+1$-minors of M.

Remark 1.2. If $d_i = 1$, for $i = 1,\ldots,d$, then the entries of M are simply the local parametrizations of the sections spanning V.

Denote by BN the following (Brill-Noether) matrix consisting of the "constant terms" of M:

$$
(1.3)\qquad \begin{bmatrix}
a_{0,1,0} & \cdots & a_{n,1,0} \\
\vdots & & \vdots \\
a_{0,1,d_1-1} & \cdots & a_{n,1,d_1-1} \\
\vdots & & \vdots \\
a_{0,k,0} & \cdots & a_{n,k,0} \\
\vdots & & \vdots \\
a_{0,k,d_k-1} & \cdots & a_{n,k,d_k-1}
\end{bmatrix}
$$

Corollary 1.3.

$D \in V_d^r$ if and only if all $d-r+1$ minors of BN vanish.

The following remarks will be useful:

Remark 1.4.

Regard $S_1,\ldots,S_d$ as the d elementary symmetric functions in d variables $T_1,\ldots,T_d$, and let $W_j(\underline{S})$ be as in Formula (1.1). Then

$$W_j(S_1(T_1,\ldots,T_d),\ldots,S_d(T_1,\ldots,T_d))$$

is the sum of all monic monomials of degree j in $T_1,\ldots,T_d$.

Remark 1.5.

$$W_j(S_1,\ldots,S_d) = \sum S_1^{i_1} \cdot \cdots \cdot S_d^{i_d} \cdot (-1)^{\sum_{j=1}^{d} i_j(j-1)} \cdot \frac{(i_1+\cdots+i_d)!}{i_1!\cdots i_d!}$$

where the first sum is taken over those $(i_1,\ldots,i_d)$ such that $\sum_{j=1}^{d} j \cdot i_j = d.$

§2. The tangent space dimension of V^1_d at $D \in C^{(d)}$.

The varieties V^1_d are interesting since they parametrize divisors that are "special" with respect to the linear systems V.

Let $D = \sum_{i=1}^{k} d_i P_i$, where $D \in V^1_d$, and the P_i are distinct points on C. We will use Theorem 1.1. to compute the tangent space dimension of V^1_d at D. The Brill-Noether matrix BN (Formula (1.3)) consists of k groups of concecutive rows, where the i'th group (consisting of d_i rows) corresponds to the point P_i, for $i = 1,\ldots,k$.

Definition 2.1.

ℓ_i is the maximal integer $s \in \{0,\ldots,d_i-1\}$ such that the matrix consisting of all rows of BN except the s+1'th row in the i'th group, has rank d-1. If no such integer exists, set $\ell_i = -1$.

Explanation. Assume for simplicity that V is base-point free and thus maps C into some P^n. For a chosen set of local parameters of C at the P_i we can talk about derivative vectors of C at the P_i. Call the point P_i itself the O'th derivative vector of C at P_i. Then ℓ_i is the maximal integer $s \in \{0,\ldots,d_i-1\}$ such that the union of the O'th,...,$\hat{s}$'th,...,d_i-1'th derivative vectors of C at P_i and the O'th,...,d_j-1'th derivative vectors at P_j, for $j \neq i$, span a d-2 plane in P^n. If no such s exists, then $\ell_i = -1$.

Observation 2.2.

$D \in V^2_d \iff \ell_i = -1$, for $i = 1,\ldots,k$.

Definition 2.3.

Assume $D' \in C^{(d')}$, for some $d' \in N$. Denote by $V(-D')$ the linear system $V \cap H^0(C,L(-D'))$.

We now give the main result of this section:

Theorem 2.4.

The tangent space dimension of V_d^1 at D is

$$\min(d, \operatorname{rk} V(-\sum_{i=1}^{k}(d_i+\ell_i+1)P_i)+2d-n-2) \quad \text{where} \quad \operatorname{rk} V = n+1.$$

Proof.

It is enough to study the constant and linear parts of the matrix M in Formula (1.2). Since $\operatorname{rk} BN \leq d-1$, we may assume that only the $d-1$ first columns of BN are non-zero. Since we will only study the linear parts of the d-minors, we may assume that the entries in the $d-1$ first columns are constant. Assume first $D=dP$. We may drop the index i in M, and we have:

$$M=\begin{bmatrix} a_{0,0} & \cdots & a_{d-2,0} & (-1)^{d-1}a_{d-1,d}s_d & \cdots & (-1)^{d-1}a_{n,d}s_d \\ a_{0,1} & \cdots & a_{d-2,1} & (-1)^{d-2}a_{d-1,d}s_{d-1}+(-1)^{d-1}a_{d-1,d+1}s_d & \cdots & (-1)^{d-2}a_{n,d}s_{d-1}+(-1)^{d-1}a_{n,d+1}s_d \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{0,d-1} & \cdots & a_{d-2,d-1} & a_{d-1,d}s_1+\cdots+(-1)^{d-1}a_{d-1,2d-1}s_d & \cdots & a_{n,d}s_1+\cdots+(-1)^{d-1}a_{n,2d-1}s_d \end{bmatrix}$$

Here we used that the linear part of $W_j(s_1,\ldots,s_d)$ is $(-1)^{j-1}s_j$, for $j=1,\ldots,d$. See Formula (1.1) or Remark 1.5. Let D_{j-1} be the $d-1$ minor formed by the $d-1$ first columns of M (or BN) minus the j'th row. We see that ℓ is the largest integer j such that $D_j \neq 0$ if such an integer exists (See Definition 2.1.).

The linear parts of the $n+2-d$ relations cutting out V_d^1 are (up to signs):

$$(a_{i,d}D_\ell)s_{d-\ell} - (a_{i,d+1}D_\ell + a_{i,d}D_{\ell-1})s_{d-\ell+1} + \cdots + (-1)^\ell(a_{i,d+\ell}D_\ell + \cdots + a_{i,d}D_0)s_d$$

for $i = d-1,\ldots,n$.

The coefficient matrix of these relations in $s_1,\ldots,s_d$ is easily seen to have the same rank as

$$N = \begin{bmatrix} a_{d-1,d} & \cdots & a_{d-1,d+\ell} \\ \vdots & & \vdots \\ a_{n,d} & \cdots & \alpha_{n,d+\ell} \end{bmatrix}$$

Hence the tangent space dimension of V_d^1 at D is $d-\mathrm{rk}\,N$ if $\ell\geq 0$ and d otherwise. Assume first $\ell\geq 0$. Let us find

$$\mathrm{rk}\,V(-(\ell+d+1)P).$$

Since $\ell\geq 0$, Observation 2.2. gives that the matrix BN has rank exactly $d-1$, and therefore a section contained in $V(-dP)$ must be of the form

$$c_{d-1}X_{d-1} + \cdots + c_nX_n, \text{ where the } c_j \in K, \text{ and}$$

where X_j is the section corresponding to the $j+1$'th column of M. The conditions that such a section should be contained in $V(-(\ell+d+1)P)$ are:

$$\begin{aligned} a_{d-1,d}c_{d-1} &+ \cdots + a_{n,d}c_n = 0 \\ &\vdots \\ a_{d-1,d+\ell}c_{d-1} &+ \cdots + a_{n,d+\ell}c_n = 0. \end{aligned}$$

These equations in the variables $c_{d-1},\ldots,c_n$ give rise to a coefficient matrix, which is the transpose of N.

Hence $\mathrm{rk}\,V(-(\ell+d+1)P) = n-d+2 - \mathrm{rk}\,N$, and we deduce that the tangent space dimension of V_d^1 at D is

$$d - \text{rk } N = 2d-n-2 + \text{rk } V(-(\ell+d+1)P)$$

Since $\text{rk } V(-(\ell+d+1)P) \leqslant \text{rk } V(-dP) = n-d+2$, our tangent space dimension is at most

$(2d-n-2) + (n-d+2) = d$. Hence the theorem holds when $D = dP$, and $\ell \geqslant 0$.

When $D=dP$ and $\ell=-1$, the tangent space dimension is d since all the D_j are zero. On the other hand:

$$2d-n-2 + \text{rk } V(-(\ell+d+1)P) = 2d-n-2 + \text{rk } V(-dP)$$

$$= (2d-n-2) + (n+1-\text{rk } BN) \geqslant d+1 \quad , \text{ since } \text{rk } BN \leqslant d-2.$$

Hence d is the minimum of d and $2d-n-2 + \text{rk } V(-(\ell+d+1)P)$. Our proof is now complete in the case $D=dP$. The general case follows easily using the same argument for each group of d_i rows of M.

§3. A local study of V_n^1, where $\text{rk } V = n+1 \geqslant 4$

In [J], Theorem 2.3.1., we gave a multiplicity formula for trisecant lines to a space curve. In this section we will generalize this formula.

Let $D \in C^{(n)}$ be a point of V_n^1, where $\text{rk } V = n+1 \geqslant 4$.

Assume:

1.) For each $D' \in C^{(n-1)}$, such that $D' \leqslant D$, we have $D' \notin V_{n-1}^1$

2.) If $D = \sum_{i=1}^{k} n_i P_i$ (all $n_i > 0$), then $D+P_i \notin V_{n+1}^2$, for $i=1,\dots,k$

3.) Char $K = 0$, and $K = \bar{K}$.

Proposition 3.1.

Under Assumptions 1.), 2.), 3.) we have:

a) The tangent space dimension of V_n^1 at D is $\mathrm{rk}\, V(-2D) + n-2$, where $\mathrm{rk}\, V(-2D)$ is 0 or 1.

b) $\dim O_{V_n^1, D} = n-2$

c) The multiplicity of V_n^1 at D is the largest integer s such that $\mathrm{rk}\, V(-sD) \geq 1$. (with equality if V_n^1 is singular at D).

<u>Proof</u>: Let ℓ_i, for $i=1,\dots,k$, be the integers described in Definition 2.1. Assumption 1.) gives $\ell_i = n_i - 1$ for all i. The tangent space dimension formula in Part a) is then a special case of Theorem 2.4., and it holds also when char $K > 0$.

Assumption 2.) gives that $\mathrm{rk}\, V(-2D)$ is 0 or 1, because if $\mathrm{rk}\, V(-2D) \geq 2$, then $2D \in V_{2n}^{n+1}$, and then $D + P_i \in V_{n+1}^2$ for <u>all</u> $i \in \{1,\dots,k\}$. Hence a) holds.

By general facts about determinantial varieties we have $\dim O_{V_n^1, D} \geq n-2$.

If $\mathrm{rk}\, V(-2D) = 0$, then the tangent space dimension of V_n^1 at D is $n-2$ by a). Hence $\dim \hat{O}_{V_n^1, D} \leq n-2$, and b) follows. Furthermore V_n^1 is non-singular at D in this case. Hence the multiplicity of V_n^1 at D is 1. Since $\mathrm{rk}\, V(-2D)=0$, and $\mathrm{rk}\, V(-1 \cdot D) \geq 2 \geq 1$, the number given in c) is also 1. Hence c) follows when $\mathrm{rk}\, V(-2D)=0$.

It remains to prove b) and c) when $\mathrm{rk}\, V(-2D)=1$. Let V be generated by the sections $\{X_0, \dots, X_n\}$.

$\mathrm{rk}\, V(-D) \geq 2$ since $D \in V_n^1$, and $\mathrm{rk}\, V(-D) \leq 2$ since $D \notin V_n^2$ by Assumption 1.).

Hence $\mathrm{rk}\, V(-D) = 2$, and we may assume that X_{n-1} and X_n generate $V(-D)$. This means that the entries in the 2 last columns of the BN-matrix (Formula (1.3)) are zero.

We may assume that X_n generates $V(-2D)$ since $\mathrm{rk}\ V(-2D)=1$. We will also assume that $D = nP$. The proof of the general case is a slight generalization of this special case, essentially only involving more indices. At the end of the proof we will add a few words about how this generalization can be made. When $D=nP$, the matrix M is of the following form:

$$\begin{bmatrix} a_{0,0}+\cdots & a_{n-2,0}+\cdots & (-1)^{n-1}a_{n-1,n}s_n+\cdots & a_{n,mn}W_{mn}+\cdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{0,n-1}+\cdots & a_{n-2,n-1}+\cdots & a_{n-1,n}s_1+\cdots+(-1)^{n-1}a_{n-1,2n-1}s_n+\cdots & a_{n,mn}W_{mn-n+1}+\cdots \end{bmatrix}$$

where $m = \max\{s \mid \mathrm{rk}\ V(-sD)=1\} \geqslant 2$, and $W_j = W_j(\underline{s})$ as in Formula (1.1).

To set up the column to the right (corresponding to X_n) we have used

$$a_{n,0} = a_{n,1} = \cdot \cdot \cdot = a_{n,mn-1} = 0$$

which is true since $X_n \in V(-mD)$. In particular this column contains no linear terms in $\{\underline{s}\}$.

In the n'th column (corresponding to X_{n-1}) we have listed all linear terms in $\{\underline{s}\}$. Observe that $a_{n-1,n} \neq 0$, because $a_{n-1,n} = 0$ implies $(n+1)P \in V^2_{n+1}$, which contradicts Assumption 2.).

Summing up we see that there is at most one n-minor of M that contains linear terms, namely the one obtained by disregarding the column corresponding to X_n.

Denote by $R_j(s_1,\ldots,s_n)$ the n-minor of M obtained by disregarding the column corresponding to X_j, for $j=0,\ldots,n$.

By Assumption 1.) the n-1-minor obtained from the n-1 first columns of the BN-matrix minus the bottom row is non-zero. This observation together with the fact that $a_{n-1,n}$ is non-zero enables us to use the relation $R_n(s_1,\ldots,s_n)$ to express s_1 as a function

$$f(s_2,\ldots,s_n)$$

We see that:

$$\hat{O}_{V_n^1,D} \cong K[[s_2,\ldots,s_n]]/(\bar{R}_0,\ldots,\bar{R}_{n-1})$$

where $\bar{R}_j = R_j(f(s_2,\ldots,s_n),\ s_2,\ldots,s_n)$ for $j=0,\ldots,n-1$. Denote by $\underline{M}$ the maximal ideal of the last ring. The dimension of $\hat{O}_{V_n^1,D}$ is n-2 if there is a relation between the images of $s_2,\ldots,s_n$ modulo $\underline{M}^r$ for some r. Then the multiplicity m' of V_n^1 at D is the smallest integer r such that there is such a relation. We will show

$$m' = m = \max\{s \mid rkV(-sD) = 1\}.$$

Using Remark 1.5. one sees that the entries in the column corresponding to X_n contain no terms of degree less than m in $s_1,\ldots,s_n$.

Hence the relations $\bar{R}_1,\ldots,\bar{R}_{n-2}$ contain no terms of degree less than m+1 in $s_2,\ldots,s_n$ since all the constant terms in the column corresponding to X_{n-1} are zero.

We also see that the relation $\bar{R}_{n-1}$ contains no terms of degree less than m.

To show b) and c) it is therefore enough to show that the homogeneous part of degree m of $\bar{R}_{n-1}$ does not vanish identically. Denote by D_j the n-1 minor obtained from the n-1 first columns of the BN-matrix while disregarding the j+1'th row, for $j=0,\ldots,n-1$.

Using Formula (1.2) and Remark 1.5. we see that $R_{n-1}(s_1,\dots,s_n)$ is (up to a possible shift of sign):

$$
\begin{aligned}
& s_n^m(D_0\cdot a_{n,mn}+D_1\cdot a_{n,mn+1}+\cdots+D_{n-1}\cdot a_{n,mn+n-1}) \\
& - s_n^{m-1}s_{n-1}(\quad D_1\cdot a_{n,mn}+\cdots+D_{n-1}\cdot a_{n,mn+n-2})\cdot m \\
& \quad \vdots \\
& + (-1)^{n-1}s_n^{m-1}s_1(\qquad D_{n-1}\cdot a_{n,mn})\cdot m \\
& + (-1)^{n-1}s_n^{m-2}s_{n-1}s_2(\qquad D_{n-1}\cdot a_{n,nm})\cdot m(m-1)
\end{aligned}
$$

+ other terms of degree m + terms of degree $m+1$ or more.

There is no $s_n^{m-2}s_{n-1}s_1$-term by Remark 1.5. The relation $\bar{R}_{n-1}$ is obtained by substituting $s_1 = f(s_2,\dots,s_n)$ in the above relation. If the homogeneous part of degree m vanishes, it implies in particular that the terms involving

$$s_n^m,\ s_n^{m-1}s_{n-1},\dots,s_n^{m-1}s_2,s_n^{m-2}s_{n-1}s_2$$

vanish. This gives the following coefficient matrix in the "variables" $a_{n,mn},\dots,a_{n,mn+n-1}$:

$$
N = \begin{bmatrix}
\star & D_1 & \cdots & D_{n-2} & D_{n-1} \\
\star & -mD_2 & \cdots & -mD_{n-1} & \\
 & & \iddots & & \\
\star & (-1)^{m-2}mD_{n-1} & & & 0 \\
(-1)^{n-1}m(m-1)D_{n-1} & & & &
\end{bmatrix}
$$

The stars depend on $f(s_2,\dots,s_n)$

Assumption 1.) implies $D_{n-1} \neq 0$. Hence $\det N \neq 0$ when $\mathrm{Char}\, K = 0$. Hence the homogeneous part of $\bar{R}_{n-1}$ of degree m vanishes identically only if

$$a_{n,mn} = \cdots = a_{n,mn+n-1} = 0.$$

But this implies $\mathrm{rk}\, V(-(m+1)D) = 1$ which contradicts the definition of m. This completes the proof of b) and c) when $D=nP$. We see that the proof does not work if $n=2$, since we need $s_n^{m-2} s_{n-1} s_2$ to be different from $s_n^{m-1} s_1$.

In the general case $D = \sum_{i=1}^{k} n_i P_i$ essentially the same argument works when $n = \sum_{i=1}^{k} n_i \geq 3$. We always get one and only one relation between the $s_{i,j}$ modulo $(\underline{s})^2$, and we use this relation to express one of the $s_{i,1}$, say $s_{1,1}$, as a linear function in the other $s_{i,j}$ modulo $(\underline{s})^2$. This will follow from Assumption 2.). One can always assume $n_1 = \max_i \{n_i\}$. Then one splits into 3 cases; $n_1 \geq 3$, $n_1 = 2$, $n_1 = 1$. In each case one ends up with a skew-triangular coefficient matrix analogous to N, with D_{i,n_i-1}'s on the skew diagonal. All D_{i,n_i-1} are non-zero by Assumption 1.), and one gets a contradiction the same way as in the case $D=nP$. Hence b) and c) hold in general.

<u>Definition 3.2</u>.

For a variety X and a point P in X the tangent cone $\mathcal{T}_P(X)$ of X at P is

$$\mathrm{Spec}\left(\bigoplus_{i=0}^{\infty} m^i/m^{i+1}\right)$$

where m is the maximal ideal of the local ring $O_{X,P}$.

The projectivized tangent cone $P\mathcal{T}_P(X)$ of X at P is

$$\mathrm{Proj}\left(\bigoplus_{i=0}^{\infty} m^i/m^{i+1}\right).$$

Corollary 3.3.

Under Assumptions 1.), 2.), 3.) the projectived tangent cone $P\mathcal{T}_D(V_n^1)$ is a hypersurface of degree m in P^{n-2}, where

$$m = \max\{s \mid rk\ V(-sD) \geq 1\}.$$

Proof. Corollary 3.3. follows from the proof of Proposition 3.1.

§4. The tangent cone $\mathcal{T}_D(V_n^1)$, where $rk\ V = n+1 \geq 4$.

In this section we will not always prove our assertions. Our goal is to give a geometrical interpretation of $\mathcal{T}_D(V_n^1)$ (or $P\mathcal{T}_D(V_n^1)$) described at the end of §3.

In §3 we studied a point D in V_n^1, where $rk\ V = n+1 \geq 4$. Under Assumptions 1.), 2.), 3.) of §3 we gave a description of the dimension, embedding dimension and multiplicity of V_n^1 at D.

A question which then arises naturally is: When is the projectivized tangent cone $P\mathcal{T}_D(V_n^1)$ singular? If $n=3$ and V_n^1 is a curve, then $P\mathcal{T}_D(V_n^1)$ is singular if V_n^1 does not have normal crossings at D; we also say that V_n^1 possesses a non-ordinary singularity at D in this case. In [J] we gave necessary and sufficient local conditions on C for determining whether the trisecant curve (essentially V_3^1) possesses non-ordinary singularities or not. We want to generalize these conditions to apply to any V_n^1, $n \geq 3$, where $rk\ V = n+1$.

In order to do this we assume:

2!). V is base-point free and $D+P \notin V_{n+1}^2$ for any point $P \in C$.

Assumption 2!) is of course a strengthening of Assumption 2.) of §3; but this strengthening is of no importance for the local geometry of

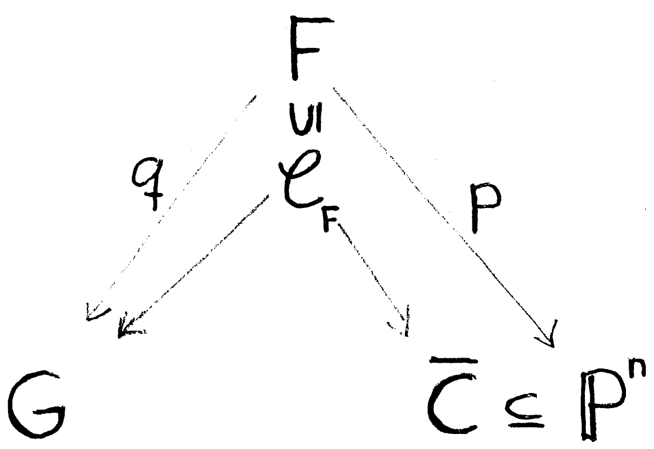
V_n^1 at D. Whatever local result we prove for V_n^1 at D under Assumptions 1.), 2'.), 3.) will also hold under Assumptions 1.), 2.), 3.). This is true because the matrix M (Formula (1.2)) is only dependent on the behaviour of V at the points $P_1,\dots,P_k$, and because any base point of V is outside $\{P_1,\dots,P_k\}$ by Assumption 1.).

Under Assumption 2'.) V defines a map $\phi: C \to \bar{C} \subset P^n$. Let $G = G(n-2,n)$ be the Grassmannian, which parametrizes the n-2 planes in P^n.

For a n-2 plane H denote by [H] the corresponding point in G. Denote by F the incidence variety

$$\{([H], P) \in G \times P^n \mid P \in H\}.$$

Consider the following diagram:



Here p and q are the natural projection maps from F to P^n and G respectively, and $\mathcal{C}_F = p^{-1}(\bar{C})$.

Let Sec be the subvariety of G cut out by the sheaf of O_G-ideals:

$$F^{n-1}(q_* O_{\mathcal{C}_F}),$$

that is the sheaf of n-1'th Fitting ideals of the O_G-sheaf $q_* O_{\mathcal{C}_F}$. Then Sec parametrizes n-2 planes that are n-secant to C. This definition of Sec is taken from [GP], where the case n=3 is treated.

Assume $D \in V_n^1$, and that Assumptions 1.), 2!), 3.) hold. Then D spans a unique n-2 plane; that is $P_1,\ldots,P_k$ and the d_i-1 first derivative vectors of $\bar{C}$ at P_i, for $i=1,\ldots,k$, span a unique n-2 plane H.

We make the following claim:

$$(4.1) \qquad \mathcal{T}_D(V_n^1) \simeq \mathcal{T}_{[H]}(\mathrm{Sec}).$$

In fact we strongly believe:

$$(4.2) \qquad \hat{O}_{V_n^1, D} \simeq \hat{O}_{\mathrm{Sec},[H]}.$$

We have not made any attempts to prove (4.2), but we have proved (4.1) when D consists of n distinct points.

To find $\mathcal{T}_D(V_n^1)$ one simply calculates the leading forms of the relations $R_{n-1}(\underline{s})$ and $R_n(\underline{s})$ described in the proof of Proposition 3.1. In [J] an explicit description of $\mathcal{T}_{[H]}(\mathrm{Sec})$ is given in the case where n=3, whether D consists of 3 distinct points or not.

It is easy, but a little painstaking, to generalize this explicit description to arbitrary $n \geq 3$, when the n points of D are distinct. Comparing the 2 tangent cones one sees that they are isomorphic.

We omit the very technical calculations here. In principle the same method should work when the n points are not distinct.

We will assume that Formula (4.1) is always true under Assumptions 1.), 2'.), 3).

<u>Definition 4.1</u>.

For a curve C and and a hypersurface M in P^n, denote by $I(P, C\cap M)$ the usual intersection number between C and M at P.

From Formula (4.1) and Proposition 3.1., a) we see:
Sec is singular at $[H]$ <=> V_n^1 is singular at D <=> There exists a unique hyperplane $\mathcal{H}$ in P^n with

$$I(P_i, \bar{C}\cap\mathcal{H}) \geq 2n_i, \text{ for } i=1,\dots,k.$$

We have $\mathrm{Sec} \subset G \subset P^S$ for some large S. Making explicit calculations analogous to those in [GP] and [J] one finds that the embeded (compactified) tangent space in P^S to Sec at $[H]$ is

$$\check{\mathcal{H}} \subset G \subset P^S,$$

where $\check{\mathcal{H}}$ is the $n-1$ plane in G, which parametrizes the $n-2$ planes in the hyperplane $\mathcal{H} \subset P^n$.

Hence the embedded tangent cone in P^S to Sec at $[H]$ is a union of an $n-3$ dimensional family of lines in $\check{\mathcal{H}}$. Each point of the projectivized tangent cone $P\mathcal{T}_{[H]}(\mathrm{Sec})$ or $P\mathcal{T}_D(V_n^1)$ corresponds to one such line.

A line L in $\check{\mathcal{H}}$ through $[H]$ is a nesting of a 1-dimensional family of $n-2$ planes in $\mathcal{H}$ containing a fixed $n-3$ plane h_L contained in H.

Hence each point of $P\mathcal{T}_{[H]}(\mathrm{Sec})$ and $P\mathcal{T}_D(V_n^1)$ corresponds to an $n-3$ plane h_L in the $n-2$-plane H. Denote by $[h]$ the point in $\check{H}$

corresponding to an n-3-plane h, where $\check{H}$ is the n-2 plane which parametrizes the n-3 planes in H.

By Corollary 3.3. $P\mathcal{T}_D(V_n^1)$ is a hypersurface of degree $m=\max\{s \mid \mathrm{rk}\ V(-sD)\geq 1\}$ in P^{n-2}. From the above discussion it is clear that a natural geometrical interpretation of this P^{n-2} is $\check{H}$, and that

$P\mathcal{T}_D(V_n^1) \cong \{[h_L] \mid L$ is a line in $\check{\mathcal{H}}$ through $[H]$, such that L is contained in the embedded tangent cone to Sec at $[H]\}$.

Two problems now arise in a natural way:

(i) Find those n-3 planes h in H such that $[h] \in P\mathcal{T}_{[H]}(\mathrm{Sec})$.

(ii) Find those n-3 planes h in H such that $[h]$ is a singular point of $P\mathcal{T}_{[H]}(\mathrm{Sec})$.

We state without proofs the solutions to problems (i) and (ii) (Results 4.2. and 4.3. respectively). Result 4.2. is a generalization of Theorem 2.3.2. of [J], and Result 4.3. is a generalization of Theorem 2.3.3. in [J].

We have proved Results 4.2. and 4.3. in the case where D consists of n distinct points, but we omit the technical details here.

<u>Result 4.2</u>.

Under Assumptions 1.), 2!), 3) we have $[h] \in P\mathcal{T}_{[H]}(\mathrm{Sec})$ if and only if there exists a hypersurface M in P^n such that

a) deg M=m+1, and M has a singularity of multiplicity at least m at all points of h.

b) $I(P_i, M\cap\bar{C}) \geq (m+1)n_i$, for all $P_i \in H \cap C$

c) $m \cdot H \subseteq M \cap$, i.e. $I(M) \subseteq (I(\mathcal{H}) + I(H)^m)$, and $H \not\subseteq \mathrm{Sing}(M)$

d) The equation defining M in P^n is equal to the equation of a cone of degree m+1 with h contained in its vertex set, modulo the square of the ideal defining H.

Remark: M can be taken to be a union of a 1-dimensional family of n-2 planes containing H. Thus M gives rise to a curve Γ in G. The tangent line to Γ at [H] is L, where $h = h_L$.

Result 4.3.

Under Assumptions 1.), 2'.), 3.) we have: [h] is a singular point of $P\mathcal{T}_{[H]}(\mathrm{Sec})$ if and only if there exists a hypersurface N in P^n such that:

a) N is a cone of degree m, and h is contained in the vertex set of N.

b) $I(P_i, \bar{C} \cap N) \geq (m+1)n_i$, for $i=1,\ldots,k$

c) $H \not\subseteq \mathrm{Sing}(N)$

Corollary 4.4.

Under Assumptions 1.), 2.), 3.) we have: $P\mathcal{T}_D(V_n^1)$ is singular if and only if there exists a cone N and an n-3 plane h as described in Result 4.3., a), b), c).

§5. Stationary bisecants for a space curve.

In §5 we assume char K = 0, and $K = \bar{K}$. Let C be a non-singular curve in P^3, and let P_1 and P_2 be points on C. The line $\overline{P_1P_2}$ is usually called a stationary bisecant if the tangents to C at P_1 and P_2 meet. In general there is a 1-dimensional family of stationary bisecants for a space curve. We will define a scheme in $C^{(2)}$, which essentially parametrizes divisors $P_1 + P_2$ with P_1 and P_2 as described.

Some divisors 2P may also occur as points on this scheme in $C^{(2)}$ since tangent lines are in some sense bisecants.

Let C be mapped into P^3 by evaluating sections of some linear system V of rank 4. Consider the map:

(5.1) $\quad i: \; C^{(2)} \to C^{(4)}$, where $i(D) = 2D$

for divisors D in $C^{(2)}$.

Definition 5.1.

The scheme of stationary bisecants for C with respect to V is $i^{-1}(V_4^1)$.

Remark 5.2.

Clearly $D \in i^{-1}(V_4^1) \iff 2D \in V_4^1$. If $P_1 \neq P_2$, then $P_1 + P_2 \in i^{-1}(V_4^1) \iff$ the tangent lines to C at P_1 and P_2 meet.

We also have:

$2P \in i^{-1}(V_4^1) \iff P$ is a flex on C, or the osculating plane of C at P is hyperosculating.

It will follow from the proofs of Propositions 5.3. and 5.6. that $i^{-1}(V_4^1)$ is either a curve or empty.

We will use Theorem 1.1. to determine the multiplicity of $i^{-1}(V_4^1)$ at an arbitrary point D (in $C^{(2)}$) in terms of the local geometry of C at the secant points in P^3. The cases $D=2P$ and $D = P_1 + P_2$ $(P_1 \neq P_2)$ will be treated separately. As before we denote by $I(Q, C \cap F)$ the intersection multiplicity between a curve C and a surface F at a point Q in P^3.

The multiplicity of $i^{-1}(V_4^1)$ at $D=2P$

Let L be the tangent line to C at the point P. Set $m_2 = \max\{\ell \mid \ell P \in V_\ell^{\ell-2}\}$, or equivalently: $m_2 = I(P, C \cap H)$ for a general member H of the pencil of planes containing L. If P is not a flex on C, then $m_2 = 2$. Set $m_3 = \max\{\ell \mid \ell P \in V_\ell^{\ell-3}\}$, or equivalently:

$m_3 = \max\limits_{H \supseteq L}\{I(P, C \cap H)\}$. Clearly $m_3 \geq m_2 + 1$.

We now give our main result in the case $D=2P$:

Proposition 5.3.

The multiplicity of the curve $i^{-1}(V_4^1)$ at $2P$ is $[\frac{m_2+m_3}{2}]-2$, where $[x]$ means the integral part of the real number x.

Proof:

Let t be a local parameter for C at P. Without loss of generality we may assume that C is parametrized locally at P as:

$$X_0 = 1$$
$$X_1 = t$$
$$X_2 = \sum_{j\geq m_2} \alpha_j t^j \quad , \quad \alpha_{m_2} \neq 0$$
$$X_3 = \sum_{j\geq m_3} \beta_j t^j \quad , \quad \beta_{m_3} \neq 0.$$

Let s_1, s_2, s_3, s_4 be local parameters for $C^{(4)}$ at $4P$, where the s_k are the k'th elementary functions in t_1, t_2, t_3, t_4; 4 replicas of t.

By Theorem 1.1., we have

$$\hat{O}_{V_4^1, 4P} = K[[s_1,s_2,s_3,s_4]]/(\det M),$$

where

$$M = \begin{bmatrix} 1 & s_1 & \sum_j \alpha_j W_j(\underline{s}) & \sum_j \beta_j W_j(s) \\ 0 & 1 & \sum_j \alpha_j W_{j-1}(\underline{s}) & \sum_j \beta_j W_{j-1}(s) \\ 0 & 0 & \sum_j \alpha_j W_{j-2}(\underline{s}) & \sum_j \beta_j W_{j-2}(s) \\ 0 & 0 & \sum_j \alpha_j W_{j-3}(\underline{s}) & \sum_j \beta_j W_{j-3}(s) \end{bmatrix}$$

We see that

(5.2) $$\det M = \sum_{j\geq m_2} \alpha_j W_{j-2}(\underline{s}) \cdot \sum_{j\geq m_3} \beta_j W_{j-3}(\underline{s}) - \sum_{j\geq m_2} \alpha_j W_{j-3}(\underline{s}) \cdot \sum_{j\geq m_3} W_{j-2}(\underline{s})$$

Let S_1 and S_2 be local parameters of $C^{(2)}$ at $2P$, where the S_k are the k'th symmetric functions in T_1, T_2; 2 formal replicas of t.

The map (5.1) induces a map

$$i^*: K[[s_1,s_2,s_3,s_4]] \to K[[S_1,S_2]].$$

Clearly $\hat{O}_{i^{-1}(V_4^1),2P} \cong K[[S_1,S_2]]/(R)$ where R is the power series obtained by substituting i^*s_k for s_k in (5.2), for $k=1,2,3,4$. The multiplicity $\mathrm{mult}_{2P}(i^{-1}(V_4^1))$ is the lowest value e_1+e_2 for any term $S_1^{e_1}S_2^{e_2}$ occuring in R. We will first find the i^*s_k. Let $s_k = s_k(t_1,t_2,t_3,t_4)$; that is: Regard s_k as the k'th elementary symmetric function in 4 replicas of t, for $k=1,\ldots,4$.

We define

$$\phi_k(T_1,T_2) = s_k(T_1,T_1, T_2,T_2).$$

Clearly $\phi_k(T_1,T_2)$ is symmetric in T_1, T_2, for $k=1,\ldots,k$. Hence there are unique functions $\phi_k(S_1,S_2)$ such that $\phi_k(S_1(T_1,T_2), S_2(T_1,T_2) = \phi_k(T_1,T_2)$ for $i=1,\ldots,k$.

One sees that $i^*s_k(S_1,S_2)= \phi_k(S_1,S_2)$ for all k.

We then obtain:

$i^*s_1 = 2S_1,\quad i^*s_2 = S_1^2 + 4S_2$
$i^*s_3 = 2S_1S_2 \quad i^*s_4 = S_2^2.$

We have:

$$R = \sum_{j\geqslant m_2} \alpha_j(i^*W_{j-2}) \cdot \sum_{j\geqslant m_3} \beta_j(i^*W_{j-3}) - \sum_{j\geqslant m_2} \alpha_j(i^*W_{j-3}) \cdot \sum_{j\geqslant m_3} \beta_j(i^*W_{j-2}), \tag{5.3}$$

where $i^*W_\ell = W_\ell(i^*s_1,\dots,i^*s_4)$ for all ℓ.

The next task is to describe the i^*W_ℓ. First we remark that each of the rings $K[[s_1,s_2,s_3,s_4]]$ and $K[[S_1,S_2]]$ is graded in 2 ways:

We define:

$$\deg_1 s_k = 1,\ \deg_2 s_k = k, \quad \text{for } k=1,\dots,4.$$

$$\deg_1 S_k = 1,\ \deg_2 S_k = k, \quad \text{for } k=1,2.$$

One sees that the $W_j(s_1,s_2,s_3,s_4)$ and the $i^*W_j(S_1,S_2)$ are homogeneous in the sense that:

$$\deg_2 W_j(S_1,S_2,S_3,S_4) = \deg_2 i^*W_j(S_1,S_2) = j.$$

This follows from Remark 1.4. combined with the fact that the map i^* is $\deg_2$-preserving.

<u>Definition</u>.

Let c_j for $j=0,1,2,\dots$ be the unique integers such that

$$i^*W_j(S_1,S_2) \equiv c_j S_2^{\frac{j}{2}} \quad \text{mod } S_1 \quad \text{when } j \text{ is even,}$$

$$i^*W_j(S_1,S_2) \equiv c_j S_1 S_2^{\frac{j-1}{2}} \quad \text{mod } S_1^2 \quad \text{when } j \text{ is odd.}$$

Clearly the terms $c_j S_2^{j/2}$ or $c_j S_1 S_2^{\frac{j-1}{2}}$ are the leading forms of the $i^*W_j(S_1,S_2)$ with respect to the $\deg_1$-grading if the c_j are non-zero.

We now give a useful technical lemma.

Lemma 5.4.

a) $c_0 = 1$, $c_1 = 2$, and $|\frac{c_{j+2}}{c}| > 3$ for all $j \geq 0$. In particular $c_j \neq 0$ for $j \geq 0$.

b) The c_{4n} and c_{4n+1} are positive integers, and the c_{4n+2} and c_{4n+3} are negative integers, and $|\frac{c_{j+2}}{c_j}| > |\frac{c_{j+3}}{c_{j+1}}| < 4$ for all non-negative odd integers j.

Proof of Lemma 5.4.

Clearly $c_0 = 1$.

Consider the formula:

$$W_j(s_1,\dots,s_4) = \begin{bmatrix} s_1 & s_2 & \cdot & \cdot & \cdot & s_j \\ 1 & s_1 & \cdot & \cdot & \cdot & s_{j-1} \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & & s_1 \end{bmatrix}$$

where $s_j = 0$ for $j \geq 5$. We expand the matrix along the first row and obtain the recursion formula:

$$W_j(\underline{s}) = s_1W_{j-1}(\underline{s}) - s_2W_{j-2}(\underline{s}) + s_3W_{j-3}(\underline{s}) - s_4W_{j-4}(\underline{s})$$

Using the map i^* we get

$$i^*W_j(\underline{S}) = 2S_1i^*W_{j-1}(\underline{S}) - (S_1^2+4S_2)i^*W_{j-2}(\underline{S})$$
$$+ 2S_1S_2i^*W_{j-3}(\underline{S}) - S_2^2i^*W_{j-4}(\underline{S}).$$

for all integers $j \geq 1$.

In particular we obtain:

(5.4)
$$c_j = -4c_{j-2} - c_{j-4}, \quad \text{when } j \text{ is even}$$
$$c_j = 2c_{j-1} - 4c_{j-2} + 2c_{j-3} - c_{j-4}, \quad \text{when } j \text{ is odd.}$$

For $r=0,1,2,\ldots$ denote by $P(r)$ the following assertion:

All statements in Lemma 5.4 hold for all c_j with $j=4r$, $4r+1$, $4r+2$, $4r+3$.

It is enough to prove $P(r)$ for all r by induction. The case $r=0$ is verified by direct calculation. The induction step follows easily using Formula (5.4).

We now return to the proof of Proposition 5.3, and we split into 4 cases:

Case 1. m_2 odd, m_3 even.
We will find the leading form of the relation R (Formula (5.3)) with respect to the $\deg_1$-grading. The first 2 terms of R are:

$$\alpha_{m_2}\beta_{m_3}(i^*W_{m_2-2}i^*W_{m_3-3} - i^*W_{m_2-3}i^*W_{m_3-2}).$$

The other terms are of the form

$$\alpha_h\cdot\beta_j\; i^*W_k\cdot i^*W_\ell,\ \text{where}\ \ k+\ell \geqslant m_2 + m_3 - 4.$$

We conclude that $\deg_1 M(S_1,S_2) \geqslant \frac{m_2+m_3-3}{2}$ for all monomials $M(S_1,S_2)$ arising from these terms. This is true since $\deg_2 i^*W_j = j$ for all j, and since $\deg_1 M(S_1,S_2) \geqslant \frac{\deg_2 M(S_1,S_2)}{2}$.

The same conclusion also holds for all monomials arising from the term $i^*W_{m_2-2}i^*W_{m_3-3}$.

By Lemma 5.4. a) the leading form of the product $-\alpha_{m_2}\beta_{m_3}i^*W_{m_2-3}i^*W_{m_3-2}$, and hence of R, is:

$$-\alpha_{m_2}\beta_{m_3}c_{m_2-3}c_{m_3-3}S_2^{\frac{m_2+m_3-5}{2}}$$

where c_{m_2-3}, c_{m_3-2} (and of course α_{m_2}, β_{m_3}) are non-zero.

Hence the multiplicity of $i^{-1}(V_4^1)$ at $2P$ is $m = \frac{m_2 + m_3 - 5}{2} = [\frac{m_2 + m_3}{2}] - 2$, and the leading form of R is S_2^m (up to a multiplicative constant).

Case 2. m_2 even, m_3 odd.

Same proof and conclusion as in Case 1, except that the leading form $\alpha_{m_2}\beta_{m_3}c_{m_2-2}c_{m_3-3}S_2^m$ arises from the term $\alpha_{m_2}\beta_{m_3}i^*W_{m_2-2}i^*W_{m_3-3}$.

Case 3. m_2 and m_3 even.

In a similar way as in Case 1 we see that the leading form of R with respect to the $\deg_1$-grading is:

$$\alpha_{m_2}\beta_{m_3}(c_{m_2-2}c_{m_3-3} - c_{m_2-3}c_{m_3-2})S_1S_2^{\frac{m_2+m_3}{2}-3}$$

$$+ (\alpha_{m_2}\beta_{m_3+1} - \alpha_{m_2+1}\beta_{m_3})c_{m_2-2}c_{m_3-2}S_2^{\frac{m_2+m_3}{2}-2} .$$

provided this form does not vanish identically. It is enough to show that $c_{m_2-2}c_{m_3-3} \neq c_{m_2-3}c_{m_3-2}$ since α_{m_2} and β_{m_3} are non-zero.

We have

$$|c_{m_2-2}\, c_{m_3-3}| = |c_{m_2-2}c_{m_2-3}| \cdot (|\frac{c_{m_2-1}}{c_{m_2-3}}| \cdot \cdots \cdot |\frac{c_{m_3-3}}{c_{m_3-5}}|)$$

and

$$|c_{m_2-3}c_{m_3-2}| = |c_{m_2-3}c_{m_2-2}| \cdot (|\frac{c_{m_2}}{c_{m_2-2}}| \cdot \cdots \cdot |\frac{c_{m_3-2}}{c_{m_3-4}}|)$$

By Lemma 5.4. b. the first value is strictly larger than the last value.

Hence the multiplicity of $i^{-1}(V_4^1)$ at $2P$ is $m = \frac{m_2 + m_3}{2} - 2$ and the leading form of R is

(5.5) $$(S_1 + kS_2) \cdot S_2^{\frac{m_2 + m_3}{2} - 3}$$

up to a non-zero multiplicative constant, where k is another constant.

Case 4. m_2 and m_3 odd

This case is treated in essentially the same way as Case 3, and the conclusion is the same.

Corollary 5.5.

If P is not a flex on C, then the multiplicity of $i^{-1}(V_4^1)$ at $2P$ is

$$[\frac{m_3}{2}] - 1, \text{ where}$$

$m_3 = I(P, C \cap H)$, for the osculating plane H of C at P.

The multiplicity of the curve $i^{-1}(V_4^1)$ at $D = P_1 + P_2$.

Assume $P_1 \neq P_2$, and let L be the line $\overline{P_1P_2}$. Set $n_i = I(P_i, C \cap H)$, for $i=1,2$, where H is a general member of the pencil of planes containing L. We may assume $n_1 \geq n_2$.

Let r be the maximal integer such that there exists a plane H with

$$I(P_i, C \cap H) \geq n_i + r, \text{ for } i=1,2.$$

Let r_2 be the maximal integer such that there exists a plane H_2 containing L with

$$I(P_2, C \cap H_2) = n_2 + r_2.$$

Proposition 5.6.

The multiplicity of the curve $i^{-1}(V_4^1)$ at $P_1 + P_2$ is:

$$\min(n_1+n_2+r-2,\ 2n_2+r_2-1)$$

Proof:

Choose coordinates X_0, X_1, X_2, X_3 for P^3, and let t_i be a local parameter at P_i, for $i=1,2$. Without loss of generality we choose

$$X_0 = 1$$
$$X_1 = t_i + k_i$$
$$X_2 = \sum_{j \geq n_i} \alpha_{i,j} t_i^j$$
$$X_3 = \sum_{j \geq n_i+r} \beta_{i,j} t_i^j$$

as local parametrizations at P_i, for $i=1,2$.

By the definitions of n_1, n_2, and r, we may assume that α_{i,n_1} and α_{2,n_2} are non-zero, and that β_{1,n_1+r} or β_{2,n_2+r} is non-zero.

We see that the line $L = \overline{P_1P_2}$ has equations $X_2 = X_3 = 0$, and that $P_i = (1,k_i,0,0)$ for $i=1,2$, with $k_1 \neq k_2$.

The unique plane (if any) which intersects C a least $n_i + 1$ times at P_i, for $i=1,2$, has equation $X_3 = 0$. This is also the equation of H_2.

By Theorem 1.1., we have:

$$\hat{O}_{V_4^1,\,2P_1+2P_2} \cong K[[s_{1,1},\ s_{1,2},\ s_{2,1},\ s_{2,2}]]/(\det M).,$$

where

$$M = \begin{bmatrix} 1 & k_1 + s_{1,1} & \sum\limits_{j \geq n_1} \alpha_{1,j} W_j(s_{1,1}, s_{1,2}) & \sum\limits_{j \geq n_1+r} \beta_{1,j} W_j(s_{1,1}, s_{1,2}) \\ 0 & 1 & \sum\limits_{j \geq n_1} \alpha_{1,j} W_{j-1}(s_{1,1}, s_{1,2}) & \sum\limits_{j \geq n_1+r} \beta_{1,j} W_{j-1}(s_{1,1}, s_{1,2}) \\ 1 & k_2 + s_{2,1} & \sum\limits_{j \geq n_2} \alpha_{2,j} W_j(s_{2,1}, s_{2,2}) & \sum\limits_{j \geq n_2+r} \beta_{2,j} W_j(s_{2,1}, s_{2,2}) \\ 0 & 1 & \sum\limits_{j \geq n_2} \alpha_{2,j} W_{j-1}(s_{2,1}, s_{2,2}) & \sum\limits_{j \geq n_2+r} \beta_{2,j} W_{j-1}(s_{2,1}, s_{2,2}) \end{bmatrix}$$

The map $i: C^{(2)} \to C^{(4)}$, where $i(D) = 2D$ induces a map

$$i^*: \hat{O}_{C^{(4)}, 2P_1+2P_2} \to \hat{O}_{C^{(2)}, P_1+P_2}.$$

Now

$$\hat{O}_{C^{(4)}, 2P_1+2P_2} \simeq \hat{O}_{C^{(2)}, 2P_1} \underset{K}{\otimes} \hat{O}_{C^{(2)}, 2P_2}$$

$$\simeq K[[s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}]],$$

where the $s_{\ell,j}$ can be regarded as formal, algebraically independent, variables.

$s_{\ell,j}$ can also be regarded as the j'th elementary function in 2 replicas $t_{\ell,1}, t_{\ell,2}$ of the local parameter t_ℓ of C at P_ℓ, for $\ell=1,2$, $j=1,2$.

Furthermore:

$$\hat{O}_{C^{(2)}, P_1+P_2} \simeq \hat{O}_{C,P_1} \underset{K}{\otimes} \hat{O}_{C,P_2} \simeq K[[t_1, t_2]],$$

Hence we regard i^* as a map

$$i^*: K[[s_{1,1}, s_{1,2}, s_{2,1}, s_{2,2}]] \to K[[t_1, t_2]].$$

We have: $\hat{O}_{i^{-1}(V_4^1), P_1+P_2} \simeq K[[t_1, t_2]]/\det M(i^* s_{1,1}, \ldots, i^* s_{2,2})$.

Clearly $i^* s_{\ell,j} = s_{\ell,j}(t_\ell, t_\ell)$, $\ell=1,2$, $j=1,2$. From Remark 1.4. We then obtain:

$$i^*W_j(t_\ell) = W_j(i^*s_{\ell,1}, i^*s_{\ell,2}) = (j+1)t_\ell^j$$

This implies that

$$\hat{O}_{i^{-1}(V_4^1),\ P_1+P_2} \simeq K[[t_1,t_2]]/(R)$$

where R is the determinant of the matrix obtained from M by substituting $W_j(s_{\ell,1}, s_{\ell,2})$ by $(j+1)t_\ell^j$ for $\ell=1,2,\ j\geq 0$.

Calculation gives that the leading form of R is:

(5.6)

$$(k_1-k_2)[n_1(n_2+r)\alpha_{1,n_1}\beta_{2,n_2+r}t_2^r - n_2(n_1+r)\alpha_{2,n_2}\beta_{1,n_1+r}t_1^r]\cdot t_1^{n_1-1}\cdot t_2^{n_2-1}$$

or

$$(5.7)\qquad r_2\alpha_{2,n_2}\beta_{2,n_2+r_2}\cdot t_2^{2n_2+r_2-1}$$

or the sum of these forms.

One must check that neither of the forms vanishes identically as a polynomial in t_1, t_2, and that the forms do not cancel each other. Clearly (5.7) does not vanish. (5.7) cancels (5.6) only if $n_1=1$, but then $n_2=1$ also, and the forms have different degrees. Hence they do not cancel each other. For the form (5.6) we have 2 cases:

a) r=0. Then the form vanishes iff

$$\alpha_{1,n_1}\beta_{2,n_2} - \alpha_{2,n_2}\beta_{1,n_1} = 0$$

But the last expression is zero if and only if there is a plane H, with $I(P_i, C\cap H)\geq n_i+1$, for $i=1,2$. This would contradict the definition of r, so the form does not vanish.

b) r>0. The form does not vanish since

(i) $k_1 \neq k_2$ (ii) α_{1,n_1} and α_{2,n_2} are non-zero

(iii) β_{1,n_1+r} or β_{2,n_2+r} is non-zero.

Hence the multiplicity of $i^{-1}(V_4^1)$ at $P_1 + P_2$ is equal to the degree of the leading form of R:

$$\min(n_1+n_2-2+r,\ 2n_2+r_2-1)$$

This gives the proposition.

Corollary 5.7.

If a stationary secant $\overline{P_1P_2}$ is not a tangent to C at any of the points P_1, P_2, then the multiplicity of $i^{-1}(V_4^1)$ at $P_1 + P_2$ is

$$r = \min(I(P_1, C\cap H),\ I(P_2, C\cap H)) - 1$$

where H is the plane spanned by the tangent lines to C at P_1 and P_2.

Comment: Assume

a) No plane intersects C more than 4 times at any point.

b) C has no bitangents.

c) C has no flexes.

d) No plane is osculating at more than one point of C.

e) For each tangential trisecant line to C tangent to C at say P_1 and intersecting C transversally at say P_2, the osculating plane at P_1 does not contain the tangent to C at P_2.

Then it follows from Propositions 5.3. and 5.6. that the curve $i^{-1}(V_4^1)$ is non-singular.

A non-singular space curve has only finitely many tangential trisecants, flexes, bitangents, and hyperosculating or biosculating planes.

Hence it follows that the curve (scheme) $i^{-1}(V_4^1)$ is always reduced.

This curve might however be reducible. As an example of this, take C as the complete intersection of two quadric surfaces. Then C is contained in 4 quadric cones, and each generatrix of each such cone is a stationary bisecant line. Hence $i^{-1}(V_4^1)$ has (at least) 4 components in this case.

<u>A geometrical interpretation of the tangent cone $\mathcal{T}_D(i^{-1}(V_4^1)$</u>

In Definition 3.2. we described the (projectivized) tangent cone of a variety at a point. The tangent cone of the curve $i^{-1}(V_4^1)$ at a point D is determined by the leading form of the relation R as given in Formula (5.3) in the case D=2P, or as in Formula (5.6) and (5.7) where the leading form is given explicitly in the case $D = P_1 + P_2$, $P_1 \neq P_2$.

In both cases the tangent cone is determined by a homogeneous polynomial of degree m in 2 variables, where m is the multiplicity of $i^{-1}(V_4^1)$ at D. This polynomial splits into m linear factors. It turns out that in many cases each linear factor in the leading form corresponds to a point on the secant line L with a certain geometrical significance. Clearly each linear factor corresponds to a point of the projectivized tangent cone $P\mathcal{T}_D(i^{-1}(V_4^1))$. Hence we have an analogy to Result 4.2. in these cases. We would like to explain this more closely.

As usual we denote by $\ell(L)$ the point in the Grassmannian G=G(1,3) corresponding to a line L. Set

$B = \{\ell(L) | L$ satisfies a) or b) below$\}$

a) $L \cap C = \{P_1, P_2\}$, and L is not a tangent line to C.

b) $L \cap C = \{P\}$, and L is a tangent, but not a flex tangent line to C at P.

By the Trisecant lemma the closure $\bar{B}$ is a surface in G. It is a standard fact that $\bar{B}$ is locally isomorphic to $C^{(2)}$ at points of B under the map that sends the secant (tangent) line $\ell(L)$ to the divisor $P_1 + P_2 (2P)$. Moreover $\bar{B}$ is non-singular at points of B.

Let S be the subcurve of $\bar{B}$ corresponding to stationary bisecants in the sense described earlier. Then S is locally isomorphic to $i^{-1}(V_4^1)$ at points of $S \cap B$.

Consider the Plücker embedding $G \subseteq P^5$. It is a well-known fact; see for example [G-P], p. 16, that the points of $S \cap B$ are exactly those points of B such that the embedded tangent planes to $\bar{B}$ in P^5 are globally contained in G (in fact as β-planes). For a point $\ell(L)$ on $S \cap B$, this tangent plane is $\check{H}$, where H is the stationary plane in P^3 spanned by the divisor $2D$ on C.

This information implies that if not C is contained in a cone consisting of stationary bisecant lines, then the family of stationary bisecant lines envelope another curve $\mathcal{C}$ in P^3. The points of $\mathcal{C}$ are those where 2 concecutive stationary bisecants meet.

Considering the stationary bisecants as dual lines, the same family envelopes a curve Γ in $\check{P}^3$.

The following is easily verified.

1) C is on a cone consisting of stationary bisecant lines <=> A component of $\mathcal{C}$ degenerates to a point <=> A component of Γ is plane.

2) $\mathcal{C}$ and Γ are dual to each other, that is Γ parametrizes the osculating planes of $\mathcal{C}$, and vice versa.
3) Γ parametrizes the stationary bisecant planes of C.

Since $i^{-1}(V_4^1)$ is locally isomorphic to S at points of $S \cap B$, we can study the tangent cone to S at $\ell(L)$ instead of that of $i^{-1}(V_4^1)$ at D. Since the embedded tangent space of B at $\ell(L)$ is the dual plane $\check{H}$, we can embed $\mathcal{T}_{\ell(L)}(S)$ as a union of m lines in $\check{H}$ through the point $\ell(L)$. But a line in $\check{H} \subset G$ through $\ell(L)$ corresponds to a pencil of lines in $H \subset P^3$ through some point Q of L. Such a point Q of L corresponds to a point where L meets a concecutive stationary bisecant. Furthermore the points Q of L arising this way are exactly the points of $L \cap \mathcal{C}$ arising from the local branch(es) of S.

This means that the explicit calculations of the leading forms performed earlier in §5 tell us how the points of $L \cap \mathcal{C}$ are located in Cases a) and b).

Case a.

$L \cap C = \{P_1, P_2\}$, L is not a tangent line. Set $r = \min(I(P_1, C \cap H), I(P_2, C \cap H)-1)$ for the stationary plane H. By Formula (5.6) the leading form in t_1, t_2 is (up to a constant)

$$\alpha_{1,1}\beta_{2,r+1}t_2^r - \alpha_{2,1}\beta_{1,r+1}t_1^r$$

Hence the multiplicity m is r, and we get r distinct points of $L \cap \mathcal{C}$ outside C unless either $\beta_{1,r+1}$ or $\beta_{2,r+1}$ is zero. If, say, $\beta_{1,r+1} = 0$, which means $I(P_1, C \cap H) \geqslant r+1$, then all r points of $L \cap \mathcal{C}$ collapse to one point. It turns out that this single point is P_2. See Result 5.8. below, or Remark 5.9.

Case b. $L \cap C = \{P\}$, L is tangent to C at P, but P is not a flex. We recall the definition $m_3 = I(P, C \cap H)$, where H is the osculating (stationary) plane of C at P.

We recall that the leading form in S_1, S_2 is $S_2^{\frac{m_3-3}{2}}$ when m_3 is odd and $(S_1+kS_2)S^{\frac{m_3}{2}-2}$ when m_3 is even.

It turns out that the factor S_2 corresponds to the (secant) point P of $C \cap L$, while the factor $S_1 + kS_2$ corresponds to a point outside P. "In general", when $m_3 = 4$, we get only the last factor.

In cases a. and b. we have another description of the points of $L \cap \mathcal{C}$ arising from the local branch(es) of S. Denote by m the multiplicity of S at $\ell(L)$.

Result 5.8.

$Q \in L$ is a point of $\mathcal{C}$ iff there exists a cone N of degree $m+1$ with vertex at Q such that $\mathrm{Sing}(N) \not\ni L$ and such that

Case a. $I(P_i, C \cap N) \geqslant m+2$, for $i=1,2$

Case b. $I(P, C \cap N) \geqslant 2m + 4$.

Idea of proof: Let F be the surface in P^3 swept out by the stationary bisecant lines. Let C' be a dummy curve on F transversal to the ruling around L. Regard L as a singular tri-secant to $C \cup C'$. The point $\ell(L)$ is contained in a non-reduced component of the trisecant curve in G. Then apply Result 4.3. in the case $n=3$.

Remark 5.9. Recall the local parametrizations of C introduced in the proof of Propositon 5.6. Referring to these parametrizations, Result 5.8 translates in Case a) to:

$Q = (1,k,0,0)$ is a point on $L\cap$ iff

$$\left(\frac{k_2 - k}{k_1 - k}\right)^r = \frac{\beta_{1,r+1}}{\beta_{2,r+1}} \cdot \frac{\alpha_{2,1}^{r+1}}{\alpha_{1,1}^{r+1}}.$$

A similar result can be obtained in Case b).

We might return to a more detailed study of the curves $\mathcal{C}$, S, [in another paper. With the information we have now it is easy to compute the "expected" genera, degrees, and numbers of cusps of these curves.

§6. Singularities of plane curves.

Assume rk V = 3, and that V is base point free. Thus V defines a map

$$\phi: C \to \bar{C} \subseteq P^2$$

We can "measure" the singularities of $\bar{C}$ by studying the scheme V_2^1 in $C^{(2)}$. This scheme may consist of 2 kinds of points:

1) Divisors $P_1 + P_2$, where $P_1 \neq P_2$
2) Divisors $2P$.

The first ones correspond to nodes of C; the latter ones to cusps. If V_2^1 is finite, it is well known that its total length is

$$\tfrac{1}{2}(d-1)(d-2) - g,$$

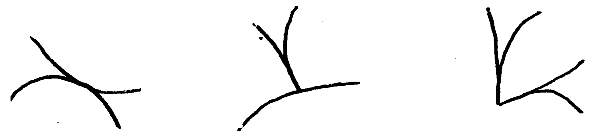
where $d = \deg \bar{C} = \deg L$, and $g = \text{genus}(C)$.

Tangent space dimensions of V_2^1.

Assume $D = P_1 + P_2 \in V_2^1$, where $P_1 \neq P_2$. We see from Theorem 2.4. that the tangent space dimension of V_2^1 at D is:

A least 1 iff the two branches of $\bar{C}$ at $\phi(P_1) = \phi(P_2)$ have a common tangent line <u>or</u> $2P_i \in V_2^1$ for i=1 or 2.

2 iff both $2P_1$ and $2P_2$ are contained in V_2^1.

$\bar{C}$: 

Tangent space dimension : 1 1 2.

Assume $D = 2P \in V_2^1$. We see from Theorem 2.4. that the tangent space dimension of V_2^1 at D is:

At least 1 iff the unique tangent line L of $\bar{C}$ at $\phi(P)$ intersects the branch of $\bar{C}$ at least 4 times at $\phi(P)$.

2 iff $4P \in V_4^3$; which means that the multiplicity of the branch of $\bar{C}$ at $\phi(P)$ is at least 4.

$\bar{C}$: 

Tangent space dimension : 1 1 2

<u>The multiplicity of V_2^1 at $D=2P$.</u>

From now on we will concentrate on divisors of the type $2P$. We will not prove any thing essentially new, but we will show how our set-up fits in well with traditional results.

Detnote by $\mathrm{Mult}_D V_2^1$ the multiplicity or local length of V_2^1 at a divisor D. Clearly the δ-invariant of $\bar{C}$ at $Q \in P^2$ is $\sum \mathrm{Mult}_D V_2^1$, where the sum is taken over those divors $P_1 + P_2$ and $2P$ such that $\phi(P_1) = \phi(P_2) = Q$ and $\phi(P) = Q$. We will show how to find the $\mathrm{Mult}_{2P} V_2^1$, when $\mathrm{Char}K = 0$, and $K = \bar{K}$.

Choose

$$X_r = \sum_{j=0}^{\infty} \alpha_{r,j} t^j \quad , \text{ for } r=0,1,2$$

as local parametrizations at P of the sections spanning V. We may assume $X_0 \equiv 1$. The matrix M from Formula (1.1) is:

$$\begin{bmatrix} 1 & \sum_{j\geq 0} \alpha_{1,j} W_j(s_1,s_2) & \sum_{j\geq 0} \alpha_{2,j} W_j(s_1,s_2) \\ 0 & \sum_{j\geq 1} \alpha_{1,j} W_{j-1}(s_1,s_2) & \sum_{j\geq 1} \alpha_{2,j} W_{j-1}(s_1,s_2) \end{bmatrix}$$

We assume $P = (1,0,0)$ and obtain

$$\hat{O}_{V_2^1, 2P} = K[[s_1,s_2]]/I$$

where

$$I = \left(\sum_{j\geq 2} \alpha_{1,j} W_{j-1}(s_1,s_2), \sum_{j\geq 2} \alpha_{2,j} W_{j-1}(s_1,s_2) \right)$$

We have used that $\alpha_{1,1} = \alpha_{2,1} = 0$ by assumption. When Char $K = 0$, it is a standard fact that we may simplify our local parametrizations:

$$X_0 = 1, \; X_1 = t^n, \; X_2 = \sum_{j\geq n+1} \alpha_{2,j} t^j,$$

where $n\geq 2$ is the multiplicity at $\phi(P)$ of the branch of $\bar{C}$ in question. The ideal I reduces to

$$\left(W_{n-1}(s_1,s_2), \sum_{j\geq n+1} \alpha_{2,j} W_{j-1}(s_1,s_2)\right).$$

We see that $\mathrm{Mult}_{2P} V_2^1$ = colength I is equal to the intersection number of 2 algebroid curves at the origin in the s_1, s_2-plane. We will compute this number (Result 6.1.).

Considering s_1, s_2 as elementary symmetric functions in two formal replicas t_1, t_2, we have by Remark 1.4:

$$W_{n-1}(s_1(t_1, t_2), s_2(t_1, t_2)) = \prod_{r=1}^{n-1} (t_1 - \varepsilon_{n,r} t_2)$$

where $\varepsilon_{n,r} = e^{\frac{2\pi r i}{n}}$.

By standard arithmetic this gives:

(6.1)

$$W_{n-1}(s_1,s_2) = \prod_{r=1}^{\frac{n-1}{2}} (s_1^2 - k_{n,r} s_2), \text{ when } n \text{ is odd}$$

$$W_{n-1}(s_1,s_2) = s_1 \cdot \prod_{r=1}^{\frac{n-2}{2}} (s_1^2 - k_{n,r} s_2), \text{ when } n \text{ is even}$$

where $k_{n,r} = 2 + \varepsilon_{n,r} + \varepsilon_{n,r}^{-1}$

In any case $\mathrm{Mult}_{2P}V_2^1$ is the sum of the intersection numbers obained by intersecting the algebroid curve with equation

$$\sum_{j\geq n+1} \alpha_{2,j} W_{j-1}(s_1,s_2) \qquad \text{with each of the}$$

curves corresponding to the factors of $W_{n-1}(s_1,s_2)$ (at the origin).

Formula (6.1) implies:

a) $W_{j-1}(s_1, \frac{s_1^2}{k_{n,r}}) \equiv 0$ iff

$\varepsilon_{n,r}$ is a primitive m'th root of unity for an m dividing j

b) $W_{j-1}(0,s_2) \equiv 0$ iff j is even.

For each $m\geq 2$ we define

$$B_m = \min\{\ell \mid m \text{ does not divide } \ell, \text{ and } \alpha_{2,\ell}\neq 0\}.$$

$$r_m = \#\{\text{primitive m'th roots of unity}\},$$

or recursively: $r_m = m-1 - \sum r_{m_i}$, where the sum is taken over all m_i that divide m, except 1 and m. We then obtain:

<u>Result 6.1</u>.

$$\mathrm{Mult}_{2P}V_2^1 = \sum_{i=1}^{s} \frac{r_{m_i}(B_{m_i}-1)}{2}$$

where $m_1,\ldots,m_s$ are the positive integers (except 1) dividing n.

§7 <u>A note on Weierstrass points</u>.

Let V be a linear system of rank $r+1$ and degree d on a curve C. We will use Theorem 1.1. to prove a well-known formula for the weight (multiplicity) of a rank $\ell+1$ Wronskian point of V, $0\leq\ell\leq r$. A rank $r+1$ Wronskian point is a Weierstrass point.

First we will define our terms, without making any assumptions on the characteristic of K.

Consider the map:

$$\phi_\ell : C \to C^{(\ell+1)}$$

where $\phi_\ell(P) = (\ell+1)P$, for $P \in C$.

Definition 7.1.

a) We say that V is classical if $\phi_\ell^{-1}(V^1_{\ell+1})$ is a finite set for $0 \leq \ell \leq r$.

b) Assume V is classical. We define the (finite) rk $\ell+1$ Wronskian scheme of V as

$$T_\ell = \phi_\ell^{-1}(V^1_{\ell+1}).$$

c) We define the (finite) Weierstraß scheme of V as T_r. The points of T_r are denoted by Weierstrass points of V.

Let P be an arbitrary point of C, and let t be a local parameter of C at P. Then there are uniquely determined integers (not depending on the choice of t) $h_0 < h_1 < \cdots < h_r$ such that there are sections $X_0,\ldots,X_r$ spanning V with local parametrizations

$$X_0 = \sum_{j \geq h_0} \alpha_{0,j} t^j, \quad \ldots, \quad X_r = \sum_{j \geq h_r} \alpha_{r,j} t^j,$$

with $\alpha_{i,h_i} \neq 0$, for $i=0,\ldots,r$. The integers $h_o,\ldots,h_r$ are called the Hermite invariants of V at P. If V is classical, then $h_i = i$, for $i=0,\ldots,r$, for all but a finite set of points on C.

We now give our result:

Proposition 7.2.

Assume $\mathrm{char} K = 0$ or $\mathrm{char} K \geq \ell+1$, and that V is classical. Then the multiplicity of T_ℓ at P is $\sum_{i=0}^{\ell} (h_i - i) = \sum_{i=0}^{\ell} h_i - \frac{\ell(\ell+1)}{2}$.

Comment: This is essentially Theorem 15, ii of [L2]. In [L2], Theorem 15, i, one proves that if $\mathrm{Char}K = 0$, or $\mathrm{char}K \geq d+1$, then V is classical.

Proof: By Theorem 1.1. we have

$$\hat{O}_{V^1_{\ell+1},(\ell+1)P} \approx K[[s_1,\dots,s_{\ell+1}]]/J$$

where J is generated by the $\ell+1$-minors of

$$M = \begin{bmatrix} \sum_{j\geq h_0} \alpha_{0,j}W_j(\underline{s}) & \cdots\cdots & \sum_{j\geq h_r} \alpha_{r,j}W_j(s) \\ \vdots & & \vdots \\ \sum_{j\geq h_0} \alpha_{0,j}W_{j-\ell}(\underline{s}) & \cdots\cdots & \sum_{j\geq h_r} \alpha_{r,j}W_{j-\ell}(s) \end{bmatrix}$$

The map $\phi_\ell : C \to C^{(\ell+1)}$ induces a map

$$\phi_\ell^*: \hat{O}_{C^{(\ell+1)},(\ell+1)P} \approx K[[s_1,\dots,s_{\ell+1}]] \to \hat{O}_{C,P} \simeq K[[t]]$$

such that for $k=1,\dots,\ell+1$, we have:

$\phi_\ell^*(s_k) = s_k(t,\dots,t)$ where s_k is the k'th elementary symmetric function in $\ell+1$ variables. From Remark 1.4. we have

$$\phi_\ell^*(W_j(\underline{s})) = W_j(\phi_\ell^* s_1,\dots,\phi_\ell^* s_{\ell+1}) = \binom{j+\ell}{\ell} \cdot t^j \quad , \text{ for all } j.$$

This implies that

$$\hat{O}_{T_\ell,P} \simeq K[[t]]/\phi_\ell^*(J)$$

where $\phi_\ell^*(J)$ is generated by the $\ell+1$-minors of the matrix:

$$(7.1) \qquad \begin{bmatrix} \sum_{j\geq h_0} \alpha_{0,j}\binom{j+\ell}{\ell}t^j & \cdots\cdots & \sum_{j\geq h_r} \alpha_{r,j}\binom{j+\ell}{\ell}t^j \\ \vdots & & \vdots \\ \sum_{j\geq h_0} \alpha_{0,j}\binom{j}{\ell}t^{j-\ell} & \cdots\cdots & \sum_{j\geq h_r} \alpha_{r,j}\binom{j}{\ell}t^{j-\ell} \end{bmatrix}$$

The multiplicity of T_ℓ at P is the lowest number m such that there is a term t^m in one of the minors generating $\phi_\ell^*(J)$.

The $\ell+1$-minor consisting of the $\ell+1$ first columns of (7.1) can be written as

$$\sum_{j\geq m} c_j t^j,$$

where $m = h_0 + (h_1-1) + \cdots + (h_\ell-\ell) = \sum_{i=0}^{\ell}(h_i-i)$. Clearly no terms t^n, with $n<m$, is contained in any of the generators of $\phi_\ell^* J$. Hence we have proved the proposition if we can show that c_m is non-zero. We have

$$c_m = \alpha_{0,h_0}\cdot\ldots\cdot\alpha_{\ell,h_\ell}\cdot\begin{vmatrix}\binom{h_0+\ell}{\ell} & \cdots\cdots & \binom{h_\ell+\ell}{\ell}\\ \binom{h_0-1+\ell}{\ell} & & \binom{h_\ell-1+\ell}{\ell}\\ \vdots & & \vdots\\ \binom{h_0}{\ell} & \cdots\cdots & \binom{h_\ell}{\ell}\end{vmatrix}$$

The proposition follows from the following lemma:

<u>Lemma 7.3.</u>

$$\begin{vmatrix}\binom{h_0+\ell}{\ell} & \cdots\cdots & \binom{h_\ell+\ell}{\ell}\\ \vdots & & \vdots\\ \binom{h_0}{\ell} & \cdots\cdots & \binom{h_\ell}{\ell}\end{vmatrix} = \begin{vmatrix}\binom{h_0}{0} & \cdots\cdots & \binom{h_\ell}{0}\\ \binom{h_0}{1} & \cdots\cdots & \binom{h_\ell}{1}\\ \vdots & & \vdots\\ \binom{h_0}{\ell} & \cdots\cdots & \binom{h_\ell}{\ell}\end{vmatrix}$$

<u>Comment:</u> In [L1], Lemma 9, one shows that the determinant to the right is $\prod_{0\leq j<i\leq\ell}(h_i-h_j)\cdot\frac{1}{\prod_{i=1}^{\ell} i!}$, which is non-zero.

<u>Proof of Lemma 7.3.:</u>

In the first row set

$$\binom{h_j+\ell}{\ell} = \binom{h_j+\ell-1}{\ell-1} + \binom{h_j+\ell-1}{\ell} \qquad \text{for } 0 \leq j \leq \ell.$$

Since the entries in the second row are $\binom{h_j+\ell-1}{\ell}$, these terms can be deleted in the first row. In this way the entries in row nr. k+1 can be changed from $\binom{h_j+\ell-k}{\ell}$ to $\binom{h_j+\ell-k-1}{\ell-1}$ for $k=0,\ldots,\ell-1$ and $j=0,\ldots,\ell$.

Then start at the top again, and treat all but the 2 last rows the same way once more.

When the top row has been treated this way ℓ times, we end up with the desired determinant.

This completes the proof of Lemma 7.3. and also of Proposition 7.2.

Corollary 7.4.

Assume char K = 0, or charK $\geq$ r+1. Then the multiplicity of P as a Weierstraß point is

$$\sum_{i=0}^{r} (h_i - i)$$

Remark: It is a well known fact that the total length of T_r, that is the sum of the multiplicites of the Weierstraß points, is:

$$((g-1)r+d)(r+1), \text{ where } g \text{ is the genus of } C.$$

This follows from [A-C-G-H], p. 345 and p. 358, when $K = \mathbf{C}$.

Non-classical linear systems.

What happens if we impose no restrictions on char K? This question has been answered in a very satisfactory way in [L2], and we would be happy to reproduce some of the results in [L2] using our set-up. It seems however that our methods are to crude when $0 < \text{charK} < d$. Still we will add a few words about this case.

Let $h_0,\dots,h_r$ be the Hermite invariants of V at a point P of C. On an open set of C the Hermite invariants are constant with values $b_0,\dots,b_r$. When char $K=0$ or char$K > d$, we have $b_i=i$, for $i=0,\dots,r$. When $2 \leq$ char$K \leq d$, we have $i\leq b_i<b_{i+1}$ for $0\leq i\leq r-1$, and b_i might or might not be equal to i for all i. In this case we have:

$$\phi_{b_\ell}^{-1}(V_{b_\ell+1}^{b_\ell-\ell+1}) \text{ is a finite set, for } 0\leq \ell\leq r,$$

in analogy with Definition 7.1.a.

$\phi_{b_\ell}^{-1}(V_{b_\ell+1}^{b_\ell-\ell+1})$ is also defined as a finite scheme, which we denote by T_ℓ, and

$$R = \hat{O}_{T_\ell,P} \simeq K[[t]]/\phi_{b_\ell}^{*}(J)$$

where $\phi_{b_\ell}^{*}(J)$ is generated by the $\ell+1$ minors of the following $(b_\ell+1)\times(r+1)$ matrix:

$$\begin{bmatrix} \sum_{j\geq h_0} \alpha_{0,j}\binom{j+b_\ell}{\ell}t^j & \cdots\cdots\cdots & \sum_{j\geq h_r} \alpha_{r,j}\binom{j+b_\ell}{\ell}t^j \\ \vdots & & \vdots \\ \sum_{j\geq h_0} \alpha_{0,j}\binom{j}{b_\ell}t^{j-b_\ell} & \cdots\cdots\cdots & \sum_{j\geq h_r} \alpha_{r,j}\binom{j}{b_\ell}t^{j-b_\ell} \end{bmatrix}$$

(As usual $\binom{a}{b}=0$ if $b>a$).

The multiplicity of T_ℓ at P is the length of the ring R. One sees that $P\in T_\ell$ iff $\alpha_{\ell,j}=0$ for $j<b_\ell$, that is iff $h_s\geq b_s+1$.

Set-theoretically we have: P is a rank $\ell+1$ Wronskian point in the sense of [L2] iff $P \in \bigcup_{k=0}^{\ell} T_k$. In [L2], Example 1, p.64, one shows that it is possible that $P\notin T_\ell$, but $P\in T_k$ for some $k<\ell$. Hence the multiplicity we have described for a point of T_ℓ is different from the multiplicity described in [L2] for a rank $\ell+1$ Wronskian point.

Both multiplicities are however well defined.

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