Some qualitative properties of 2×2 systems of conservation laws of mixed type.[1]

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Abstract. We study qualitative features of the initial value problem $z_t + F(z)_x = 0, z(x,0) = z_0, x \in \mathbf{R}$, where $z(x,t) \in \mathbf{R}^2$, with Riemann inital data, viz. $z_0(x) = z_l$ if $x < 0$ and $z_0(x) = z_r$ if $x > 0$. In particular we are interested in the case when the system changes type. It is proved that if both z_l and z_r are in the hyperbolic region, then the solution will not enter the elliptic region. If z_l and z_r are in the elliptic region, and the elliptic region is convex, then part of the solution has to be outside the elliptic region.

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1. Introduction. In this note we analyze certain qualitative properties of the 2×2 system of partial differential equations in one dimension on the form

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = 0 \tag{1.1}$$

with $u = u(x,t)$, $v = v(x,t)$, $x \in \mathbf{R}$. In pariticular we are interested in the initial value problem with initial data, i.e.

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{cases} \binom{u_l}{v_l}, & \text{for } x < 0 \\ \binom{u_r}{v_r}, & \text{for } x > 0 \end{cases} \tag{1.2}$$

where u_l, u_r, v_l, v_r are constants.

The system (1.1),(1.2) arises as a model for a diverse range of physical phenomena from traffic flow [**2**] to three–phase flow in porous media [**18**]. Common for these applications is that one obtains from very general assumptions from a physical point of view a system of mixed type, i.e. there is a region $E \subset \mathbf{R}^2$ of phase space where the 2×2 matrix

$$dF = \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix} \tag{1.3}$$

has no real eigenvalues. The system is then called elliptic in E.

Consider e.g. the case of three–phase flow in porous media where the unknown functions u and v denote saturations, i.e. relative volume fractions of two of the phases, e.g. oil and water respectivly. A recent numerical study [**1**] gave as a result with realistic physical data that there in fact is a small compact region E in phase space, quite surprisingly the Riemann problem (1.1),(1.2) turned out to be rather well–behaved numerically in this situation.

Subsequent mathematical analysis [**17**],[**7**],[**10**],[**11**] showed that one in general has to expect mixed type behaviour in this case. Also in applications to elastic bars and Van der Waal fluids [**8**],[**19**],[**13**],[**14**],[**15**],[**9**] there is mixed type behaviour.

Parallel to this development there has been a detailed study of certain model problems with very simple flux functions (f,g) with elliptic behaviour in a compact region E which has revealed a very complicated structure of the solution to the Riemann problem [**5**],[**6**]. In general one must expect nonuniqueness of the solution for Riemann problems, see [**3**].

We prove two theorems. In the first theorem we prove that the Hugoniot locus of a point in a convex elliptic region E does not intersect that component of E. In the second theorem we prove that if the initial data is outside E, then the solution will remain outside E.

2. Qualitative properties. We write (1.1) as

$$z_t + F(z)_x = 0 \tag{2.1}$$

where $z = \binom{u}{v}$ and $F = \binom{f}{g}$, with Riemann initial data

$$z(x,0) = \begin{cases} z_l, & \text{for } x < 0 \\ z_r, & \text{for } x > 0. \end{cases} \tag{2.2}$$

We assume that f and g are real differentiable functions such that the Jacobian dF has real eigenvelues exept in components of $\mathbf{R}^2$, each of which are convex. Let

$$E = \left\{ z \in \mathbf{R}^2 | \lambda_j(z) \notin \mathbf{R} \right\}. \tag{2.3}$$

A shock solution is a solution of the form

$$z(x,t) = \begin{cases} z_l, & \text{for } x < st \\ z_r, & \text{for } x > st. \end{cases} \tag{2.4}$$

where the shock speed s must satisfy the Rankine–Hugoniot relation

$$s(z_l - z_r) = F(z_l) - F(z_r). \tag{2.5}$$

The Hugoniot locus of z_l is the set of points satisfying

$$H_{z_l} = \left\{ z \in \mathbf{R}^2 | \exists s \in \mathbf{R}, s(z_l - z) = F(z_l) - F(z) \right\}. \tag{2.6}$$

The other basic ingredient in the solution of the Riemann problem is rarefaction waves. These are smooth solutions of the form $z = z(s/t)$ that satisfy (2.1). The value $z(\xi)$ must be an integral curve of r_j, $j = 1,2$ where r_j is a right eigenvector of dF corresponding to λ_j. ξ is the speed of the wave; $\xi = \lambda_j(z(x/t))$, therefore λ_j has to increase with ξ as z moves from left to right in the solution of the Riemann problem. Note that no rarefaction wave can intersect E since the eigenvectors are not defined there.

For a system of non–strictly hyperbolic conservation laws, the Riemann problem does not in general posess a unique solution, and by making the entropy condition sufficiently lax in order to obtain existence of a solution, one risks losing uniqueness. However it is believed that the correct entropy condition which singles out the correct physical solution is that the solution should be the limit as $\epsilon \to 0$ of the solution of the associated parabolic equation

$$z_t^\epsilon + F(z^\epsilon)_x = \epsilon z_{xx}^\epsilon \qquad \epsilon > 0. \tag{2.7}$$

We then say that the shock has a viscous profile, see however [4]. Let now z_l, z_r be two states that can be connected with a shock of speed s. We seek solutions of the form

$$z^\epsilon = z^\epsilon\left(\frac{x - st}{\epsilon}\right) = z^\epsilon(\xi) \tag{2.8}$$

and then obtain

$$-s\frac{d}{d\xi}z^\epsilon + \frac{d}{d\xi}F(z^\epsilon) = \frac{d^2}{d\xi^2}z^\epsilon \tag{2.9}$$

which can be integrated to give

(2.10) $$\frac{d}{d\xi}z^\epsilon = F(z^\epsilon) - sz^\epsilon + A$$

where A is a constant of integration. If $z^\epsilon(\xi)$ converges to the correct solution we must have

(2.11) $$\lim_{\xi\to-\infty} z^\epsilon(\xi) = z_l \qquad \lim_{\xi\to\infty} z^\epsilon(\xi) = z_r$$

(provided the derivatives converge sufficiently fast) which implies

(2.12) $$\frac{d}{d\xi}z^\epsilon = \big(F(z^\epsilon) - F(z_l)\big) - s(z^\epsilon - z_l) = \gamma(z^\epsilon).$$

We see that z_l and z_r are fixpoints for this field, and if it admits an orbit *from* z_l *to* z_r we say that the shock has a viscous profile. The associated eigenvalues of this field are

(2.13) $$\lambda_j(z) - s \qquad j = 1, 2.$$

We can now classify the various possibilities for a shock,as in Table 1, according to what kind of fixpoints z_l and z_r are.

If z_l is a sink or z_r is a source we cannot have any orbit from z_l to z_r, which leaves only four possibilities. Assume $z \in E$ and that E has convex components, let E_z denote the component of E that contains z. Then we have

THEOREM 1. *If $z_l \in E$, then*

(2.14) $$H_{z_l} \cap E_{z_l} = \{z_l\}$$

and if $z_r \in E$, then

(2.15) $$H_{z_r} \cap E_{z_r} = \{z_r\}.$$

PROOF: We will show (2.14), (2.15) will follow by symmetry. Let $z_r \in E_{z_l}$, we will show that $s(z_r - z_l) = F(z_r) - F(z_l)$ cannot be satisfied. Let

(2.16) $$\alpha(t) = tz_r + (1-t)z_l \quad t \in (0,1)$$

be the straight line between z_r and z_l which is contained in E_{z_l} by convexity. Let

(2.17) $$\beta(t) = F\big(\alpha(t)\big).$$

By the mean value theorem we must have a $\tilde{t} \in (0,1)$ such that

(2.18) $$\gamma'(\tilde{t}) = k\big(\gamma(1) - \gamma(0)\big) = k\big(F(z_r) - F(z_l)\big) = \tilde{k}(z_r - z_l),$$

but

(2.19) $$\gamma'(t) = dF\big(\alpha(t)\big)\alpha'(t) = dF\big(\alpha(t)\big)(z_r - z_l).$$

Thus dF has a real eigenvalue at the point $\alpha(\tilde{t})$, and therefore this point cannot be in E. Therefore the Hugoniot relation cannot be satisfied for the pair z_l, z_r. ∎

This implies that if $z_l \in E$ and $\{z_l, z_r\}$ are the initial values of a Riemann problem, then, the state immediatly adjacent to z_l (z_r) in the solution will be outside of E_{z_l} (E_{z_r}). This is so since this state must either be a point on a rarefaction or a shock. Rarefaction curves do not enter E, and we have just shown that neither does the Hugoniot locus.

THEOREM 2. *Consider a solution $z = z(x,t)$ of (2.1) with Riemann initial data (2.2).*

(1) *If $z_l, z_r \notin E$, then also $z(x,t) \notin E$ for all x and t.*
(2) *If $z_l \in E$ or $z_r \in E$ and $z(\tilde{t},\tilde{x}) \in E$ for some $(\tilde{t},\tilde{x})$ then $z(\tilde{t},\tilde{x}) \in \{z_l, z_r\}$.*

PROOF: (1) Assume $\tilde{z} = z(\tilde{t},\tilde{x}) \in E$ for some $(\tilde{t},\tilde{x})$. Then $\tilde{z}$ is the right (left) state of an admissible shock with speed $\tilde{s}_l$ ($\tilde{s}_r$) and left (right) state $\tilde{z}_l$ ($\tilde{z}_r$). In E the eigenvalues constitute a pair of complex conjugates and $\tilde{z}$ is a source (sink) if $\mathrm{Re}(\lambda_j(\tilde{z})) - \tilde{s}_l > 0$ ($\mathrm{Re}(\lambda_j(\tilde{z})) - \tilde{s}_r < 0$). Hence we obtain

$$\tilde{s}_l \geq \lambda_j(\tilde{z}) \geq \tilde{s}_r \tag{2.20}$$

which contradicts the fact that $\tilde{z}_r$ is to the right of $\tilde{z}_l$. (2) is similar to (1). ∎

These two theorems state that if the initial values in a Riemann problem is inside a component of a convex elliptic region, then the solution will contain values outside this region if the entropy condition is based on the "vanishing viscosity" approach. Furthermore if the initial values are outside the elliptic region, then the solution will not enter this region. If one then has an initial function with sufficiently small total variation taking values outside E, then the function F can be redefined in E and the random choice method can be used to prove existence of a weak solution as in the purely hyperbolic case. This also implies that initial values in E will disappear after some time when using numerical schemes that are based on solving Riemann problems, such as random choice methods or front tracking methods. It is the authors belief that, if a solution exists, this is also the case for this solution.

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z_l \ z_r	source	saddle	sink
source	right transport	Lax-1	compressive
saddle	1-expansive	crossing	Lax-2
sink	expansive	2-expansive	left transport

Table 1.