A WEIGHTED SOBOLEV INEQUALITY AND HARMONIC MEASURE ASSOCIATED TO QUASIREGULAR FUNCTIONS

Bernt Øksendal

Abstract

A weighted Sobolev inequality in $\mathbb{R}^n$ of the form

$$\int_U |u(x)|^2 \rho(x)dx \leq C \cdot \int_U |\nabla u(x)|^2 \rho(x)dx, \qquad u \in C_0^\infty(U)$$

is established, in the case when $\rho = J_\phi^{1-\frac{2}{n}}$, J_ϕ being the Jacobian determinant of a quasiregular function ϕ on a bounded domain $U \subset \mathbb{R}^n$. This gives the existence in general of the harmonic measure of the diffusion X_t associated to ϕ. As an application a new result about boundary values of quasiregular functions is proved.

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§1. Introduction.

Let U be a bounded domain in $\mathbb{R}^n$, $n>2$ and let $\phi:U\to\mathbb{R}^n$ be a non-constant quasiregular function. (For definition and basic properties see e.g. [6] and [10]). It was proved in [9] that there exists a diffusion X_t in U with law P^x and life time $\zeta<\infty$ such that if B_t is a Brownian motion in $\mathbb{R}^n$ then the process

$$M_t=\begin{cases}\phi(X_t); & t<\zeta\\ B_{t-\zeta}\ ; & t>\zeta\end{cases} \tag{1.1}$$

(with the natural probability law) is also a Brownian motion. For details and applications see [9]. Of course, for the applications it is crucial to have information about the behaviour of X_t. It is easy to see that X_t must approach ∂U as $t\to\zeta$, in the sense that $X_t(\omega)$ leaves every compact $K\subset U$ for good eventually, for a.a.ω. However, an important problem left open in [9] was the existence in general of the limit

$$X_\zeta \overset{\text{def}}{=} \lim_{t\to\zeta} X_t \quad \text{a.s. } P^x \quad \text{for} \quad \text{q.a.} \quad x\in U \tag{1.2}$$

if we only assume that

$$\zeta<\infty \quad \text{a.s. } P^x \quad \text{for} \quad \text{q.a.} \quad x\in U, \tag{1.3}$$

where "q.a." denotes quasi-all with respect to the capacity associated to X_t (see [4], §3.1).

Property (1.3) holds for example if the exit time $\tau_{\phi(U)}$ from $\phi(U)$ for Brownian motion in $\mathbb{R}^n$ is finite a.s. (see (3.1) in [9]). In [9, Lemma 3.3] the existence of X_ζ is proved under additional assumptions on ϕ and U. The purpose of this paper is to prove that

X_ζ exists without any other conditions on ϕ and U than (1.3). (Theorem 3.2). A main inqredient of the proof is a weighted Sobolev inequality of independent interest (Theorem 2.2).

The existence of X_ζ enables us to define the harmonic measure $\lambda = \lambda^X_{a,U}$ associated to the process X_t (i.e. to the function ϕ) with respect to a point $a \in U$ by

(1.4) $$\lambda^X_{a,U}(H) = P^a[X_\zeta \in H], \quad \text{for } H \subset \partial U,$$

where P^a is the probability law of X_t starting at a.

The general existence of X_ζ is important for example in the study of boundary values of ϕ. To illustrate this we give an application regarding asymptotic values (Theorem 4.3), which partially extends a result of Martio and Rickman [7, Theorem 5.11].

The method of proof is an interplay between stochastic arguments and fundamental results about degenerate elliptic equations, quasiregular functions and A_p-weights. I believe that this interplay can be very fruitful and it deserves to be investigated further.

§2. A weighted Sobolev inequality.

The process X_t is constructed as the Hunt process associated to the following (densely defined) Dirichlet form on $H = L^2(U; J_\phi dx)$

(2.1) $$\mathcal{E}_\phi(u,v) = \int (\nabla u)^T \cdot F \cdot \nabla v \, dx \; ; \quad u,v \in C_0^\infty(U) \subset L^2(U; J_\phi dx)$$

Here dx denotes Lebesgue measure in R^n, $C_0^\infty(U)$ denotes the set of infinitely differensiable functions with compact support in U, $(\cdot)^T$ denotes (matrix) transposed and F is the $n \times n$ matrix

(2.2) $$F = \frac{1}{2} J_\phi \cdot (\phi')^{-1} \cdot ((\phi')^{-1})^T,$$

where $\phi' = [\frac{\partial \phi_i}{\partial x_j}]_{i,j}$ is the derivative (matrix) of ϕ and

(2.3) $\quad J_\phi = \det \ \phi'$

is the Jacobian of ϕ.

It is known that $J_\phi > 0$ a.e. (with respect to Lebesgue measure dx in $\mathbb{R}^n$), so $(\phi')^{-1}$ exists a.e.

One way of expressing that X_t is associated to $\mathcal{E}_\phi$ is to say that the generator A of X_t is related to $\mathcal{E}_\phi$ by

(2.4) $\quad \mathcal{E}_\phi(u,v) = -(Au,V)_H \ ; \quad u \in D(A),\ v \in \mathfrak{D}(\mathcal{E}_\phi)$

where $(\cdot,\cdot)_H$ denotes inner product in H and $\mathfrak{D}(A), \mathfrak{D}(\mathcal{E}_\phi)$ are the domains of definition of A, $\mathcal{E}_\phi$, respectively. See [4] for more information about Dirichlet forms and associated processes.

Before we proceed let us explain the main idea of our approach: From now on we assume that (1.3) holds. Suppose $X_\zeta = \lim_{t \to \zeta} X_t$ exists. Then we can solve in a stochastic sense the Dirichlet problem

(2.5) $$\begin{aligned} Au &= 0 \quad \text{in} \quad U \\ u &= f \quad \text{on} \quad \partial U \end{aligned}$$

where $f \in C(\partial U)$ is a given bounded function. The solution is given by

(2.6) $\quad u(x) = E^x[f(X_\zeta)], \quad x \in U$

where E^x denotes expectation wrt. P^x. More precisely, this function u solves the problem in the following sense:

(2.7) $\quad \mathfrak{A}u = 0 \quad \text{in } U$

and

(2.8) $\quad \lim_{t \to \zeta} u(X_t) = f(X_\zeta) \quad$ a.s.,

where $\mathfrak{A}$ denotes the characteristic operator of X_t (see e.g. [8], Ch. VIII).

Thus the existence of X_ζ is closely related to the solution of the Dirichlet problem. The idea is first to establish the solution of this problem (2.5) in a distribution sense and then prove that this implies that X_ξ exists (and hence the distribution solution must in fact coincide with the stochastic solution (2.6)).

Returning to (2.4) we see that by using (2.1) and (2.2) we can express A in distribution sense by

(2.9) $\quad J_\phi \cdot A(u) = \operatorname{div}(F \cdot \nabla u), \quad u \in \mathfrak{D}(A).$

In particular, if we define

(2.10) $\quad Lu = \operatorname{div}(F \cdot \nabla u)$

then clearly

(2.11) $\quad Lu = 0 \iff Au = 0.$

Since ϕ is quasiregular there exists a constant $K < \infty$, called the deformation constant such that

(2.12) $\quad \frac{1}{K} J_\phi(x) \leq \|\phi'(x)\|^n \leq K J_\phi(x) \qquad$ for a.a.x

and

(2.13) $\quad \frac{1}{K} J_\phi^{-1}(x) \leq \|(\phi')^{-1}(x)\|^n \leq K \cdot J_\phi^{-1}(x) \qquad$ for a.a.x

where $\| \ \|$ denotes the operator norm on R^n. Therefore, if $\xi \in \mathbb{R}^n$ we have

(2.14) $\quad \xi^T F \xi = J_\phi |((\phi')^{-1})^T \xi|^2 \sim J_\phi^{1-\frac{2}{n}} |\xi|^2,$

where $a \sim b$ means that $\frac{a}{b}$ and $\frac{b}{a}$ are bounded by constants. Hence L is a degenerate elliptic operator. In [1], [2] and [3] a class of such operators are studied. However, the assumptions posed by these authors are not satisfied here, so it is not clear to what extent their strong results remain valid for our L. More precisely, it is an interesting open problem whether the results that they obtain for the operator L above in the case when ϕ is quasiconformal carry over to our case when ϕ is just quasiregular.

We will now establish a small step in the direction of an affirmative answer to this question: A weighted Sobolev inequality for quasiregular ϕ (corresponding to Property 2 in [3]). This inequality is crucial for the existence of the solution of the Dirichlet problem for L.

First we prove an auxiliary result. From now on we put

$$\rho = J_\phi^{1-\frac{2}{n}}$$

and we let $H_0 = H_0(U,\rho)$ be the closure of $C_0^\infty(U)$ with respect to the norm

$$(2.15) \qquad \|u\|_{U,\rho}^2 = \int_U |u|^2 \rho dx + \int_U |\nabla u|^2 \rho dx$$

LEMMA 2.1. Let $w \in H_0(U,\rho)$. Then

$$\lim_{t\to\zeta} w(X_t) = 0 \qquad \text{a.s. } P^x, \text{ for a.a. } x \in U.$$

Proof. Choose $w_k \in C_0^\infty(U)$ such that $\|w-w_k\|_{U,\rho}^2 \to 0$ as $k\to\infty$. By [4, Lemma 5.1.2] we can find a subsequence w_{k_j} s.t.

$$w_{k_j}(X_t) \to w(X_t) \text{ uniformly (in t) on compact intervals a.s.}$$

P^x for a.a.x. Therefore $t \to w(X_t)$ is continuous and since $w_{k_j}(X_\zeta) = 0$ for all j we conclude that

$$0 = w(X_\zeta) = \lim_{t\uparrow\zeta} w(X_t) \qquad \text{a.s.}$$

THEOREM 2.2 (Weighted Sobolev inequality).
There exist a constant $C<\infty$ (depending only on the deformation constant K, the diameter of U and the dimension n) such that

$$(2.16) \qquad \int_U |u(x)|^2 \rho(x)dx \le C \int_U |\nabla u(x)|^2 \rho(x)dx \quad \text{for all } u \in C_0^\infty(U),$$

where $\rho = J_\phi^{1-\frac{2}{n}}$.

Proof. Assume that (2.16) does not hold. Then for all k we can find $u_k \in C_0^\infty(U)$ such that

$$(2.17) \qquad 1 = \int_U |u_k|^2 \rho dx > \int_U |\nabla u_k|^2 \rho dx$$

Hence $\nabla u_k \to 0$ in $L^2(U;\rho dx)$.

Let B_ϕ denote the branch set of ϕ, i.e. the (closed) set of points in U where ϕ is not locally a homeomorphism. By Property 5 in [3] there exists for each $z \in U \setminus B_\phi$ a ball $Q_z \subset U$ centered at z such that the weighted Poincare inequality holds in Q_z:

$$(2.18) \qquad \int_{Q_z} |u-\bar{u}^{(z)}|^2 \rho \, dx \leq C_1 \int_{Q_z} |\nabla u|^2 \rho \, dx \qquad \text{for all } u \in C^\infty(U)$$

where C_1 only depends on K, U and n. Here $\bar{u}^{(z)}$ is the weighted average value of u in Q_z, i.e.

$$\bar{u}^{(z)} = \left(\int_{Q_z} \rho dx\right)^{-1} \cdot \int_{Q_z} u \rho dx$$

The family $\{Q_z\}_{z \in B_\phi}$ covers $U \setminus B_\phi$ so we can find a countable subcollection $\{Q_j\}_{j=1}$ which covers $U \setminus B_\phi$ and such that no point of $U \setminus B_\phi$ is covered by more than $n+1$ balls [5]. Writing $\bar{u}^{(j)} = \bar{u}^{(z_j)}$ where z_j is the center of Q_j we get

$$(2.19) \qquad \sum_j \int_{Q_j} |u-\bar{u}^{(j)}|^2 \rho \, dx \leq (n+1) C_1 \cdot \int_U |\nabla u|^2 \rho \, dx$$

We now apply this to the sequence $\{u_k\}$. First note that by Hölder's inequality we have

$$|\bar{u}_k^{(j)}| \leq \int_{Q_j} |u_k|^2 \rho \, dx \leq 1 \qquad \text{for all } j,k$$

so by taking a subsequence we may assume that

$$\bar{u}_k^{(j)} \to \beta^{(j)}, \text{ say, as } k \to \infty.$$

Then by (2.17) and (2.19) applied to u_k we get that

$$\lim_{k\to\infty} \int_{Q_j} |u_k-\beta^{(j)}|^2\rho \, dx = 0 \qquad \text{for all } j.$$

Taking another subsequence we conclude that

$$(2.20) \qquad \lim_{k\to\infty} u_k(x) = \beta^{(j)} \quad \text{a.e. } (dx) \text{ on } Q_j, \quad \text{for all } j.$$

Since the topological dimension of B_ϕ is at most $n-2$ (see [10]), it follows that $U\setminus B_\phi$ must be connected ([5], Th.IV.4). Therefore (2.20) implies that

$$\beta^{(j)} = \beta^{(i)} = \beta, \quad \text{say},$$

for all i,j.

We conclude that

$$u_k \to \beta \quad \text{in } L^2(U,\rho\, dx)$$

and

$$\nabla u_k \to 0 \quad \text{in } L^2(U,\rho\, dx).$$

In other words, if we define $H_0^2(U,\rho) = H_0^2$ to be the closure of $C_0^\infty(U)$ in the norm

$$\|u\|_U^2 = \int_U |u|^2\rho \, dx + \int_U |\nabla u|^2\rho \, dx$$

we have obtained that $\beta \in H_0^2$.

However, by Lemma 2.1 this is only possible if $\beta=0$, which contradicts that from (2.17) we must have $\int_U \beta^2\rho \, dx = 1$. This contradiction proves Theorem 2.2.

<u>§3. Boundary values and harmonic measure</u>.

By the weighted Sobolev inequality Theorem 2.2 it follows that $\mathcal{E}_\phi(u,v)$ is an inner product for $H_0(U,\rho)$ and therefore there exists a <u>Green operator</u>

$$G: H_0(U,\rho)^* \to H_0(U,\rho)$$

such that

(3.1) $\mathcal{E}_\phi(G(T),v) = T(v)$

for all $T \in H_0(U,\rho)^*$ (the dual of $H_0(U,\rho)$) and all $v \in H_0(U,\rho)$.

In particular, if $f \in C^\infty(\mathbb{R}^n)$ and we define the distribution T by

(3.2) $T = Lf = \mathrm{div}(F \cdot \nabla f)$

then T can be regarded as an element of $H_0(U,\rho)^*$ by putting

(3.3) $T(v) = -\int_U \nabla f^T \cdot F \cdot \nabla v \, dx$

(see [2, p. 579-581] for details)

The variational solution g of the Dirichlet problem in U with boundary values $f|\partial U$, where $f \in C^\infty(\mathbb{R}^n)$, can then be described by

(3.4) $g = f + w$ where $w = G(T) = G(Lf) \in H_0(U,\rho)$

Note that $Lg = 0$ (in the sense that $\mathcal{E}_\phi(g,v) = 0$ for all $v \in H_0(U,\rho)$) because, by (3.1) - (3.3),

$$\mathcal{E}_\phi(g,v) = \mathcal{E}_\phi(f,v) + \mathcal{E}_\phi(G(T),v) = \mathcal{E}_\phi(f,v) - \mathcal{E}_\phi(f,v) = 0.$$

The basic idea is now to prove that

(3.5) $\lim_{t \to \zeta} g(X_t) = g^*$ exists a.s. P^x, for a.a.x

Using (3.5) and Lemma 2.1 we conclude that

$$\lim_{t \to \zeta} f(X_t) \text{ exists a.s. } P^x, \text{ for a.a.x}$$

In particular, applying this to $f(x_1,\dots,x_n) = x_k$ for $k=1,2,\dots,n$ we obtain the conclusion we seek:

(3.6) $\lim_{t \to \zeta} X_t = X_\zeta$ exists a.s. P^x, for a.a.x

Thus we aim to prove (3.5). First we establish an auxiliary result:

<u>LEMMA 3.1</u> Suppose u is A-harmonic in U, i.e. $u \in \mathcal{D}(A)$ and $Au=0$. Let $\tilde{u}$ be its quasicontinuous version. Then

$$u^* = \lim_{t \to \zeta} \tilde{u}(X_t) \tag{3.7}$$

exists a.s. P^x, for a.a. $x \in U$.

<u>Proof</u>. First note that by the decomposition theorem ([4, Theorem 5.2.2]) we can write

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t + N_t,$$

where M_t is a martingale with energy

$$e(M) = \lim_{t \to 0} \frac{1}{2t} \int_U E^x[M_t^2]\, \rho(x)dx < \infty$$

and N_t is a continuous additive functional with zero energy. Since $Au=0$ we have $N_t=0$ ([4, Theorem 5.3.4). Therefore $\tilde{u}(X_t)$ is a martingale. Thus Lemma 3.1 may be regarded as a version of the martingale convergence theorem. However, since the version we need involves the left limit at ζ and not at ∞, an extra argument is needed to complete the proof:

If (3.7) does not hold, then we can find real numbers $a<b$ such that

(3.8) $P^x[\tilde{u}(X_t)$ crosses the interval (a,b) infinitely many times for $t<\zeta] > 0$

on a set of x-values of positive Lebesgue measure.

We now define a sequence $\{\tau_n\}$ of stopping times as follows:
Let τ_1 be the first time t that $u(X_t)$ reaches the value a, τ_2 the first time $t>\tau_1$ that $u(X_t)=b$, τ_3 the first $t>\tau_2$ such that $u(X_t)=a$ and so on. Then $\tau_n \uparrow \zeta$. Fix $T>0$ and put $\sigma_n = \tau_n \wedge T$. Then for a.a.x

$$E^x[\tilde{u}(X_{\sigma_n})^2] = E^x[M_{\sigma_n}^2] \le E^x[M_T^2] < \infty \quad \text{for all } n.$$

Hence by the martingale convergence theorem [11]

$$\lim_{n\to\infty} \tilde{u}(X_{\sigma_n}) \quad \text{exists a.s. } P^x, \quad \text{for a.a.} x$$

This holds for arbitrary T and therefore gives a contradiction to (3.8). That proves the lemma.

We have now proved the main result of this section:

THEOREM 3.2

$$X_\zeta = \lim_{t\to\zeta} X_t \quad \text{exists a.s. } P^x, \quad \text{for a.a. } x\in U.$$

§4. Applications.

In [7] Martio and Rickman proved that if ϕ is quasimeromorphic on the open unit ball B of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \phi(B)$ has positive n-capacity, then ϕ has asymptotic values (which may be infinite) at a dense set of points $y\in\partial B$. (That ϕ has an asymptotic value at y means that there exists a path γ: $[0,1] \to \bar{B}$ with $\gamma[0,1)\subset B$, $\gamma(1)=y$ and such that

$$\lim_{t\to 1} \phi(\gamma(t)) \quad \text{exists)}$$

A natural question is whether the set of points y where ϕ has asymptotic values can be described more closely and for other sets than the ball. The next result shows that this can reduced to the problem of estimating the null sets of the X-harmonic measure λ^X:

COROLLARY 4.1 Let $\phi: U\to \mathbb{R}^n$ be quasiregular, where as before U is a bounded domain and (1.3) holds. Then ϕ has (finite) asymptotic values at a.a. $y\in\partial U$ with respect to λ^X.
Moreover, if ϕ is non-constant then the set of asymptotic values has positive classical capacity in $\mathbb{R}^n$ (The classical capacity in $\mathbb{R}^n$ is the capacity associated to the kernel $|x|^{2-n}$ if $n>2$ and $\log\frac{1}{|x|}$ if $n=2$).

Proof. Since the existence of $X_\zeta = \lim_{t \to \zeta} X_t$ is established we can apply the argument of [9, Theorem 3.5].

The following result shows that λ^X is always positive on non-empty open "radial" subsets of ∂U:

THEOREM 4.2. Suppose the exit time $\tau_{\phi(U)}$ from $\phi(U)$ of Brownian motion is finite a.s. Let $V \neq \emptyset$ be an open subset of ∂U with the property that there exists a closed cone K with vertex in U such that

$$K \cap \partial U \subset V.$$

Then $\lambda^X(V) > 0$.

Proof. The idea of the proof is based on the construction a function $\hat{\phi}$ from ϕ by repeated reflections about hyperplanes through the vertex of the cone. To describe the reflection operation, let us assume for simplicity that the vertex is at the origin and that the hyperplane is the boundary of the halfspace $H = \{(x_1, \ldots, x_n) \in \mathbb{R}^n;\ x_1 > 0\}$. Define

$$\psi(x) = \begin{cases} \phi(x) & \text{if} \quad x \in U \cap H \\ \phi(-x_1, x_2, \ldots, x_n) & \text{if} \quad \tilde{x} = (-x_1, x_2, \ldots, x_n) \in U \cap H \end{cases}$$

Then ψ satisfies all the requirements of a quasiregular function on $\tilde{U} = \{x; x \in U \cap H \text{ or } \tilde{x} \in U \cap H\}$, except that the sign of J_ψ is negative a.e. for $x_1 < 0$. However, as pointed out in [9, p. 280] it is still true that the process $\tilde{X}_t$ associated to the Dirichlet form

$$\mathcal{E}_\psi(u,v) = \frac{1}{2} \int_{\tilde{U}} \nabla u^T \cdot |J_\psi| \cdot (\psi')^{-1} \cdot ((\psi')^{-1})^T \nabla v \, dx; \quad u, v \in C_0^\infty(\tilde{U})$$

is mapped into Brownian motion by ψ.

Assume that $\lambda^X(V) = 0$.

Note that $\tilde{X}_t$ coincide in law with X_t up to the first exit time from $U \cap H$. So if

$$P^x[X_\zeta \in V] = 0 \quad \text{for q.a. } x \in U \cap H$$

then

$$P^x[\tilde{X}_{\tilde{\zeta}} \in V] = 0 \quad \text{for q.a. } x \in U \cap H$$

Moreover, by symmetry we also have

$$P^x[\tilde{X}_{\tilde{\zeta}} \in \tilde{V}] = 0, \quad \text{where } \tilde{V} = \{x; \tilde{x} \in V\},$$

for q.a. x such that $\tilde{x} \in U \cap H$ and consequently for q.a. $x \in U \cap H$ by the strong Markov property.

We conclude that if $\lambda^X(V)=0$, then $\lambda^{\tilde{X}}(V \cup \tilde{V})=0$. Repeating this construction with suitable hyperplanes $P_1, \ldots, P_k$ through the origin we can obtain a function $\hat{\phi}$ on a set $\hat{U}$ whose boundary is contained in the finite union, W, of V and its corresponding reflections $\tilde{V}_1, \ldots, \tilde{V}_k$, such that its corresponding process $\hat{X}_t$ satisfies

$$P^x[\hat{X}_{\hat{\zeta}} \in W] = 0 \quad \text{for q.a. } x \in \hat{U}$$

Since the life time $\hat{\zeta}$ of $\hat{X}_t$ must be finite (by our assumption on $\tau_{\phi(U)}$), this is a contradiction. So the statement of the lemma must hold.

We can now prove the following partial extension of Martio and Rickman's asymptotic value theorem:

THEOREM 4.3. Suppose $\tau_{\phi(U)} < \infty$ a.s. and that each point $y \in \partial U$ has an open neighbourhood V with the property that there exists a closed cone K with vertex in U such that $K \cap \partial U \subset V$. Then ϕ has (finite) asymptotic values at a dense set of points $y \in \partial U$.

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Dept. of Mathematics
University of Oslo
Box 1053, Blindern
N-0316 Oslo 3
NORWAY