ON ACTIONS OF AMENABLE GROUPS ON II_1- FACTORS.

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Abstract: Given a II_1-factor M with separable predual and α a free action of a countable amenable discrete group G on M, we show that the crossed product $M \times_\alpha G$ has property Γ (resp. is McDuff) when M itself has property Γ (resp. is McDuff).

1. Introduction

Let M denote a II_1-factor with separable predual and normalized trace τ. As usual, the Hilbert norm $\|\cdot\|_2$ on M given by τ is defined by $\|x\|_2=\tau(x^*x)^{\frac{1}{2}}$, $x\in M$. Recall that M is said to have property Γ (of Murray and von Neumann [14]) when for any $x_1,\dots,x_n\in M$, $\varepsilon>0$, there exists a unitary $u\in M$ such that $\tau(u)=0$ and $\|[x_i,u]\|_2<\varepsilon$, $i=1,\dots,n$. Property Γ plays an important role in the theory of II_1-factors and has been characterized in many ways. As a sample we refer to [4], [6] and [8]. Now, let $\alpha:G\to Aut(M)$ denote an action of a countable discrete group G on M which is free, i.e. each α_g is outer, $g\neq 1$. Then consider the resulting crossed product $M \times_\alpha G$, which is well known to be a II_1-factor. The main purpose of this note is to etablish the following result, believed to be true by Popa (cf.[19;p.32] or [20;3.3.2]).

Theorem A: If G is amenable and M has property Γ, then $M \times_\alpha G$ has also property Γ.

Theorem A has previously been obtained for G finite [3;th.1] and for $G=\mathbb{Z}$ [19;p.32]. On the other hand, it is elementary to produce examples of free actions of nonamenable groups on II_1-factors with property Γ such that the resulting crossed products also have property Γ. For an example of a free action of $\mathbb{Z}$ on a II_1-factor without property Γ such that the crossed product has property Γ we refer to [17;prop. 4.3]. For other connected results, see [12] and [13].

Another interesting property for II_1-factors, which is stronger than property Γ, is that of being McDuff (see for example [5], [6] and [9]). Recall that M is called McDuff if M is $*$-isomorphic to $M \bar{\otimes} R$, where R denotes the hyperfinite II_1-factor. In order to prove theorem A, we first present a proof of the following theorem, which in essence may be attributed to Ocneanu:

<u>Theorem B</u>: If G is amenable and M is McDuff, then $M \times_{\alpha} G$ is McDuff too.

When G is finite, theorem B is a consequence of [18;prop. 1.11 ii)]. Examples of free actions of nonamenable groups on McDuff II_1-factors such that the resulting crossed products are McDuff are easy to construct. For an example of a free action of $\mathbb{Z}$ on a non McDuff II_1-factor (with property Γ) such that the crossed product is McDuff, we refer to [11]. Further, we show that the remaining part of theorem A, modulo theorem B, is true:

<u>Theorem C</u>: If G is amenable and M has property Γ without being McDuff, then $M \times_{\alpha} G$ has property Γ.

We begin this paper with a section (§2) devoted to a review of some facts about cocycle crossed actions and regular extensions ([2], [16], [23] and [24]). Our main interest lies in a folklore result about decomposition of crossed products, which we need explicitely in §3 where theorem B and C are proved. Our proof of theorem B relies heavily on two deep results of Ocneanu [16;th. 1.1 and th. 1.2], which themselves rely on techniques and results developed by Ornstein and Weiss, McDuff, Jones and Connes among others. We note that theorem B may also be deduced from an assertion stated without proof by Ocneanu (see [16;p.6, the assertion following th. 1.2]). However, we propose a slightly different approach, which we hope is of independent interest. On the other hand, the main idea in the proof of theorem C is to invoke in a suitable way a result of Schmidt [22;th.2.4], which itself is an outgrowth of the Connes-Feldman-Weiss theorem.

We follow standard notation and terminology, as may be found for example in [13]. Otherwise, the reader may consult [7] and [16]. We quote here som notation.

Suppose we are given a von Neumann algebra N acting on a Hilbert space $\mathcal{H}$ and a discrete group H. Then

$\mathrm{Aut}(N)$ = the group of *-automorphisms of N,

$U(N)$ = the group of unitaries in N,

$B(\mathcal{H})$ = the bounded linear operators acting on $\mathcal{H}$,

$\ell^2(H,\mathcal{H})$ = the Hilbert space of all $\mathcal{H}$-valued functions ξ on H such that $\sum_{h\in H} \|\xi(h)\|^2 < +\infty$,

$\ell^2(H) = \ell^2(H,\mathbb{C})$,

$\mathrm{Aut}(H)$ = the group of automorphisms of H.

When $u \in U(B(\mathcal{H}))$ is such that $uNu^*=N$, $\mathrm{ad}(u)$ denotes the *-automorphism of N implemented by u. Finally, when $\alpha: H\to \mathrm{Aut}(H)$ denotes an action of H on N with resulting crossed product $N \times_\alpha H$, we sometimes identify N with its canonical copy in $N \times_\alpha H$.

2. Cocycle crossed actions and regular extensions.

Let N denote a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A cocycle crossed action of a discrete group K on N is a pair (β,u), where $\beta: K\to \mathrm{Aut}(N)$ and $u: K\times K\to U(N)$ satisfy for $k,\ell,m \in K$

$$\beta_k\beta_\ell = \mathrm{ad}(u(k,\ell))\beta_{k\ell},$$

$$u(k,\ell)\, u(k\ell,m) = \beta_k(u(\ell,m))\, u(k,\ell m),$$

$$u(1,\ell) = u(k,1) = 1.$$

The <u>regular extension</u> of N by K, say $N \times_{(\beta,u)} K$, is then defined as the von Neumann algebra acting on $\ell^2(K,\mathcal{H})$ generated by $\pi_\beta(N)$ and $\lambda_u(K)$, where π_β is the faithful normal representation of N on $\ell^2(K,\mathcal{H})$ defined by

$$(\pi_\beta(x)\ \xi)(\ell) = \beta_{\ell^{-1}}(x)\ \xi(\ell),$$

while, for each $k \in K$, $\lambda_u(k)$ is the unitary operator on $\ell^2(K,\mathcal{H})$ defined by

$$(\lambda_u(k)\xi)(\ell) = u(\ell^{-1},k)\xi(k^{-1}\ell), \quad (x\in N,\ \xi\in\ell^2(K,\mathcal{H}),\ \ \ell\in K).$$

Accordingly, when $u\equiv 1$, i.e. when β is an action of K on N, the regular extension amounts to the ordinary crossed product $N \times_\beta K$.

One checks easily that the covariance formula

$$\pi_\beta(\beta_k(x)) = \mathrm{ad}(\lambda_u(k))\ (\pi_\beta(x))$$

holds for all $k\in K$, $x\in N$, and also that

$$\lambda_u(k)\lambda_u(\ell) = \pi_\beta(u(k,\ell))\ \lambda_u(k\ell)$$

holds for all $k,\ell\in K$.

Further, one may proceed as in [25;prop. 3.4] (cf.[2:Th.5]) to verify the following proposition, which assures that the algebraic structure of $N \times_{(\beta,u)} K$ is independent of the Hilbert space $\mathcal{H}$.

<u>Proposition 1</u>: Suppose $\theta: N\to P$ is a *-isomorphism between two von Neumann algebras N and P, and that (β,u) is a cocycle crossed action of a discrete group K on N. Define $\bar\beta_k = \theta\ \beta_k\ \theta^{-1}\ \in \mathrm{Aut}(P)$ and $\bar u(k,\ell) = \theta(u(k,\ell)) \in U(P)$ for all $k,\ell\in K$. Then $(\bar\beta,\bar u)$ is a cocycle crossed action of K on P, and there exists a *-isomorphism $\tilde\theta: N \times_{(\beta,u)} K \to P \times_{(\bar\beta,\bar u)} K$ such that

$$\pi_{\bar\beta}\ (\theta(x)) = \tilde\theta\ (\pi_\beta(x)) \qquad (x\in N),$$

$$\lambda_{\bar u}\ (k) = \tilde\theta\ (\lambda_u(k)) \qquad (k\in K).$$

Now, given a map $v\colon K \to U(N)$ with $v_1 = 1$, the <u>perturbation of (β,u) by v</u> is by definition the pair $(\tilde{\beta},\tilde{u})$ obtained by setting

$$\tilde{\beta}_k = \mathrm{ad}(v_k)\ \beta_k,$$

$$\tilde{u}(k,\ell) = v_k \beta_k(v_\ell)\ u(k,\ell)\ v_{k\ell}^*\ , \quad (k,\ell \in K).$$

One readily verifies that $(\tilde{\beta},\tilde{u})$ is a cocycle crossed action of K on N. We say that (β,u) is a <u>coboundary</u> (of v) when $\tilde{u} \equiv 1$.

The next proposition is well known; indeed it is merely a restatement of a part of [24;prop. 5.1.2.].

<u>Proposition 2</u>: If $(\tilde{\beta},\tilde{u})$ is a perturbation (by v) of a cocycle crossed action (β,u) of a discrete group K on a von Neumann algebra N, then

$$N \times_{(\tilde{\beta},\tilde{u})} K \text{ is } {*}\text{-isomorphic to } N \times_{(\beta,u)} K.$$

Our main interest in this section is to show how cocycle crossed actions and regular extensions naturally appear when decomposing crossed products. For group von Neumann algebras, this has been treated in [24;prop. 3.17] (and in [2;th. 11]). When the acting group in a given crossed product may be decomposed as a semi-direct product, the expected decomposition of the crossed product as a "double" crossed product has been pointed out in [1;th. 4.3] and [21;th. 2.4]. As we have not been able to find a suitable reference in the literature for the general situation, and we need an explicit version in the next section, we now sketch a proof of such a result. It generalizes slightly [15;th. 3].

<u>Proposition 3</u>: Let $1 \to H \to G \overset{\pi}{\to} K \to 1$ denote an exact sequence of discrete groups and $\alpha: G \to \mathrm{Aut}(M)$ an action of G on a von Neumann algebra M acting on a Hilbert space $\mathcal{H}$. Identify H with its copy in G and set $N = M \times_{\alpha|H} H$, where $\alpha \mid H$ denotes the restriction of α to H on M. Then there exists a cocycle crossed action (β, u) of K on N such that

$$M \times_\alpha G \text{ is } *\text{-isomorphic to } N \times_{(\beta,u)} K.$$

<u>Proof</u>: For each $k \in K$, $k \neq 1$, choose $n_k \in G$ such that $\pi(n_k)=k$, and set $n_1=1$. Then define $\sigma: K \to \mathrm{Aut}(H)$ by $\sigma_k(h) = n_k h n_k^{-1}$ $(h \in H)$, and $v: K \times K \to H$ by $v(k,\ell) = n_k n_\ell n_{k\ell}^{-1}$ $(k,\ell \in K)$. One verifies that (σ, v) satisfies for $k,\ell,m \in K$

$$\sigma_k \sigma_\ell = \mathrm{ad}(v(k,\ell))\sigma_{k\ell},$$
$$v(k,\ell)v(k\ell,m) = \sigma_k(v(\ell,m))\, v(k,\ell m),$$
$$v(k,1) = v(1,\ell) = 1.$$

Write γ for $\alpha|H$, so that $N = M \times_\gamma H$. Then denote by $\{\pi_\gamma(x),\ \lambda(h);\ x \in M,\ h \in H\}$ (resp. $\{\pi_\alpha(x),\ \bar{\lambda}(g);\ x \in M,\ g \in G\}$) the generators of N on $\ell^2(H, \mathcal{H})$ (resp. $M \times_\alpha G$ on $\ell^2(G, \mathcal{H})$).

<u>Claim 1</u>: For each $k \in K$, there exists $\beta_k \in \mathrm{Aut}(N)$ such that

i) $\beta_k(\pi_\gamma(x)) = \pi_\gamma(\alpha_{n_k}(x))$ $(x \in M)$,

ii) $\beta_k(\lambda(h)) = \lambda(\sigma_k(h))$ $(h \in H)$.

Assume first that α is implemented by a unitary representation $g \to a(g)$ of G on $\mathcal{H}$.

Then define $\beta_k \in \mathrm{Aut}(B(\mathcal{H} \otimes \ell^2(H)))$ by

$$\beta_k = \mathrm{ad}(a(n_k) \otimes d_k) \quad (k \in K),$$

where d_k is the unitary operator on $\ell^2(H)$ defined by

$$(d_k\xi)(h) = \xi(\sigma_k^{-1}(h)) \quad (\xi \in \ell^2(H),\ h \in H).$$

Identifying canonically $\mathcal{H}\otimes\ell^2(H)$ with $\ell^2(H,\mathcal{H})$, one checks, essentially as in [21; lem. 2.3], that each β_k satisfies i) and ii) above. Hence the desired β_k's are obtained by restriction to N.

If α is not implemented on $\mathcal{H}$, then $\bar{\alpha}:G\to\mathrm{Aut}(\pi_\alpha(M))$ defined by $\bar{\alpha}_g=\pi_\alpha\ \alpha_g\ \pi_\alpha^{-1}$ $(g\in G)$ is implemented by $g\to\bar{\lambda}(g)$ on $\ell^2(G,\mathcal{H})$. Accordingly, there exists $\bar{\beta}_k\in\mathrm{Aut}(\pi_\alpha(M)\ x_{\bar{\gamma}}\ H)$ satisfying the analogues of i) and ii) for each $k\in K$, where $\bar{\gamma}$ denotes the restriction of $\bar{\alpha}$ to H. Now, a straightforward application of proposition 1 (with $P=\pi_\alpha(M)$, $\theta=\pi_\alpha$, $\beta=\gamma$ and $u\equiv 1$) gives the existence of the desired β_k's on N (by setting $\beta_k=\tilde{\theta}^{-1}\ \bar{\beta}_k\ \tilde{\theta}$) and claim 1 is etablished.

Define $u(k,\ell)\in U(N)$ by

$$u(k,\ell)=\lambda(v(k,\ell))\qquad (k,\ell\in K).$$

With the help of claim 1 and the cocycle equations for (σ,v), it is elementary to check that the induced pair (β,u) is a cocycle crossed action of K on N.

The regular extension $N\ x_{(\beta,u)}K$, which acts on $\ell^2(K,\ell^2(H,\mathcal{H}))$ is then clearly generated by

$$\{\pi_\beta(\pi_\gamma(x)),\ \pi_\beta(\lambda(h)),\ \lambda_u(k);\ x\in M,\ h\in H,\ k\in K\}.$$

Define $W: \ell^2(K,\ell^2(H,\mathcal{H}))\to\ell^2(G,\mathcal{H})$ by

$$(W\xi)(g)=[\xi(\pi(g))]\,(n_{\pi(g^{-1})}\,g)\qquad (\xi\in\ell^2(K,\ell^2(H,\mathcal{H})),\ g\in G).$$

Plainly, W is a unitary operator and $W^*:\ell^2(G,\mathcal{H})\to\ell^2(K,\ \ell^2(H,\mathcal{H}))$ is given by

$$[(W^*f)(k)]\,(h)=f(n^{-1}_{k^{-1}}\,h)\qquad (f\in\ell^2(G,\mathcal{H}),\ k\in K,\ h\in H).$$

Since G is generated by $\{h,\ n_k\ ;\ h\in H,\ k\in K\}$, the proof of the proposition is clearly achieved as soon as one etablishes the following:

<u>Claim 2</u>: i) $W\,\pi_\beta(\pi_\gamma(x))W^* = \pi_\alpha(x) \quad (x\in M)$

ii) $W\pi_\beta(\lambda(h))W^* = \bar{\lambda}(h) \quad (h\in H)$

iii) $W\,\lambda_u(k)W^* = \bar{\lambda}(n_k) \quad (k\in K)$

We leave the proof of ii) to the reader and prove i) and iii). Let $x\in M$, $k\in K$, $g\in G$ and $\xi\in\ell^2(G,\mathcal{K})$, and set $\ell=\pi(g)\in K$. Then

$$
\begin{aligned}
(W\pi_\beta(\pi_\gamma(x))W^*\xi)(g) &= [(\pi_\beta(\pi_\gamma(x))W^*\xi)(\ell)]\,(n_{\ell^{-1}}\,g)\\
&= [\beta_{\ell^{-1}}(\pi_\gamma(x))\;W^*\xi(\ell)]\,(n_{\ell^{-1}}\,g)\\
&= [\pi_\gamma(\alpha_{n_{\ell^{-1}}}(x))\;W^*\xi(\ell)]\,(n_{\ell^{-1}}\,g)\\
&= \gamma_{(n_{\ell^{-1}}g)^{-1}}\,(\alpha_{n_{\ell^{-1}}}(x))\;[W^*\xi(\ell)]\,(n_{\ell^{-1}}g)\\
&= \alpha_{g^{-1}}(x)\;\xi\,(n^{-1}_{\ell^{-1}}\;n_{\ell^{-1}}\;g)\\
&= (\pi_\alpha(x)\;\xi)\;(g),
\end{aligned}
$$

which proves i).

Further

$$
\begin{aligned}
(W\,\lambda_u(k)W^*\xi)(g) &= [(\lambda_u(k)W^*\xi)(\ell)]\,(n_{\ell^{-1}}\,g)\\
&= [u(\ell^{-1},k)\;W^*\xi(k^{-1}\ell)]\,(n_{\ell^{-1}}\,g)\\
&= [\lambda(v(\ell^{-1},k))\;W^*\xi(k^{-1}\ell)]\,(n_{\ell^{-1}}\,g)\\
&= [W^*\xi(k^{-1}\ell)]\,((n_{\ell^{-1}}\,n_k\,n^{-1}_{\ell^{-1}k})^{-1}\;n_{\ell^{-1}}\,g)\\
&= \xi(n^{-1}_{(k^{-1}\ell)^{-1}}\;n_{\ell^{-1}k}\;n_k^{-1}\;g)\\
&= (\bar{\lambda}(n_k)\xi)\;(g),
\end{aligned}
$$

which proves iii).

QED.

We note that crossed products by locally compact (separable) groups may be handled in the same way, with some minor modifications following [24], but we leave this to the reader.

3. Proofs of theorem B and C.

In this section, we suppose that we are given a II_1-factor M with separable predual and normalized trace τ, and a free action α of a countable discrete group G on M.

We recall that $\theta \in Aut(M)$ is called centrally trivial, $\theta \in Ct(M)$, if for any centralizing sequence (x_n) in M, i.e. which is norm bounded and satisfies that $\|[x_n, y]\|_2 \to 0$ $(n \to +\infty)$ for any $y \in M$, one has that $\|\theta(x_n) - x_n\|_2 \to 0$ $(n \to +\infty)$, (cf. [5] and [16]). Further, α is called centrally trivial (resp. centrally free) on M when $\alpha_g \in Ct(M)$ (resp. $\alpha_g \notin Ct(M)$) for each $g \in G$, $g \neq 1$.

Lemma 4: Suppose that α is centrally trivial on M. Then

a) each central sequence in M identifies with a central sequence in $M \times_\alpha G$.

b) $M \times_\alpha G$ is McDuff when M is McDuff.

Proof: a) follows immediately from the covariance formula in $M \times_\alpha G$ and the assumption on α, while

b) is a direct consequence of a) and McDuff's theorem [9;th. 3].

QED.

Lemma 5: If G is amenable, α is centally free on M and M is McDuff, then $M \times_\alpha G$ is McDuff.

Proof: By combining [16;th. 1.2] and [25;Cor. 3.6], we have $M \times_\alpha G$ is $*$-isomorphic to $(M \bar{\otimes} R) \times_{\alpha \otimes idR} G$, the latter being clearly McDuff.

QED

<u>Theorem B</u>: If G is amenable and M is McDuff, then $M \times_\alpha G$ is McDuff.

<u>Proof</u>: Define $H = \{h \in G \mid \alpha_h \in Ct(M)\}$. Since $Ct(M)$ is normal in $Aut(M)$, H is a normal subgroup of G and we may define $K = G/H$. Since G is amenable, K is itself amenable (cf. [10;th. 1.2.4]). Let γ denote the restriction of α to H on M. Trivially, γ is a free action of H on M and by lemma 4b), $N = M \times_\gamma G$ is McDuff. Further, proposition 3 says that there exists a cocycle crossed action (β,u) of K on N such that $M \times_\alpha G$ is $*$-isomorphic to $N \times_{(\beta,u)} K$. We now claim that (β,u) is centrally free on N, i.e. $\beta_k \in Ct(N)$ for each $k \in K$, $k \neq 1$.
Indeed, let $k \in K$, $k \neq 1$. From the proof of proposition 3, there exists $n_k \in G$, $n_k \notin H$, such that $\beta_k(\pi_\gamma(x)) = \pi_\gamma(\alpha_{n_k}(x))$ for all $x \in M$. By definition of H, α_{n_k} is centrally free on M, i.e. there exists a central sequence (x_i) in M such that $\|\alpha_{n_k}(x_i) - x_i\|_2 \not\to 0$ $(i \to +\infty)$. Then $(\pi_\gamma(x_i))$ is a central sequence in N (cf. lemma 4a)) such that

$$\|\beta_k(\pi_\gamma(x_i)) - \pi_\gamma(x_i)\|_2 = \|\pi_\gamma(\alpha_{n_k}(x_i) - \pi_\gamma(x_i)\|_2$$
$$= \|\alpha_{n_k}(x_i) - x_i\|_2 \not\to 0 \quad (i \to +\infty).$$

Hence β_k is centrally free on N.
Now, by appealing to [16;th. 1.1], we have that (β,u) is a coboundary; hence we may perturb (β,u) to a centrally free action $\tilde{\beta}$ of K on N, and, by proposition 2, we have that $N \times_{(\beta,u)} K$ is $*$-isomorphic to $N \times_{\tilde{\beta}} K$. By lemma 5, $N \times_{\tilde{\beta}} K$ is McDuff.
Altogether, this show that $M \times_\alpha G$ is McDuff.

QED.

We now turn to the proof of theorem C. We pick a free ultrafilter ω on $\mathbb{N}$ and denote by M^ω the ultraproduct algebra of M (which is a II_1-factor) and by τ^ω its canonical trace. Set $M_\omega = M' \cap M^\omega$ (the relative commutant of the canonical copy of M in M^ω) and let $\tilde{\alpha}: G \to \mathrm{Aut}(M_\omega)$ denote the induced action defined $\tilde{\alpha}_g((x_n)) = (\alpha_g(x_n))$, $g \in G$. For more information on this, we refer to [4], [5], [9], [16] and [19].

<u>Lemma 6</u>: If G is finitely generated and amenable, and M has property Γ without being McDuff, then $M \times_\alpha G$ has property Γ.

<u>Proof</u>: Denote by $g_1, \ldots, g_r$ the generators of G $(r < +\infty)$. Since M has property Γ without being McDuff, we have that M_ω is non-trivial completely non-atomic abelian von Neumann algebra (cf. [4] and [9]). If $\tilde{\alpha}$ is not ergodic on M_ω, then let q be a $\tilde{\alpha}$-fixed non-scalar element in M_ω. From the covariance formula in $M \times_\alpha G$, one obtains easily that $q \in (M \times_\alpha G)' \cap (M \times_\alpha G)^\omega$. Since $q \notin \mathbb{C}$, this implies that $M \times_\alpha G$ has property Γ (by [4]).
Suppose next (for the sake of obtaining a contradiction) that $\tilde{\alpha}$ is ergodic on M_ω. By [22;th. 2.4], $\tilde{\alpha}$ is then not strongly ergodic on M_ω, i.e. there exists a sequence of projections (p_i) in M_ω such that $\tau^\omega(p_i) = \frac{1}{2}$ $(i \in \mathbb{N})$ and $\|\tilde{\alpha}_g(p_i) - p_i\|_2 \to 0$ $(i \to +\infty)$ for all $g \in G$. It should be noted that we here, in fact, apply [22;th. 2.4] on a countably generated $\tilde{\alpha}$-invariant completely non atomic von Neumann subalgebra of M_ω, such as the one generated by $\{\tilde{\alpha}_g(a), g \in G\}$ for an $a \in M_\omega$ with infinite spectrum.
By taking a subsequence of (p_i) and renaming, there exists a sequence (q_n) in M_ω such that, given $n \in \mathbb{N}$, then

$$\|\tilde{\alpha}_{g_j}(q_n) - q_n\|_2 < \frac{1}{n}, \quad j=1,\ldots,r.$$

Now we may represent each q_n by a sequence of projections in M, $q_n = (q_{n,m})_m$, with $\tau(q_{n,m}) = \frac{1}{2}$ for all $n,m \in \mathbb{N}$ (cf. [5] or [9]). For each $n \in \mathbb{N}$, we may then choose $m_n \in \mathbb{N}$ such that

$$\|\alpha_{g_j}(q_{n,m_n}) - q_{n,m_n}\|_2 < \frac{1}{n}, \quad j=1,\dots,r, \text{ and}$$

$$\|[q_{n,m_n}, y_k]\|_2 < \frac{1}{n}, \quad k=1,\dots,n, \text{ where } (y_k) \text{ is a}$$

$\|\cdot\|_2$-dense sequence in the unit ball of M fixed from the beginning.

Let so $q = (q_{n,m_n})_n \in M^\omega$. From the above inequalities, one obtains easily that $q \in M_\omega$ and $\tilde{\alpha}_{g_j}(q) = q$, $j=1,\dots,r$. Futher, q is a projection with $\tau^\omega(q) = \frac{1}{2}$. Hence q is a non-scalar element in M_ω, which is $\tilde{\alpha}$-fixed since $g_1,\dots,g_r$ generates G. This contradicts the assumption of ergodicity on $\tilde{\alpha}$.

QED.

<u>Theorem C</u>: If G is amenable and M has property Γ without being McDuff, then $M \times_\alpha G$ has property Γ.

<u>Proof</u>: Since G is countable, we may write $G = \bigcup_{j \in \mathbb{N}} G_j$, where (G_j) is an increasing sequence of finitely generated subgroups of G; by amenability of G, each G_j is amenable (cf. [10;th. 1.2.5]). Set $N_j = M \times_{\alpha|G_j} G_j$ (identified as a subfactor of $M \times_\alpha G$) for each $j \in \mathbb{N}$. By Lemma 6, each N_j has property Γ. Since (N_j) is an increasing sequence of subfactors of N such that $M \times_\alpha G = \overline{\bigcup_{j \in \mathbb{N}} N_j}^{\|\cdot\|_2}$, it follows from [19;th. 1.4.1i)] that $M \times_\alpha G$ has property Γ.

QED.

Our proof of lemma 6 is inspired by the proof of [19;th. 1.4.1 iii)], where Popa shows that theorem A is valid when $G = \mathbb{Z}$. His idea is to apply the Rokhlin-type theorem of Connes to $\tilde{\alpha}$ in M_ω. This argument requires α to be centrally free on M, but one reduces easily to this case.

Also, a more direct proof of theorem A in the same spirit would clearly be available if the analogue of Schmidt's result could be shown in the non-abelian case.

Acknowledgements. We would like to thank Erling Størmer for stimulating discussions during the preparation of this note and the Norwegian Research Council (NAVF - D.00.01.194) for financial support.

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