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USING RANDOM MOTION TO STUDY QUASIREGULAR FUNCTIONS

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ABSTRACT. If ϕ is a non-constant quasiregular function on a domain U in $\mathbb{R}^n$ then one can construct a continuous strong Markov process X_t on U which is mapped by ϕ into n-dimensional Brownian motion. We give an outline of this construction, together with some applications. This stochastic approach leads to several interesting questions both regarding the processes involved and regarding the relations to other, non-stochastic methods, such as non-linear potential theory, degenerate elliptic equations and A_p-weights.

§0. Introduction.

The quasiregular functions (also called the functions with bounded deformation) may be regarded as relatives of the more familiar complex analytic functions. The two function families share many properties, especially in the two-dimensional case (identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$). For a survey see [21]. In view of the many successful applications of stochastic calculus in the study of analytic functions it is therefore natural to ask if one can use stochastic methods in the investigation of quasiregular functions as well.

The answer to this question is yes [16]. In this survey we first outline how the stochastic process enters the scene and we mention some applications (§1). Then we discuss the relations between this approach and 2 other methods, which both are important in the study of quasiregular functions:

a) Non-linear potential theory (§2)

b) Degenerate elliptic linear equations and A_p-weights (§3).

We believe that the stochastic approach will be a useful addition to these methods. Moreover, this approach leads to a number of interesting questions, both regarding the properties of the processes in question and regarding the relation between the 3 methods. We will state some of these questions later in this article. For other interesting problems see [16].

§1. Quasiregular functions and Dirichlet forms.

Recall that a quasiregular function ϕ on a connected open set $U \subset \mathbb{R}^n$ is a continuous function $\phi: U \to \mathbb{R}^n$ which is absolutely continuous on almost every straight line segment in U with partial derivatives which are locally in L^n (with respect to Lebesgue measure) and such that there exists a constant $K < \infty$ such that

$$\|\phi'(x)\|^n \leq K \cdot J_\phi(x) \quad \text{for a.a. } x \in U \tag{1.1}$$

with respect to n-dimensional Lebesgue measure m. (Such functions ϕ are also called K-quasiregular). Here $\|\phi'(x)\|$ denotes the norm of the linear map $\phi'(x): \mathbb{R}^n \to \mathbb{R}^n$ given by the matrix

$$\phi'(x) = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial\phi_n}{\partial x_1} & \cdots & \frac{\partial\phi_n}{\partial x_n} \end{bmatrix} = \left[\frac{\partial\phi_i}{\partial x_j}\right]_{i,j} \tag{1.2}$$

and $J_\phi(x) = \det(\phi'(x))$ is the Jacobian of ϕ at x.

The geometric interpretation of (1.1) is the following: The linear map $\phi'(x)$ maps the unit ball D in $\mathbb{R}^n$ into an ellipsoid E whose maximal half axis has the length $\|\phi'(x)\|$. On the other hand the volume of E is $J_\phi(x)$ times the volume of D. Therefore (1.1) means that ϕ has a (uniformly) bounded distortion in U. This was the description originally used by Rešetnjak [18] who began a systematic study of these functions in the 1960's. See [13] and [21] for more information and other references.

In the plane (i.e. if $n=2$) we can regard the quasiregular functions as generalizations of the analytic functions. This is because a function f is analytic or conjugate analytic if and only if

$$\|f'(x)\|^2 = J_f(x) \quad \text{for a.a. } x$$

i.e. if and only f is quasiregular with $K=1$.

One of the most important connections between stochastic processes and analytic functions is the following result:

THEOREM 1: If $f: W \subset \mathbb{C}^n \to \mathbb{C}$ is analytic then f maps Brownian motion in W into a time change of Brownian motion in $\mathbb{C}$. In fact, f maps any conformal martingale diffusion in W into a time change of planar Brownian motion, and this property characterizes the analytic or conjugate analytic functions among all C^2 functions from W into $\mathbb{C}$.

This theorem, which in its simplest form dates back to P. Lévy (1948) has a long history. See e.g. [1] or [20] for references and more information.

For analytic mappings $f: W \subset \mathbb{C}^n \to \mathbb{C}^m$ with $m \geq 2$ the situation is more complicated. It is no longer the case that f maps Brownian motion in W into a time change of Brownian motion in $\mathbb{C}^m$. However, it is possible to show that if $\max_{z \in W}(\text{rank } f'(z)) = m$, then there exists a conformal martingale diffusion in W which is mapped into a time change of Brownian motion in $\mathbb{C}^m$ by f. See Uböe [20].

If one tries to obtain a related result for a quasiregular function ϕ on $U \subset \mathbb{R}^n$, the first problem one encounters is the lack of smoothness of ϕ. This prohibits the use of powerful stochastic techniques like the Ito formula. Fortunately it turns out that there is a convenient alternative approach to stochastic processes which does not need much differentiability: The theory of Dirichlet forms (See [8] for an account of this theory). More explicitly, we proceed as follows (for details, see [16]):

Assume that ϕ is non-constant. Then $J_\phi > 0$ a.e. (m) in U [13] and therefore the $n \times n$ matrix

$$(1.3) \qquad S = S_\phi = J_\phi \cdot (\phi')^{-1}((\phi')^{-1})^T$$

(where T denotes transposed) is defined a.e. (m) in U. Now define the following symmetric bilinear form

$$(1.4) \qquad \mathcal{E}(u,v) = \mathcal{E}_\phi(u,v) = \tfrac{1}{2} \int_U (\nabla u)^T \cdot S \cdot \nabla v \cdot dm(x) \quad \text{for } u,v \in C_0^\infty(U)$$

where we use matrix notation (regarding ∇u as an $n \times 1$ matrix).

Regarded as a densely defined bilinear form on $H \times H$, where $H = L^2(U; J_\phi dx)$ one can prove that $\mathcal{E}_\phi$ is closable, Markovian and regular [16]. Therefore [8] there exists a Hunt process $(X_t, \Omega, P^x, \zeta)$ whose generator $A: \mathcal{D}(A) \subset H \to H$ coincides with the generator A of $\mathcal{E}_\phi$, i.e.

(1.5) $$\mathcal{E}_\phi(u,v) = -(Au,v)_H \qquad \text{for } u \in \mathcal{D}(A),\ v \in \mathcal{D}(\mathcal{E})$$

where $(\cdot,\cdot)$ denotes the inner product in H. This means that

(1.6) $$Au = \tfrac{1}{2} \cdot J_\phi^{-1} \operatorname{div}(S\nabla u) \quad \text{in the sense of distributions.}$$

Here P^x denote the probability law of $X_t(\omega)$; $w \in \Omega$, $t \geq 0$ and $\zeta \leq \infty$ is the life time of the process. Moreover, from the form of $\mathcal{E}_\phi$ we know that X_t has continuous paths and that no killing occurs inside U.

A Borel set $F \subset U$ is called <u>X_t-exceptional</u> if for a.a. (m) starting points $x \in U$ the probability that X_t hits F is zero. This is equivalent to the reguirement that

$$c_{\mathcal{E}}(F) = 0,$$

where $c_{\mathcal{E}}$ denotes the $\mathcal{E}$-capacity:

$$c_{\mathcal{E}}(W) = \inf\{(f,f)_H + \mathcal{E}(f,f);\ f \geq 1 \ \text{ on } \ W\}$$

if $W \subset U$ is open and

(1.7) $$c_{\mathcal{E}}(H) = \inf\{c_{\mathcal{E}}(W);\ W \ \text{ open, } W \supset H\}$$

for Borel sets $F \subset U$. (See [8]). In particular, we note that if $c_{\mathcal{E}}(F) = 0$ then F has zero n-dimensional Lebesgue measure. In the following the term <u>quasi-everywhere</u> (q.e.) means everywhere except possibly on a set of $c_{\mathcal{E}}$-capacity zero.

The main result is now that for quasi-all starting points $x \in U$ for X_t the function ϕ maps X_t into Brownian motion in $\mathbb{R}^n$ starting at $\phi(x)$, without time change. The main idea of the proof is the following:

Let $\hat{A} = \frac{1}{2}\Delta$ be the n-dimensional Laplacian, which coincides with the generator of Brownian motion B_t in $\mathbb{R}^n$. Choose an open set $W \subset\subset U$ (i.e. $\bar{W}$ is compact and $\bar{W} \subset U$). Then for each $y \in \phi(W)$ there exists a neighbourhood V_y of y such that each component W_j of $\phi^{-1}(V_y)$ which intersects w is a normal domain [13]. Choose $f \in C_0^2(V_y)$, i.e. f is twice continuously differentiable with compact support in V_y. Then one verifies by direct calculation based on (1.3) - (1.6) that

(1.8) $$A[(f\circ\phi)\cdot\chi_{W_j}] = (\hat{A}[f]\circ\phi)\cdot\chi_{W_j}$$

Using Dynkin's formula one can deduce from this the following:

THEOREM 2 [16]. Let $\phi: U\subset\mathbb{R}^n\to\mathbb{R}^n$ be non-constant quasiregular. Then for quasi-all $x\in U$ we have that if $X_t = X_t^x$ starts at x then the process

(1.9) $$Z_t = \begin{cases}\phi(X_t^x) & ; t<\zeta\\ B_{t-\zeta}^{\phi^*} & ; t\geq\zeta\end{cases}$$

(with the natural probability law $\hat{P}$ induced from X_t and B_t) coincides with Brownian motion in $\mathbb{R}^n$ starting from $\phi(x)$. Here

(1.10) $$\phi^*(\omega) = \lim_{t\to\zeta}\phi(X_t)$$

which exists a.s. on $\{\omega;\ \zeta(\omega)<\infty\}$.
(The notation B_s^y indicates that B_s starts at y).

This result may be regarded as the quasiregular analogue of Theorem 1. Even though the process X_t depends on ϕ, one can use the explicit description of the corresponding Dirichlet from to obtain information about X_t and then apply this in the investigation of ϕ.

For example, since X_t is not killed while in U we know that

(1.11) $$X_t\to\partial U \text{ if } t\to\zeta \quad \text{a.s. on } \{\zeta<\infty\},$$

in the sense that X_t leaves every given compact subset of U eventually, a.s. on $\{\zeta<\infty\}$. Moreover, if we let $\hat{\tau}_{\phi(U)}$ denote the first exit time from the open set $\phi(U)\subset\mathbb{R}^n$ of the n-dimensional Brownian motion Z_t, we see from Theorem 2 that

(1.12) $$\zeta\leq\hat{\tau}_{\phi(U)} \quad \text{a.s. } \hat{P}^y \text{ for all } y\in\phi(U),$$

where $\hat{P}^y$ denotes the law of Z_t starting at y.

Therefore we get the following:

COROLLARY 1. (Stochastic boundary value theorem).

Let $\phi: U \subset \mathbb{R}^n \to \mathbb{R}^n$ be quasiregular, non-constant. Assume that

(1.13) $\hat{\tau}_{\phi(U)} < \infty$ a.s. $\hat{P}^y$ for all $y \in \phi(U)$.

Then

(1.14) $\lim_{t \to \zeta} \phi(X_t)$ exists a.s. P^x for q.a. $x \in U$

Remark. A sufficient condition for (1.13) is that

$$\mathrm{Vol}(\phi(U)) < \infty$$

For other conditions (in terms of capacities) see [1].

We say that $w \in \mathbb{R}^n$ is an asymptotic value of ϕ at $z \in \partial U$ if there exists a curve γ in U terminating at z such that

$$\lim_{\substack{x \to z \\ x \in \gamma}} \phi(x) = w.$$

Unfortunately (1.14) does not directly give the existence of asymptotic values, since it is not clear that

(1.15) $\lim_{t \to \zeta} X_t \overset{\text{def}}{=} X_\zeta$ exists a.e. on $\{\zeta < \infty\}$

Sufficient conditions for (1.15) to hold are given in [16].

PROBLEM 1. Does (1.15) always hold?

If (1.13) and (1.15) hold we define the X_t-harmonic measure λ_x^X on ∂U by

$$\lambda_x^X(H) = P^x[X_\zeta \in H] \quad \text{for } H \subset \partial U.$$

COROLLARY 2. Suppose (1.13) and (1.15) hold. Then ϕ has asymptotic values at λ_x^X-almost all points $z \in \partial U$ and the set of asymptotic values constitute a non-polar set in $\mathbb{R}^n$ (in the classical sense), for q.a. $x \in U$.

In view of this result a prominent question is the following:

PROBLEM 2. Suppose (1.13) and (1.15) hold. What are the metric properties of λ_x^X? When is the closed support of λ_x^X equal to the whole boundary ∂U?

In the case $n=2$ the situation becomes much simpler. Then the generator A of $\mathcal{E}$ gets the form

$$(1.16) \qquad Af = J_\phi^{-1} \cdot \mathrm{div}(J_\phi \cdot \Gamma\Gamma^T \nabla f), \text{ where } \Gamma = (\phi')^{-1},$$

and since ϕ is quasiregular there exist constants K_1, K_2 such that

$$K_1 \le J_\phi \Gamma\Gamma^T \le K_2 \qquad \text{on } U.$$

Therefore the operator

$$(1.17) \qquad \tilde{A}f = \mathrm{div}(J_\phi \Gamma\Gamma^T \nabla f)$$

is uniformly elliptic in U if $n=2$.
Now let $\tilde{X}_t$ be the diffusion in U with generator $\tilde{A}$. Then, since

$$(1.18) \qquad A = J_\phi^{-1}\tilde{A}$$

we can represent X_t as a time change of $\tilde{X}_t$:

$$(1.19) \qquad X_t = \tilde{X}_{\alpha_t},$$

where

$$\alpha_t = \inf\{s;\ \beta_s > t\}, \quad \beta_s = \int_0^s J_\phi(\tilde{X}_r)dr$$

Now we can apply all known properties of uniformly elliptic diffusions to our process $\tilde{X}$ and then carry these over to X_t via the time change. For example, in [17] this has been used to establish boundary convergence of quasiregular functions on planar domains along the paths of η-conditional $\tilde{X}_t$-paths, i.e. pahts $\tilde{X}_t$ conditioned to exit at specified boundary points $\eta \in \partial U$. Moreover, this type of conditional convergence at η implies the classical non-tangential convergence if U is the unit disc [17, Theorem 4.1]. Using this together with known metric properties of the (elliptic) harmonic measure $\lambda^{\tilde{X}}$ of $\tilde{X}_t$ (see [12]) we obtain the following Fatou-type theorem for planar quasiregular functions:

THEOREM 3 [17]. Let U be the open unit disc D in $\mathbb{R}^2$. Suppose $\phi: D \to \mathbb{C}$ belongs to $H^p_{QR}(D)$ for some $p>0$, i.e.

$$\sup_{\tau<\zeta} E^x[|\phi(X_\tau)|^p] < \infty,$$

the sup being taken over all X-stopping times $\tau<\zeta$. (For example, this holds for all p if Area $(\phi(D))<\infty$). Then there exists $\alpha>0$ (depending only on ϕ) such that in every interval $J \subset \partial D$ there is a subset $F \subset J$ of positive α-dimensional Hausdorff measure such that the non-tangential limits of ϕ exist at every point of F.

PROBLEM 3. Can a similar Fatou-type theorem be proved in higher dimensions?

§2. Non-linear potential theory.

Using a variational argument Rešetnjak [18] proved that each component $u = \phi_k$ of a quasiregular function ϕ on a domain U in $\mathbb{R}^n$ satisfies the following non-linear (but elliptic) equation (in distribution sense)

$$\text{(2.1)} \qquad \operatorname{div}\left(\left(J_\phi^{\frac{2}{n}} \nabla u^T \Gamma\Gamma^T \nabla u\right)^{\frac{n}{2}-1} J_\phi^{\frac{2}{n}} \Gamma\Gamma^T \nabla u\right) = 0,$$

where as before $\Gamma = (\phi')^{-1}$.

A number of important properties of ϕ follow from this by using the general theory for solutions u of elliptic quasilinear equations. For example, ϕ is Hölder continuous and satisfies the Harnack inequalities (Serrin [19]) and therefore ϕ satisfies the Liouville property (Rešetnjak [18]).

In comparison with (2.1) one could say that the stochastic approach is based on the fact that each component $u=\phi_k$ of ϕ is a solution of the in general non-elliptic (but linear) equation

$$\text{(2.2)} \qquad \tilde{A}u = \operatorname{div}(J_\phi \Gamma\Gamma^T \nabla u) = 0 \quad , \quad \Gamma = (\phi')^{-1}$$

To see that (2.2) holds we apply (1.8) to the case when the function $f \in C_0^\infty(V_y)$ locally at y has the form

$$f(x) = x_k \qquad \text{if} \quad x = (x_1, \ldots, x_n)$$

Then $\frac{1}{2}\Delta f = \hat{A}f = 0$ near y and therefore $A[f\circ\phi] = A[\phi_k] = 0$ near $\phi^{-1}(y)$. Thus (2.2) holds.

Looking at ϕ as a solution of the non-linear equation (2.1) makes it natural to adopt methods from non-linear potential theory in the investigation of ϕ. In [11], [12] one studies the extremals u of variational integrals of the form

$$(2.3) \qquad I_F(u) = \int F(x,\nabla u)dm(x)$$

where the variational kernel $F(x,h)\colon U \times \mathbb{R}^n \to \mathbb{R}$ satisfies certain conditions. This is equivalent to studying weak solutions u of the equation

$$(2.4) \qquad \operatorname{div}(\nabla_h F(x,\nabla u)) = 0,$$

which is a generalization of (2.1).
One can then introduce F-harmonic measure $\omega_F(x,H)$ for $x \in U$, $H \subset \partial U$ as follows:

$$\omega_F(x,H) = \inf\{v(x);\ v \in \mathcal{V}\},$$

where $\mathcal{V}$ is the family of all superextremals v of the variational integral (2.3) with the property that

$$\varliminf_{x\to y} v(x) \geq \chi_H(y)$$

For properties of ω_F see [12].

PROBLEM 4. Choose $F = (J_\phi^{\frac{2}{n}} h^T \Gamma\Gamma^T h)^{\frac{n}{2}}$, where $\Gamma = (\phi')^{-1}$. What are the relations between F-harmonic measure ω_F and X_t-harmonic measure λ^X?

§3. Degenerate linear elliptic equations and A_p-weights.

As already noted the linear equation (2.2) is uniformly elliptic if $n=2$. But for $n>2$ the equation is degenerate elliptic. In [5], [6] and [7] the solutions of a class $\mathfrak{F}$ of degenerate elliptic equations are studied (with non-stochastic methods). This class $\mathfrak{F}$ includes equations of the form (2.2) in the case when ϕ is quasiconformal (i.e. quasiregular and a homomorphism) on a neighbourhood of $\bar{U}$. It is natural to ask if the results obtained there can be extended to the general case

when ϕ is only assumed to be quasiregular on U. This question can be approached by studying A_p-properties of J_ϕ:
Recall that if $1<p<\infty$ then a function $w(x) \geq 0$ on $U \subset \mathbb{R}^n$ is an A_p-weight i.e. a member of the class A_p (with respect to Lebesgue measure m) if

$$\sup_{Q \subset U}\left(\frac{1}{m(Q)} \int_Q w\,dx\right)\left(\frac{1}{m(Q)} \int_Q \left(\frac{1}{w}\right)^{1/(p-1)} dx\right)^{p-1} < \infty, \tag{3.1}$$

where Q denotes a cube in U.
A function $w(x) \geq 0$ is called an A_∞-weight ($w \in A_\infty$) if there exist positive constants K, δ such that

$$\frac{\int_E w\,dx}{\int_Q w\,dx} \leq K \cdot \left(\frac{m(E)}{m(Q)}\right)^\delta \tag{3.2}$$

for all cubes $Q \subset U$ and all Borel sets $E \subset Q$.

The concept was first introduced by B. Muckenhoupt [14] who discovered that boundedness in the $w(x)$-weighted L^p-norm (i.e. in $L^p(wdm)$) of the maximal function operator was equivalent to the A_p-condition on the weight w. Subsequently many other interesting connections to the A_p-property have been found. For example, B. Dahlberg proved that the classical harmonic measure for a Lipschitz domain D in $\mathbb{R}^n$ is absolutely continuous with respect to surface measure σ on ∂D and the Radon-Nikodym derivative is an A_2-weight with respect to σ [3]. For more information on A_p-weights see [2], [9] or [14].

In [5], [6] and [7] equations of the form

$$\operatorname{div}(S\nabla u) = 0$$

are studied under the following assumption on the symmetric semielliptic $n\times n$ matrix $S = S(x)$:

(3.3) The lowest eigenvalue $\lambda(x)$ of $S(x)$ is an A_2-weight

(As mentioned above this condition can be dispensed with if

$$S = J_\phi(\phi')^{-1}((\phi')^{-1})^T,$$

where $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is quasiconformal.)

In general, if ϕ is quasiregular on U and

(3.4) $$S = J_\phi(\phi')^{-1} \cdot ((\phi')^{-1})^T,$$
then we have

$$\lambda(x) \sim J_\phi(x)^{1-\frac{2}{n}}$$

(where $a \sim b$ means that $\frac{a}{b}$ and $\frac{b}{a}$ are bounded in U). In view of this one might ask if $J_\phi^{1-\frac{2}{n}} \in A_2$, or more generally what kind of A_p-properties (if any) J_ϕ has. Gehring [10, Theorem 1] has proved that if ψ is quasiconformal then J_ψ is an A_∞-weight. In fact, combining Gehring's result with general theory (see p. 249 in [2]) it follows that there exists $p<\infty$ (depending on ψ) such that $J_\psi \in A_p$. On the other hand the example (where $0<a<1$)

$$\psi(x) = |x|^{a-1}x; \; x \in \mathbb{R}^n$$

(which is K-quasiconformal with $K=a^{1-n}$) shows that for any given $q<\infty$ there exists a quasiconformal ψ with $J_\psi \notin A_q$. The only point where the A_q-condition breaks down even locally is the origin, and it is natural to ask if the set of such 'bad' points is always small, for example has volume 0:

PROBLEM 5. Let $\phi\colon U \to \mathbb{R}^n$ be quasiregular, non-constant. Does there exist a set $N \subset U$ with $m(N) = 0$ and such that $J^{1-\frac{2}{n}}$ is locally in A_2 outside N?

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