

Riemann integration on Brownian paths

by

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Abstract

In this paper we prove that certain stochastic integrals in \mathbb{C} are stable under different approximation methods. A kind of holomorphic-measurability condition and good L^p estimates are involved here. The result is used to prove that Ito and Stratonovich integrals coincide on a fairly large class of processes.

Introduction

Stochastic integrals are usually defined as limits of Riemann sums with chosen approximation points. A rather awkward property is that the value of the integral depends on the method of approximation.

If we choose to approximate by the values at the left end-points, we get Ito-integrals. Ito-integrals are useful in analysis because they are martingales and various good estimates are available. They are, however, not geometrically well behaved and are subject to a strange calculus.

The midpoint approximations are called Stratonovich-integrals. They are well behaved but unfortunately, they do not give good estimates.

It is the purpose of this paper to prove that in many cases the stochastic integrals are independent of the methods of approximation.

Some basic concepts and notation.

Let $W_t: \Omega \rightarrow \mathbb{C}$ denote Brownian motion in the complex plane. We want to define the integral

$$\int_0^T z_t dW_t$$

where $Z_t: \Omega \rightarrow \mathbb{C}$ is a stochastic process.

From a Riemann integration point of view the following definition is natural.

Let $0 = t_1 < t_1^* < t_2 < \dots < t_k^* < t_{k+1} = T$ be a partition of the interval $[0, T]$. We call a process $Z_t: \Omega \rightarrow \mathbb{C}$ Wiener-Riemann integrable on $[0, T]$ if there exists a random variable IZ_T s.t.

$$\sum_{j=1}^k Z_{t_j^*} (W_{t_{j+1}} - W_{t_j}) \rightarrow IZ_T \text{ in measure as } \max_j |t_{j+1} - t_j| \rightarrow 0$$

A quite strong measurability condition is needed. We call a process

\mathcal{F}_t^P -analytically adapted if for each t , Z_t is the L^P -closure of $\mathcal{C}_t = \{f_1(W_{t_1})f_2(W_{t_2}) \dots f_r(W_{t_r})\}$ where r is some positive integer f_1, f_2, \dots, f_r are polynomials and $0 < t_1, t_2, \dots, t_r < t$.

This last definition is motivated by the following: If we replace polynomials with continuous functions and L^P -limits by limits in measure, we get a definition of \mathcal{F}_t -measurability, well known from Ito-analysis.

We aim at proving the following.

Theorem 1

If for each $t \in [0, T]$ Z_t is \mathcal{F}_t^2 -analytically adapted, and there exist constants $C_1 < \infty$, $p > 2$, $C_2 < \infty$, $\alpha > 0$ s.t. for all $(s, t) \in [0, T]^2$ $E|Z_t|^p < C_1$ and $E|Z_t - Z_s|^2 < C_2|t-s|^\alpha$, then Z_t is WR-integrable on $[0, T]$.

I. The problem

We want to give meaning to the symbol $\int_0^T X_t(\omega) dB_t(\omega)$ where X_t is a process on Ω .

The integral cannot be defined in any ordinary sense since the paths of Brownian motion are of unbounded variation almost surely. We try to define it in terms of an approximation procedure. We do a partition $0 = t_1 < t_1^* < t_2 \dots < t_k^* < t_{k+1} = T$ of the interval $[0, T]$. A reasonable approximation would then be

$$\sum_{j=1}^k X_{t_j^*}(\omega) (B_{t_{j+1}}(\omega) - B_{t_j}(\omega)).$$

The problem is that the limit of this expression as the partition gets finer, may depend on the choice of t_j^* . If $t_j^* = t_j$, we get an Ito-integral. This limit can be shown to exist if we have reasonable conditions on X_t . In this paper we will study the difference between this choice and the others, i.e.

$$\sum_{j=1}^k (X_{t_j^*} - X_{t_j})(B_{t_{j+1}} - B_{t_j})$$

We would like this to approach zero as the partition gets finer. Unfortunately, this fails in general. Put $t_j^* = t_{j+1}$, and let $X_t = B_t$ then

$$\begin{aligned}
 E \left| \sum_{j=1}^k (X_{t_{j+1}} - X_{t_j})(B_{t_{j+1}} - B_{t_j}) \right| &= \sum_{j=1}^k E (B_{t_{j+1}} - B_{t_j})^2 \\
 &= \sum_{j=1}^k E (B_{t_{j+1} - t_j})^2 \\
 &= \sum_{j=1}^k |t_{j+1} - t_j| = T
 \end{aligned}$$

In fact, it is not hard to prove that $\sum_{j=1}^k (X_{t_{j+1}} - X_{t_j})(B_{t_{j+1}} - B_{t_j}) \rightarrow T$ in L^2 in this case, and that is hardly what we want to happen.

As we shall soon see, much more can be said when we look at processes in the complex plane. There are, however, a few real cases where we can prove something positive. We first look into this.

Example 1.

If $X_t: \Omega \rightarrow \mathbb{R}$ is measurable and there exist constants $\alpha > 1$ and $C < \infty$ s.t. $E|X_t - X_s|^2 < C|t-s|^\alpha$ then

$$\sum_{j=1}^k (X_{t_{j+1}} - X_{t_j})(B_{t_{j+1}} - B_{t_j}) \rightarrow 0 \text{ in } L^1 \text{ as } \max_j |t_{j+1} - t_j| \rightarrow 0$$

Proof.

$$\begin{aligned}
 E \left| \sum_{j=1}^k (X_{t_{j+1}} - X_{t_j})(B_{t_{j+1}} - B_{t_j}) \right| &< \sum_{j=1}^k E \left| (X_{t_{j+1}} - X_{t_j})(B_{t_{j+1}} - B_{t_j}) \right| \\
 &< \sum_{j=1}^k (E|X_{t_{j+1}} - X_{t_j}|^2)^{\frac{1}{2}} (E|B_{t_{j+1}} - B_{t_j}|^2)^{\frac{1}{2}} \\
 &< C \sum_{j=1}^k |t_{j+1} - t_j|^{\alpha/2} |t_{j+1} - t_j|^{\frac{1}{2}} \\
 &< C \max_j |t_{j+1} - t_j|^{\alpha/2 - \frac{1}{2}} \sum_{j=1}^k |t_{j+1} - t_j| \\
 &= CT \max_j |t_{j+1} - t_j|^{\alpha/2 - \frac{1}{2}} \rightarrow 0
 \end{aligned}$$

Note that the paths of such processes are only slightly better than Brownian paths where we have $\alpha=1$. A typical example of this kind of process is $X_t = \int_0^t b(B_s, s) ds$ where b is bounded.

The next example is rather artificial, but it illustrates what we are going to do later.

Example 2.

Let $X_t : \Omega \rightarrow \mathbb{R}$ be a process s.t. $E[|X_t - X_s|^4] < C|t-s|^2$ and assume that there exist a constant α , $0 < \alpha < 1$ s.t. for each t , X_t is

$\mathcal{F}_{\alpha t}$ -measurable, then $\sum_{j=1}^k (X_{t_j^*} - X_{t_j})(B_{t_{j+1}} - B_{t_j}) \rightarrow 0$ in L^2 as

$$\max_j |t_{j+1} - t_j| \rightarrow 0$$

Idea of proof

Write

$$E \left| \sum_j (X_{t_j^*} - X_{t_j})(B_{t_{j+1}} - B_{t_j}) \right|^2 = \sum_{i,j} E (X_{t_i^*} - X_{t_i})(B_{t_{i+1}} - B_{t_i})(X_{t_j^*} - X_{t_j})(B_{t_{j+1}} - B_{t_j})$$

Because X_t is $\mathcal{F}_{\alpha t}$ -measurable with $\alpha < 1$, we will in most terms where $i > j$ have $(B_{t_{i+1}} - B_{t_i})$ independent from the rest. Since $E(B_{t_{i+1}} - B_{t_i}) = 0$ the whole term will be zero. The same thing happens when $i < j$, and the nonzero terms in the sum are so few that the whole sum can be shown to be small. □

The conditions in this example can be weakened, but this need not concern us. The interesting point here is that if we have an $\tilde{\mathcal{F}}_t^2$ -analytically adapted process, and we estimate this by a double sum as above, the non-diagonal terms will always be identically zero. We now look into this.

II Complex moments of 2-dimensional Brownian motion

We let $W_t = B_{1t} + iB_{2t}$ where B_{1t} and B_{2t} are independent Brownian motions. We want to estimate expressions of the form

$\int_j (Z_{t_j}^* - Z_{t_{j+1}})(W_{t_{j+1}} - W_{t_j})$. To do this we start with some lemmas

Lemma 1

If $0 < t < \infty$ and m is any nonzero positive integer, then

$$E[W_t^m] = 0$$

Proof.

This is obvious from martingale theory, but there is also an elementary way to see it. Write $E[W_t^m] = E[(B_{1t} + iB_{2t})^m]$ and multiply out by the binomial theorem. Use independence and the relations $E[B_t^{2k}] = \frac{2^k t^k}{2^k k!}$ to write it out explicitly. It will look a bit messy but almost everything cancels out and the Lemma follows easily.

□

Lemma 2.

If $0 < t_1, t_2, \dots, t_k < \infty$ n_1, n_2, \dots, n_k are positive integers not all equal to zero, then

$$E(W_{t_1}^{n_1} W_{t_2}^{n_2} \dots W_{t_k}^{n_k}) = 0$$

Proof.

By induction on k . Case $k=1$ is lemma 1. Assume case $k-1$ proved. We can assume that $t_1 < t_2 < \dots < t_k$ and that not all n_1, n_2, \dots, n_{k-1} are equal to zero.

$$\begin{aligned}
 E(W_{t_1}^{n_1} W_{t_2}^{n_2} \dots W_{t_k}^{n_k}) &= E(W_{t_k}^{-W_{t_{k-1}}} + W_{t_{k-1}})^{n_k} W_{t_{k-1}}^{n_{k-1}} \dots W_{t_1}^{n_1} \\
 &= \sum_{a=0}^{n_k} \binom{n_k}{a} E[(W_{t_k}^{-W_{t_{k-1}}} + W_{t_{k-1}})^a] E[W_{t_{k-1}}^{a+n_{k-1}} \dots W_{t_1}^{n_1}]
 \end{aligned}$$

By hypothesis all $E[W_{t_{k-1}}^{a+n_{k-1}} \dots W_{t_1}^{n_1}] = 0$ and this proves the lemma. □

Lemma 3.

Let $0 < t_1 < t_2 < \dots < t_k < \infty$ and let n_1, n_2, \dots, n_k be any positive integers. If Y is \mathcal{F}_{t_1} -measurable with $E|Y|^p < \infty$ $p > 1$, then

$$E(W_{t_k}^{n_k} W_{t_{k-1}}^{n_{k-1}} \dots W_{t_1}^{n_1} Y) = E(W_{t_1}^{n_1 + n_2 + \dots + n_k} Y)$$

Proof

Induction on k again. Case $k=1$ is trivial. Assume case $k-1$ proved, then

$$\begin{aligned}
 &E((W_{t_k}^{-W_{t_{k-1}}} + W_{t_{k-1}})^{n_k} W_{t_{k-1}}^{n_{k-1}} \dots W_{t_1}^{n_1} Y) \\
 &= \sum_{a=0}^{n_k} \binom{n_k}{a} E(W_{t_k}^{-W_{t_{k-1}}} + W_{t_{k-1}})^{n_k - a} E(W_{t_{k-1}}^{a+n_{k-1}} \dots W_{t_1}^{n_1} Y)
 \end{aligned}$$

By lemma 2 $E(W_{t_k}^{-W_{t_{k-1}}} + W_{t_{k-1}})^{n_k - a} = 0$ unless $a = n_k$ so only this term remains

$$= E(W_{t_{k-1}}^{n_k + n_{k-1}} \dots W_{t_1}^{n_1} Y) = E(W_{t_1}^{n_1 + n_2 + \dots + n_k} Y) \text{ by hypothesis}$$

□

Lemma 3 provides an easy proof of the following lemma, which is the key to this theory.

Lemma 4.

Let $0 < t_1, t_2, \dots, t_k < \infty$ and let $0 < s < t$. If Y is \mathcal{F}_s -measurable with $E|Y|^p < \infty$ $p > 1$, and ϕ is any holomorphic polynomial in k complex variables, then

$$E[\phi(W_{t_1}, W_{t_2}, \dots, W_{t_k})(W_t - W_s)Y] = 0$$

Proof.

We look at each term in the polynomial expression and put all \mathcal{F}_s -measurable parts together with Y to form a \tilde{Y} . Then by lemma 3 we push all times down to s . Two equal parts will cancel each other.

□

Proposition 1.

Let Z be \mathcal{F}_r^2 -analytically adapted for some $r > 0$ and let $0 < s < t$. If Y is \mathcal{F}_s -measurable with $E|Y|^2 < \infty$ then $E[Z(W_t - W_s)Y] = 0$

Proof.

Given $\eta > 0$ choose \tilde{Z} in \mathcal{C}_r s.t. $E|Z - \tilde{Z}|^2 < \eta$.

$$|E(Z(W_t - W_s)Y)| = |E[(Z - \tilde{Z})(W_t - W_s)Y] + E(\tilde{Z}(W_t - W_s)Y)|$$

By lemma 4 the last term is zero, so by Hölder's inequality

$$< [E|Z - \tilde{Z}|^2]^{\frac{1}{2}} [E(|Y|^2)]^{\frac{1}{2}} [E(|W_t - W_s|^2)]^{\frac{1}{2}} < C\eta^{\frac{1}{2}}$$

and this proves the proposition since the constant does not depend on η .

□

We now proceed to define our integral. The definition will be consistent with the definition of the Ito-integral. Usually the Ito-integral is defined by some cutting and step-function procedure. Since this will destroy analytic adaptedness of processes, we have to use a slightly non-standard approach. Because of this we give a short discussion of Ito-integration.

III - The Ito-integral.

The definition of the Ito-integral is based on the following simple fact, the so-called Ito-isometry, here stated in a complex version.

Ito-isometry

If Y_{t_j} is \mathcal{F}_{t_j} -measurable and Y_{t_j} is in L^2 then

$$E\left[\left|\sum_j Y_{t_j} (W_{t_{j+1}} - W_{t_j})\right|^2\right] = 2E\left[\sum_j |Y_{t_j}|^2 |t_{j+1} - t_j|\right]$$

Proof.

$$E\left[\left|\sum_j Y_{t_j} (W_{t_{j+1}} - W_{t_j})\right|^2\right] = E\left[\sum_{i,j} Y_{t_i} (W_{t_{i+1}} - W_{t_i}) \overline{Y_{t_j} (W_{t_{j+1}} - W_{t_j})}\right]$$

Unless $i=j$ the term is zero by independence since $E(W_{t_{i+1}} - W_{t_i}) = 0$

$$= \sum_j E\left[|Y_{t_j}|^2 |W_{t_{j+1}} - W_{t_j}|^2\right] = \sum_j E\left[|Y_{t_j}|^2\right] E\left[|W_{t_{j+1}} - W_{t_j}|^2\right] = 2 \sum_j E\left[|Y_{t_j}|^2\right] |t_{j+1} - t_j|$$

□

Estimation lemma.

If Z_t is \mathcal{F}_t -measurable with $E[|Z_t - Z_s|^2] < C|t-s|^\alpha$
 $\alpha > 0$, $0 = t_1 < t_2 \dots < t_k = T$ $0 = s_1 < s_2 \dots < s_r = t$ are two partitions of the interval $[0, T]$ then

$$E[|\sum_{j=1}^k Z_{t_j} (W_{t_{j+1}} - W_{t_j}) - \sum_{i=1}^r Z_{s_i} (W_{s_{i+1}} - W_{s_i})|^2] \\ < C(T, Z_t) \max\{\max_i |s_{i+1} - s_i|^\alpha, \max_j |t_{j+1} - t_j|^\alpha\}$$

Proof.

This is just a standard estimate, and we will not go deeply into it. Similar proofs can be found in any standard textbook on Ito-integrals. The idea is just to use a refinement of the partitions to get a single sum, and then use the Ito-isometry on this sum. \square

Definition of the Ito-integral

If Z_t is \mathcal{F}_t -measurable with $E[|Z_t - Z_s|^2] < C|t-s|^\alpha$ $\alpha > 0$ then there exist a random variable IZ_T s.t.

$$\sum_j Z_{t_j} (W_{t_{j+1}} - W_{t_j}) \rightarrow IZ_T \text{ in } L^2 \text{ as } \max_j |t_{j+1} - t_j| \rightarrow 0$$

Proof.

Choose any sequence of partitions s.t. $\max_j |t_{j+1} - t_j| \rightarrow 0$. Then by the estimation lemma above, the corresponding sums form a Cauchy-sequence in L^2 converging to a random variable IZ_T . The estimate also proves that the closeness of approximation only depends on

$$\max_j |t_{j+1} - t_j|$$

\square

IV The Wiener-Riemann integral.

We are now in a position to prove our main theorem.

Theorem 1.

If for each $(s, t) \in [0, T]^2$, Z_t is \mathcal{F}_t^2 -analytically adapted with $E[|Z_t|^p] < C$ $p > 2$ and $E[|Z_t - Z_s|^2] < C|t-s|^\alpha$ $\alpha > 0$ then Z_t is WR-integrable on $[0, T]$.

Proof.

It is enough to estimate

$$E[|\sum_j (Z_{t_{j+1}}^* - Z_{t_j}) (W_{t_{j+1}} - W_{t_j})|^2]$$

$$= E[\sum_{i,j} (Z_{t_{i+1}}^* - Z_{t_i}) (W_{t_{i+1}} - W_{t_i}) \overline{(Z_{t_{j+1}}^* - Z_{t_j}) (W_{t_{j+1}} - W_{t_j})}]$$

If $i > j$ the term is zero by proposition 1 since $t_{j+1} < t_i$. The case $i < j$ is similar. So we are left to estimate

$$\sum_j E[|Z_{t_{j+1}}^* - Z_{t_j}|^2 |W_{t_{j+1}} - W_{t_j}|^2]$$

It suffices to show that if $r < s < t$ then

$$E[|Z_s - Z_r|^2 |W_t - W_r|^2] < C |t-r|^\gamma \quad \gamma > 1.$$

Put $A = \{\omega \mid |W_t - W_r| > |t-r|^a\}$ By Tshebysjev's inequality $P(A) < C |t-r|^{1-2a}$. Choose $a < \frac{1}{2}$ s.t. $2a + \alpha > 1$.

Then we consider the integrals

$$\int_A |Z_s - Z_r|^2 |W_t - W_r|^2 dP + \int_{A^c} |Z_s - Z_r|^2 |W_t - W_r|^2 dP$$

In the first integral we use the uniform boundedness of Z_t in L^p and Hölder's inequality. The second expression has a trivial estimate. This proves the bound with $\gamma = \min\{1+(1-2a)/C, 2a+\alpha\}$ □

Remark 1.

In its simplest form, the theory of the Ito-integral is a kind of L^2 -theory and the conditions \mathcal{Y}_t^2 -analytically adapted and $E[|Z_t - Z_s|^2] < C |t-s|^\alpha \quad \alpha > 0$ seems to be quite natural. If we want to integrate more than once, however, the condition $E[|Z_t|^p] < C \quad p > 2$

is quite awkward. It is very hard to prove L^p estimates with $p > 2$ on Ito-integrals, unless you are able to carry out the integration explicitly. In the next section we are going to study the problem locally, and then we can replace the condition with $E[|Z_{t \wedge \tau}|^p] < C$ where τ is a stopping time.

Remark 2.

It would be much more satisfactory if we could replace all the conditions in the theorem with limits in measure. It is reasonable to conjecture that a theorem like this should be true, but it seems to require a different kind of proof. Although the expected values of the non-diagonal terms are zero, the terms themselves are not necessarily small. It seems to be quite difficult to control these terms without good L^p estimates.

If we restrict ourselves to Ito and Stratonovich integration, some more machinery is available and to some extent we can extend the theorem to processes that do not have good L^p estimates. We will do this in section 6.

Remark 3.

As is clear from example 1 in section 1, we can prove WR-integrability under weaker assumptions than analytic adaptiveness. Without much work we can prove a generalization of theorem 1.

Theorem 2.

If Z_t satisfies the conditions of theorem 1 and Y_t is $\tilde{\mathcal{F}}_t$ -measurable with $|Y_t(\omega) - Y_s(\omega)| < C|t-s|^\beta$ a.s. where $\beta > \frac{1}{2}$, then the product $Y_t Z_t$ is WR-integrable.

Proof.

Put $H_t = Y_t Z_t$. It is clear from the conditions that H_t is Ito-integrable in the sense mentioned before, so it is enough to prove that

$$E \left| \sum_j (H_{t_{j+1}}^* - H_{t_j}) (W_{t_{j+1}} - W_{t_j}) \right| \rightarrow 0$$

To do this it is enough to estimate the expressions

$$E \left| \sum_j (Z_{t_{j+1}}^* - Z_{t_j}) (W_{t_{j+1}} - W_{t_j}) Y_{t_j} \right|$$

$$\sum_j E \left| (Y_{t_{j+1}}^* - Y_{t_j}) (W_{t_{j+1}} - W_{t_j}) Z_{t_j} \right|$$

$$\sum_j E \left| (Y_{t_{j+1}}^* - Y_{t_j}) (W_{t_{j+1}} - W_{t_j}) (Z_{t_j}^* - Z_{t_j}) \right|$$

The first expression we compare with its L^2 -norm and write it out in a double sum just as in theorem 1. Since Y_{t_j} is \mathcal{F}_{t_j} -measurable the non-diagonal terms are still zero, and since the Y_{t_j} -s are uniformly bounded the term goes to zero as before. To estimate the two remaining terms we use Hölder's inequality and the fact that the L^2 -norms of Z_{t_j} -s are uniformly bounded. The theorem follows easily from this. □

From the proof, it is easy to see that the conditions on Y_t can be weakened if the analytic part is "very nice" i.e. if the analytic part is in L^p all p , then it is sufficient to have something like $E|Y_t - Y_s|^3 < C|t-s|^3$; the product is still WR-integrable. If you look at a stochastic differential equation

$$Z_t = Z_0 + \int_0^t \phi(s) \Phi(Z_s) dW_s + \int_0^t b(s, Z_s) ds$$

and set up an iteration sequence $Z_k = Z_0 + \int_0^t \phi(s) \Phi(Z_{k-1}) dW_s + \int_0^t b(s, Z_{k-1}) ds$ it is possible

to carry out the integrations in a semi-explicit way to see that all iterates are WR-integrable by the result above. You need ϕ a polynomial, and good boundedness conditions on ϕ and b .

V Localization.

With a few minor adjustments it is easy to see that most of what we have done can be carried out locally. We need to use Dynkin's formula. For a discussion of this formula see Øksendal (1) p. 91.

Lemma.

If τ is a stopping time dominated a.s. by the first exit time of W_t from a compact set K , and m is a nonzero positive integer, then $E[W_\tau^m] = 0$.

Proof.

Put $f(z) = z^m$ and cut f by a C^2 function outside K . Then use Dynkin's formula. □

Since Brownian motion is a strong Markov process, $B_{\tau_1} - B_{\tau_2}$ behaves as $B_{\tau_1 - \tau_2}$. From this it is easy to see that lemma 2,3 and 4 follows as before.

We also make the following observation. If $\tau < t$ a.s. and is bounded as above, then by Dynkin's formula on $f(x) = x^{2m}$ cut by a C^2 function:

$$\begin{aligned} E[B_\tau^{2m}] &= E\left[\int_0^\tau 2m(2m-1)B_s^{2m-2} ds\right] \\ &< E\left[\int_0^t 2m(2m-1)B_s^{2m-2} ds\right] \\ &= 2E[B_t^{2m}] \end{aligned}$$

So essentially, it is valid to estimate the moments by sup-estimates on the stopping time. This proves that we have an estimate

$$E\left[\left|\sum_j Y_{t_j \wedge \tau} (W_{t_{j+1} \wedge \tau} - W_{t_j \wedge \tau})\right|^2\right] < C \sum_j E[|Y_{t_j \wedge \tau}|^2] |t_{j+1} - t_j|$$

and this estimate makes it possible to define a local Ito-integral in exactly the same way as before.

Now consider all definitions locally by replacing all t by $t \wedge \tau$ on appropriate places. Then we clearly have the following

Theorem 3.

Let τ be a stopping time bounded a.s. by the first exit time of W_t from some compact set. If for each $(s,t) \in [0,T]^2$ $Z_{t \wedge \tau}$ is $\mathcal{F}_{t \wedge \tau}^2$ analytically adapted, $E[|Z_{t \wedge \tau} - Z_{s \wedge \tau}|^2] < C|t-s|^\alpha$ $\alpha > 0$ and $E[|Z_{t \wedge \tau}|^p] < C$ then $Z_{t \wedge \tau}$ is WR-integrable on the interval $[0, T \wedge \tau]$. $p > 2$

Locally we are able to prove a theorem about the integral.

Theorem 4.

Assume that $Z_{t \wedge \tau}$ satisfies the conditions in theorem 3, and let $I_{S \wedge \tau} Z_{t \wedge \tau}$ denote the WR-integral of $Z_{t \wedge \tau}$ on $[0, S \wedge \tau]$ $S < T$. If τ is a stopping time s.t. $E[|I_{S \wedge \tau} Z_{t \wedge \tau}|^p] < C$ $p > 2$, then the stochastic pro-

cess $IZ_{S \wedge \tau}$ defined on the interval $[0, T \wedge \tau]$ is WR-integrable on this interval.

Proof.

From the isometry we used to define the Ito-integral it is easy to see that we have a limiting estimate

$$E[|IZ_{S \wedge \tau} - IZ_{R \wedge \tau}|^2] < C \int_R^S E[|Z_{t \wedge \tau}|^2] dt < C|S-R|$$

To satisfy the conditions in theorem 3, we only have to prove that the process $IZ_{S \wedge \tau}$ is $\mathcal{F}_{S \wedge \tau}^2$ -analytically adapted. Fix s , and choose a partition s.t. $\sum_j Z_{t_j^*} (W_{t_{j+1} \wedge \tau} - W_{t_j \wedge \tau})$ is close to $IZ_{S \wedge \tau}$ in L^2 . Since we have a finite sum and each term can be approximated by processes in $\mathcal{C}_{S \wedge \tau}$, it is easy to see that $IZ_{S \wedge \tau}$ can be approximated in L^2 by such processes. This proves the theorem. □

In the next section we will have to use some theory on semi-martingales and quadratic variations. Since it would take too long to explain all of this here, we refer to Kunita [2] for details.

VI. Ito and Stratonovich integration

Ito and Stratonovich integrals can be defined under much weaker assumptions than the ones discussed earlier in this paper. As in Kunita [2] we let $\int_0^T X_t dB_t$ denote Ito-integrals and we let $\int_0^T X_t \circ dB_t$ denote Stratonovich integrals. Both can be defined whenever X_t is a continuous semi-martingale, and the relation between the two is that

$$\int_0^T X_t dB_t = \int_0^T X_t \circ dB_t + \frac{1}{2} \langle X, B \rangle_T$$

where $\langle X, B \rangle_T$ denotes the joint quadratic variation of the processes

X_t and B_t . It is easy to see that there is a corresponding complex version of this relation.

Quadratic variations of complex processes.

We let ξ_t defined on $[0, T]$ be a continuous semimartingale in \mathbb{C} i.e. $\xi_t = X_t + iY_t$ where X_t and Y_t are real semimartingales. We consider the space of all such processes with the topology of uniform convergence in probability. (i.e) $\xi_t^n \rightarrow \xi_t$ if $P(\sup_{t < T} |\xi_t - \xi_t^n| > \epsilon) \rightarrow 0$.

The real quadratic variations $\langle X \rangle_t$ and $\langle Y \rangle_t$ are well defined, and the mapping $X \rightarrow \langle X \rangle_t$ is continuous on the subspace of local martingales. We define the quadratic variation of a process ξ_t by

$$\langle \xi \rangle_t = \langle X \rangle_t + \langle Y \rangle_t$$

We define the joint quadratic variation of a pair of processes (ξ_t, η_t) by

$$\langle \xi, \eta \rangle_t = \frac{1}{4} [\langle \xi + \eta \rangle_t - \langle \xi - \eta \rangle_t + i \langle i\xi - \eta \rangle_t - i \langle i\xi + \eta \rangle_t]$$

With this definition it is easy to see that the mappings $\xi \rightarrow \langle \xi, \eta \rangle_t$ with η a fixed local martingale are continuous on the subspace of local martingales. There is a much simpler way to view the joint quadratic variation. Actually we will have $\langle \xi, \eta \rangle_t =$ limit in

probability of $\sum_{j=1}^k (\xi_{t_{j+1}} - \xi_{t_j})(\overline{\eta_{t_{j+1}} - \eta_{t_j}})$ where $0 = t_1 < t_2 < \dots < t_{k+1} = t$ is a partition of $[0, t]$.

From the real variable theory and the definitions above, it is a straightforward matter to see that the above limit exists and is consistent with all other definitions. The point with all this is

$$\text{that } \int_0^T Z_t dW_t = \int_0^T Z_t \circ dW_t + \frac{1}{2} \langle Z, \bar{W} \rangle_T.$$

From this expression the following corollary is immediate.

Corollary 1.

If Z_t^n is a sequence of WR-integrable continuous local martingales, and Z_t is a local martingale s.t. $Z_t^n \rightarrow Z_t$ uniformly in probability, then the Ito integral $\int_0^T Z_t^n dW_t$ equals the Stratonovich integral $\int_0^T Z_t \circ dW_t$.

Remark.

Actually it is enough that Z_t is a semimartingale and that Z_t^n converges to the martingale part of Z_t .

Theorem 5.

If for each $(s,t) \in [0,T]^2$ Z_t is \mathcal{F}_t^p -analytically adapted $p > 2$ with $E[|Z_t|^q] < C$ $q > 2$ and $E[|Z_t - Z_s|^2] < C|t-s|^\alpha$ $\alpha > 0$, then the stochastic process IZ_t is well defined on $[0,T]$ and repeated Ito and Stratonovich integrals are all equal.

Proof.

Put $I^n Z_t = \sum_{j>0} Z_{\frac{j}{n}} (W_{t_{j+1}} - W_{t_j})$ where $t_j = \begin{cases} \frac{j}{n} & \text{if } \frac{j}{n} < t \\ t & \text{if } \frac{j}{n} > t \end{cases}$

It is straightforward to see that each $I^n Z_t$ satisfies the conditions in theorem 1, so each $I^n Z_t$ are WR-integrable. We also know that $I^n Z_t$ are martingales and that $I^n Z_T \rightarrow IZ_T$ in L^2 . By the martingale convergence theorem $I^n Z_t \rightarrow IZ_t$ uniformly in probability, so theorem 5 follows from corollary 1.

We introduce processes $I^n I^n Z_t$ in the same way. This again will be WR-integrable. If we can prove that $I^n I^n Z_T \rightarrow IIZ_T$ in L^2 then $\int_0^T IIZ_t dW_t = \int_0^T IIZ_t \circ dW_t$ will follow as above.

Given $\epsilon > 0$ we can find N_0 s.t. for all $n > N_0$

$$E|IZ_T - I^n Z_T|^2 < \frac{\epsilon}{64T}$$

Then for any partition $0 < t_1 < t_2 < \dots < t_k = T$

$$\begin{aligned} & E \left| \sum_j (IZ_{t_j} - I^n Z_{t_j})(W_{t_{j+1}} - W_{t_j}) \right|^2 \\ &= 2 \sum_j E[|IZ_{t_j} - I^n Z_{t_j}|^2] \cdot |t_{j+1} - t_j| \quad \text{By the Ito-isometry.} \\ &< 2T \sup_{t < T} E[|IZ_t - I^n Z_t|^2] \\ &< 2T E[\sup_{t < T} |IZ_t - I^n Z_t|^2] \\ &< 8T E[|IZ_T - I^n Z_T|^2] \quad \text{By a martingale inequality, see Kunita} \\ &< \frac{\epsilon}{8} \quad \text{[2] p. 151.} \end{aligned}$$

Now choose a partition $[0, \frac{1}{n}, \frac{2}{n}, \dots, T]$ s.t. $n > N_0$ and

$$E \left[\left| \sum_j IZ_{t_j} (W_{t_{j+1}} - W_{t_j}) - IIZ_T \right|^2 \right] < \frac{\epsilon}{8}$$

with this partition

$$I^n I^n Z_T = \sum_j I^n Z_{t_j} (W_{t_{j+1}} - W_{t_j}) \quad \text{and we get}$$

$$E[|I^n I^n Z_T - IIZ_T|^2] < \epsilon \quad \text{by trivial estimation.}$$

So $I^n I^n Z_T \rightarrow IIZ_T$ in L^2 and $\int_0^T IIZ_t dW_t = \int_0^T IIZ_t \circ dW_t$.

Clearly this kind of argument can be applied repeatedly as long as you please, and this proves that all repeated integrals are equal in the Ito and the Stratonovich sense.

□

References

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