Upper semi-continuity of convex functions and openness of affine maps

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Introduction

A fundamental result in the theory of convex functions [Bo, p.60] states that any locally bounded above convex function on an open convex set is continuous. Already the closed interval [0,1] shows that this conclusion is not necessarily valid if the convex set is not open. However, it is well-known [Ro, p. 84] that the interval [0,1], and more generally any closed convex polytope P, has the property that any locally bounded above convex function on P is upper semi-continuous. We came across convex sets with this property in the following way: Let K be any non-empty convex set in a locally convex topological vector space, let e be a point outside the linear subspace generated by the closure of K, let K^* be the convex envelope of K and e. Choose $x_0 \in K$ and let $\Pi(\cdot, x_0)$ be the affine projection from K^* onto K which sends e to x_0. We ask: When is $\Pi(\cdot, x_0)$ an open map whenever $x_0 \in K$? Our answer is that this is true if and only if K has the property that any locally bounded above convex function on K is upper semi-continuous. A convex set with this property will be called an *excellent convex set.* Note, that by the preceding, every open convex set and every closed convex polytope is an excellent set. In fact, we shall prove in section 1 that the closed convex polytopes are the only compact convex sets that are excellent. There is a geometric characterization of this property: A convex set K is excellent if and only if for any $x_0 \in K$ and any homothetic $h_\lambda(\cdot, x_0)$ with center x_0 and factor $\lambda \in <0,1>$, the image $h_\lambda(K, x_0)$ of K is a neighborhood of x_0 in K.(Theorem 1.13). We discovered these two charactrizations of excellent convex sets with the help of a certain function $\Lambda(\cdot, x_0)$ defined on K by

$$\Lambda(x, x_0) = \sup\{\lambda \in [0,1>: x - \lambda x_0 \in (1-\lambda)K\}.$$

This function is concave, and it turns out that the affine projection $\Pi(\cdot, x_0)$ considered above is open if and only if $\Lambda(\cdot, x_0)$ is lower semi-continuous. Furthermore, K is an excellent convex set if and only if $\Lambda(\cdot, x_0)$ is lower semi-continuous at x_0 whenever $x_0 \in K$. This last characterization is useful.

Applying it, we show that the intersection and the cartesian product of two excellent convex sets are convex sets of the same kind. From this we get that any open (in relative topology) convex subset of an excellent convex set again is an excellent set. The function $\Lambda(\cdot, x_0)$ has another noteworthy property: If K is closed, then $\Lambda(\cdot, x_0)$ is upper semi-continuous. This has as a consequence, that if K is closed, then K is an excellent set if and only if any locally bounded above and lower semi-continuous convex function on K is continuous (Theorem 1.19). This equivalence needs, however, not to be true if K is not closed. We exhibit a three-dimensional example to this effect. The closed unit balls of l_1^n and l_∞^n are polytopes, and hence excellent sets. In the infinite dimensional case, we show that the closed unit ball of c_0 is an excellent convex, whereas the closed unit ball of l_1 is not. In fact, if the closed unit ball of a normed space is an excellent set, then the closed unit ball of any finite dimensional subspace has to be a polytope. It is an open problem whether the converse of this statement is true.

It follows from the fundamental result on convex functions mentioned above, that the shape of a convex set at non-interior points is decisive in securing continuity of an arbitrary given convex funtion. To the best of our knowledge, the most accurate condition in this respect is to be found in [Bo, Chap. II, §2, Ex. 29]. Described a bit vaguely, it says that a bounded above convex function admits a limit at a 'conic' point. Motivated by this result, we shall say that a convex set K is *conic* at a non-interior point x_0 if there are an open, punctured convex cone C with x_0 as vertex, and an open convex neighborhood V of x_0 such that $V \cap C = V \cap \operatorname{int} K$. We show in section 2 that if K is closed and conic at every non-interior point, then K is an excellent convex set.

In section 3 we study polyhedral convex sets. By definition, these are convex sets that are the intersection of an affine manifold with the intersection of a finite number of closed half spaces. Our main result in this section is that a polyhedral convex set is conic at every non-interior point.

The subject matter is section 4 is to investigate when a closed locally compact convex set K will be an excellent set. We show, for instance, that K will have this property if and only if K is a strictly (in a topological sense) increasing denumerable union of polyhedral convex sets. (Theorem 4.3). As a corollary, we get as an extension of a classical theorem on topological vector spaces, that every closed locally compact excellent convex set is finite dimensional. Another corollary is that on such a set every convex function is upper semi-continuous.

In section 5 we take up some aspects of the following problem: If K is an excellent convex set and Q is another convex set, when is it true that an affine continuous surjection $\varphi : K \mapsto Q$ is an open map? Our main tool in investigating this problem is a theorem essentially found in [Ku, v. 2, p. 63]. It says, that if K and Q are metric spaces, then a correspondence $\phi : Q \mapsto 2^K$ is lower semi-continous if and only if the function

$$\delta : K \times Q \mapsto [0, \infty >: \delta(x, q) = \text{dist}\,(x, \phi(q))$$

is upper semi-continuous. If we assume that K is contained in a normed vector space and that ϕ is *convex* (see (5.2)), we can show that δ is a convex function. Now, if K and Q are excellent sets, it was mentioned above that $K \times Q$ is an excellent set as well. Hence, in this case δ is upper semi-continuous if and only if δ is locally bounded above. This gives a criterion for ϕ to be lower semi-continuous. In particular, we get a criterion for φ to be open. A consequence of this is that φ is always open if K is a *bounded* set (and K and Q are excellent sets). The same is true if we assume that K and Q are closed locally compact excellent sets. Finally, we show with the same method, that if K is an excellent set, then K is a *stable* convex set [Pa], which means that the middle point map $(a, b) \mapsto \frac{1}{2}(a + b)$ is open.

Terminology and notations

A *convex* set is always assumed to be a non-empty subset of a real locally convex Hausdorff topological vector space, and equipped with the induced topology. More specifically, we let E and F denote real locally convex Hausdorff topological vector spaces and we shall let $K \subset E$ and $Q \subset F$ denote non-empty convex subsets. An *affine manifold* in E is a translate of a linear subspace. If $a, b \in E$, then $[a, b]$ denotes the closed line segment and $< a, b >$ the open line segment between a and b. If A and B are subsets of a topological space, and $B \subset A$, then $\text{int}_A B$ denotes the interior of B in the relative topology of A. Furthermore, if $a \in A$, then $\mathcal{V}_A(a)$ denotes the family of all neighborhoods of a in the relative topology of A. Note that a *convex function* is always assumed to take real values. A *convex combination* is a finite sum of the form $a = \sum \lambda_j x_j$, where $\lambda_1, \ldots, \lambda_n \geq 0$ and $\sum \lambda_j = 1$. Finally, a map $\varphi : K \mapsto Q$ is called *affine* provided $\varphi(\lambda x + (1-\lambda)x') = \lambda\varphi(x) + (1-\lambda)\varphi(x')$ whenever $x, x' \in K$ and $\lambda \in [0, 1]$.

1 The utility of the function Λ

We establish in the present section the general results on excellent convex sets described in the introduction.

Let $K \subset E$ be a non-empty convex set, and let e be a point outside the linear subspace generated by the closure of K in E. If necessary, we can consider E as embedded in $E \times \mathbf{R}$, and choose $e = (0, 1) \in E \times \mathbf{R}$. We denote with K^* the convex envelope of K and e. Thus

$$K^* = \{\lambda e + (1 - \lambda)x : \lambda \in [0, 1],\ x \in K\} \tag{1.1}$$

Notice, that the number λ in the convex combination $y = \lambda e + (1 - \lambda)x$, where $x \in K$, is uniquely determined. In fact, if we more generally consider a convex combination of the form

$$a = \lambda_0 e + \sum_1^n \lambda_j x_j, \tag{1.2}$$

where $x_1, \ldots, x_n$ belong to the closure of K, an easy calculation shows that if λ_0 is not uniquely determined, then e belongs to the affine manifold generated by the closure of K, thereby contradicting the choice of e.

We now fix $x_0 \in K$, and denote with $\Pi(\cdot, x_0)$, or for short Π, the affine projection from K^* to K which maps e into x_0 and fixes every element of K. In other words

$$\Pi = \Pi(\cdot, x_0) : K^* \mapsto K : \lambda e + (1-\lambda)x \mapsto x + \lambda(x_0 - x). \tag{1.3}$$

Let a be any element in K. We define

$$I(a) = \{\lambda \in [0,1] : a - \lambda x_0 \in (1-\lambda)K\} \tag{1.4}$$

Note that always $0 \in I(a)$, and that $1 \in I(a)$ if and only if $a = x_0$.

Lemma 1.1 Let $a \in K$. Then

$$\Pi^{-1}(a) = a + I(a)(e - x_0). \tag{1.5}$$

Proof Let $y = \lambda e + (1-\lambda)x \in \Pi^{-1}(a)$. Hence

$$a = \Pi(y) = \lambda x_0 + (1-\lambda)x.$$

Since $x \in K$, we get $\lambda \in I(a)$. Furthermore, since $(1-\lambda)x = a - \lambda x_0$, it follows that

$$y = \lambda e + a - \lambda x_0 = a + \lambda(e - x_0) \in a + I(a)(e - x_0).$$

Assume conversely that $\lambda \in I(a)$ and put $y = a + \lambda(e - x_0)$. Then $a - \lambda x_0 = (1-\lambda)x$, where $x \in K$. Hence

$$y = a - \lambda x_0 + \lambda e = \lambda e + (1-\lambda)x.$$

This shows that $y \in K^*$, and since

$$\Pi(y) = \lambda x_0 + (1-\lambda)x = \lambda x_0 + a - \lambda x_0 = a$$

the relation (1.5) is established. □

Lemma 1.2 Let $a \in K$. If $a = x_0$, then $I(a) = [0,1]$, and if $a \neq x_0$, then $I(a)$ is an interval contained in $[0,1>$. Furthermore, if K is closed, then $I(a)$ is closed relative to $[0,1>$.

Proof. The first statement follows immediately from the definition of $I(a)$. Assume therefore $a \neq x_0$. Then $1 \notin I(a)$, and hence $I(a) \subset [0,1>$. Consider the map

$$\varphi : [0,1> \mapsto E : \varphi(\lambda) = a + \lambda(e - x_0).$$

Obviously, φ is an affine injection. Furthermore, by Lemma 1.1, $\varphi(I(a)) = \Pi^{-1}(a)$. Since Π is affine, it follows that $\varphi(I(a))$ is a convex set. Hence

$$I(a) = \varphi^{-1}(\varphi(I(a)))$$

is a convex subset of $[0,1>$, and is therefore an interval. Assume now that K is closed. Let

$$\lambda \in \overline{I(a)} \cap [0,1> .$$

Hence $\lambda = \lim \lambda_n$, where $\{\lambda_n\} \subset I(a)$. It follows that for every $n \in \mathbf{N}$

$$a - \lambda_n x_0 = (1 - \lambda_n) x_n ,$$

where $x_n \in K$. Since $\lambda < 1$, we get

$$x_n = (1 - \lambda_n)^{-1}(a - \lambda_n x_0) \to (1 - \lambda)^{-1}(a - \lambda x_0),$$

and where the limit x belongs to K. Therefore

$$a - \lambda x_0 = (1 - \lambda) x \in (1 - \lambda) K,$$

and thus $\lambda \in I(a)$. □

Definition 1.3 Let $x_0 \in K$. The function $\Lambda_K(\cdot, x_0)$ is defined on K by

$$\Lambda_K(a, x_0) = \sup\{\lambda : \lambda \in I(a)\}; \quad a \in K, \tag{1.6}$$

where $I(a)$ is given by (1.4). The element x_0 is said to be the *center* of $\Lambda_K(\cdot, x_0)$. If context makes the meaning clear, we shall use the notation $\Lambda(\cdot, x_0)$, or even the notation Λ for this function.

We remark that it is not hard to show that $\Lambda(\cdot, x_0)$ is an affine function on every line segment $[x_0, a] \subset K$.

Note, that since $0 \in I(a)$ and since $I(a)$ is an interval,

$$I(a) \subset [0, \Lambda(a)] \subset \overline{I(a)}. \tag{1.7}$$

Lemma 1.4 The function Λ is concave.

Proof Let $a_1, a_2 \in K$ and let $\mu_1, \mu_2 \in [0,1]$ with $\mu_1 + \mu_2 = 1$. We have to prove

$$\sum \mu_j \Lambda(a_j) \leq \Lambda(\sum \mu_j a_j)\,.$$

Let $j \in \{1,2\}$. Choose $\lambda_j \in [0, \Lambda(a_j)>$ (if $\Lambda(a_j) = 0$ we choose $\lambda_j = 0$). It is sufficient to prove

$$\sum \mu_j \lambda_j \leq \Lambda(\sum \mu_j a_j). \tag{1.8}$$

It follows from (1.7) that $\lambda_j \in I(a_j)$. Hence there is $x_j \in K$ such that

$$a_j = \lambda_j x_0 + (1 - \lambda_j) x_j$$

Therefore

$$\sum \mu_j a_j = (\sum \mu_j \lambda_j) x_0 + \sum \mu_j (1 - \lambda_j) x_j \tag{1.9}$$

Note, that since $\lambda_1, \lambda_2 < 1$,

$$1 \geq \sum \mu_j (1 - \lambda_j) = 1 - \sum \mu_j \lambda_j > 0.$$

Consequently, if we let $\lambda = \sum \lambda_j \mu_j$, then the element x defined by

$$x = (1 - \lambda)^{-1} \sum \mu_j (1 - \lambda_j) x_j$$

belongs to K. Since (1.9) can be written

$$\sum \mu_j a_j = \lambda x_0 + (1 - \lambda) x,$$

we conclude that

$$\sum \lambda_j \mu_j = \lambda \in I(\sum \mu_j a_j).$$

By the definition of Λ, this verifies the inequality (1.8). □

Lemma 1.5 Assume that the convex set K is closed. Then the function Λ is upper semi-continuous.

Proof We have to prove that for any $\alpha \in \mathbf{R}$, the set $\Lambda^{-1}([\alpha, \infty>)$ is closed. Obviously, we need only consider the case $0 < \alpha \leq 1$. Let a belong to the closure of the set $\Lambda^{-1}([\alpha, \infty>)$. Hence there is a net $\{a_i\}_I$ converging to a and satisfying

$$\Lambda(a_i) \geq \alpha, \quad i \in I$$

Choose $0<\beta<\alpha$. By (1.7) we get $\beta\in I(a_i)$. Hence there is $x_i\in K$ such that

$$a_i = \beta x_0 + (1-\beta)x_i$$

It follows that

$$x_i = (1-\beta)^{-1}(a_i - \beta x_0) \to (1-\beta)^{-1}(a - \beta x_0),$$

and where the limit belongs to K. Therefore

$$a - \beta x_0 \in (1-\beta)K.$$

This means that $\beta\in I(a)$. Hence $\beta\le\Lambda(a)$. Since $\beta<\alpha$ was arbitrarily chosen, we get $\alpha\le\Lambda(a)$, as desired. □

The following example shows that the conclusion of Lemma 1.5 needs not be valid if K is not a closed set.

Example 1.6 Let $K\subset\mathbf{R}^2$ be the closed unit square $[0,1]^2$, except that we have removed the interval $<\frac{1}{2},1]\times\{0\}$. Choose $x_0=(0,0)$ and let $\Lambda=\Lambda(\cdot,x_0)$. Then $\Lambda((\frac{1}{2},0))=0$, whereas

$$\lim_{n\to\infty}\Lambda(\frac{1}{2},\frac{1}{n})=\frac{1}{2}$$

Hence Λ is not upper semi-continuous at $(\frac{1}{2},0)$.

In the proof of the next proposition, we shall make use of the following well-known fact (see e.g.[Ku, vol. 1,p. 117]):

If X and Y are topological spaces, and $f:X\mapsto Y$ is a given map, then f is open if and only if for any subset B of Y

$$f^{-1}(\overline{B})\subset\overline{f^{-1}(B)}. \tag{1.10}$$

Proposition 1.7 Let $x_0\in K$. Then the affine projection

$$\Pi=\Pi(\cdot,x_0):K^*\mapsto K$$

is open if and only if the function

$$\Lambda=\Lambda(\cdot,x_0):K\mapsto[0,1]$$

is lower semi-continuous.

Proof *Assume that* Π *is open.* We have to show that the inverse image $\Lambda^{-1}(<-\infty, \alpha])$ is closed whenever $\alpha \in [0,1]$. Let a belong to the closure of this set. Applying Lemma 1.1 and (1.10), we get

$$a + I(a)(e - x_0) = \Pi^{-1}(a) \subset \overline{\Pi^{-1}(\Lambda^{-1}(<-\infty, \alpha]))}\,. \tag{1.11}$$

We claim that

$$\Pi^{-1}(\Lambda^{-1}(<-\infty, \alpha]) \subset \Lambda^{-1}(<-\infty, \alpha]) + [0, \alpha](e - x_0). \tag{1.12}$$

In fact, let

$$x \in \Lambda^{-1}(<-\infty, \alpha]).$$

Then $\Lambda(x) \leq \alpha$, and therefore $I(x) \subset [0, \alpha]$. By Lemma 1.1 this implies

$$\Pi^{-1}(x) \subset x + [0, \alpha](e - x_0) \subset \Lambda^{-1}(<-\infty, \alpha]) + [0, \alpha](e - x_0).$$

This proves (1.12). Let $\lambda \in I(a)$. By (1.11) and (1.12) there is a net $\{y_i\}_I$ converging to $a + \lambda(e - x_0)$ and where each y_i is of the form

$$y_i = x_i + \lambda_i(e - x_0), \tag{1.13}$$

where $\lambda_i \in [0, \alpha]$, $x_i \in K$ and $\Lambda(x_i) \leq \alpha$. By compactness of $[0, \alpha]$ we can assume, if necessary by considering a subnet, that the limit

$$\lim \lambda_i = \lambda' \in [0, \alpha]$$

exists. Since $x_i = y_i - \lambda_i(e - x_0)$, it follows that $x = \lim x_i$ exists, and that $x \in \overline{K}$. Applying (1.13) we get

$$a + \lambda(e - x_0) = \lim\, y_i = x + \lambda'(e - x_0)\,.$$

By our choice of e, we conclude that $\lambda = \lambda' \in [0, \alpha]$. Hence

$$\Lambda(a) = \sup\{\lambda : \lambda \in I(a)\} \leq \alpha.$$

This proves that $\Lambda^{-1}(<-\infty, \alpha])$ is closed.

We now assume that Λ *is lower semi-continuous.*

Let $B \subset K$ be given. According to (1.10), the map Π will be open if we can show that

$$\Pi^{-1}(\bar{B}) \subset \overline{\Pi^{-1}(B)}. \tag{1.14}$$

Let $a \in \overline{B}$. We claim that

$$a + I(a)(e - x_0) \subset \overline{\Pi^{-1}(B)}. \tag{1.15}$$

In fact, let $\lambda \in I(a)$. Assume first that $\lambda = 0$. Let $\{a_i\}_I$ be a net on B converging to a. Hence $\{a_i\}_I$ is contained in $\Pi^{-1}(B)$, and so a belongs to the closure of this set. Assume next that $\lambda > 0$. Hence

$$0 < \lambda \leq \Lambda(a) \leq \liminf_{x \to a} \Lambda(x) \stackrel{d}{=} \alpha. \tag{1.16}$$

Choose $\epsilon \in <0, \lambda>$. By definition of α there is a neighborhood $U(\epsilon)$ of a such that

$$0 < \lambda - \epsilon \leq \alpha - \epsilon < \Lambda(x); \quad x \in U(\epsilon). \tag{1.17}$$

Let U be any neighborhood of a and choose $x_U \in U \cap U(\epsilon) \cap B$. Applying (1.17) we get $\lambda - \epsilon \in I(x_U)$, and hence

$$x_U + (\lambda - \epsilon)(e - x_0) \in \Pi^{-1}(x_U) \subset \Pi^{-1}(B).$$

Since $a = \lim x_U$, we conclude that

$$a + (\lambda - \epsilon)(e - x_0) \in \overline{\Pi^{-1}(B)}.$$

Letting $\epsilon \to 0$, we obtain the inclusion (1.15). Applying Lemma 1.1, we have therefore proved (1.14). □

Definition 1.8 Let $\lambda > 0$ and $x_0 \in K$. We define the map $h_\lambda(\cdot, x_0)$ on the affine manifold generated by K by the formula

$$h_\lambda(x, x_0) = \lambda x + (1 - \lambda)x_0.$$

We shall call this map the *homothetie with center x_0 and coefficient λ*.

Observe that if $\lambda \in <0, 1]$, then

$$h_\lambda(K, x_0) \subset K. \tag{1.18}$$

Lemma 1.9 The function $\Lambda(\cdot, x_0)$ is continuous at x_0 if and only if the homothetic image $h_\lambda(K, x_0)$ of K is a neighborhood of x_0 in K whenever $\lambda \in <0, 1>$.

Proof By the definition of $\Lambda(\cdot, x_0)$, we get that this function is continuous at x_0 if and only if for any $\lambda \in <0, 1>$ there is a neighborhood U of x_0 such that $\lambda \in I(x)$ whenever $x \in U$. Putting $\mu = (1 - \lambda)^{-1}$, we observe that $\lambda \in I(x)$ if and only if $h_\mu(x, x_0) \in K$. Hence the property '$\lambda \in I(x)$ whenever $x \in U$' is true if and only if $h_\mu(U, x_0) \subset K$. But this inclusion is valid if and only if

$$U = h_{1-\lambda}(h_\mu(U, x_0), x_0) \subset h_{1-\lambda}(K, x_0). \qquad \square$$

Comment 1.10 Since $\Lambda(\cdot, x_0)$ takes values in $[0, 1]$, we have

$$\limsup_{x \to x_0} \Lambda(x, x_0) \leq 1 = \Lambda(x_0, x_0). \tag{1.19}$$

Hence $\Lambda(\cdot, x_0)$ is always upper semi-continuous at x_0. Therefore Lemma 1.9 expresses exactly when $\Lambda(\cdot, x_0)$ is lower semi-continuous at x_0.

We repeat from the introduction the definition of the main concept of the present paper.

Definition 1.11 The non-empty convex set $K \subset E$ is said to be *excelllent* provided every locally bounded above convex function on K is upper semi-continuous. $\square$

It was remarked in the introduction that every non-empty open convex set will be excellent. In particular, the locally convex vector space E itself is an excellent convex set.

The property of being an excellent convex set is preserved by open continuous affine maps. In fact, we have the following

Proposition 1.12 Let K and Q be convex sets, let

$$\varphi : K \mapsto Q$$

be an open continuous affine surjection. If K is an excellent set, then so is Q.

Proof Let g be a locally bounded above convex function on Q. Choose $\alpha \in \mathbf{R}$. We have to prove that $g^{-1}(<-\infty, \alpha>)$ is an open set in Q. Put $f = g \circ \varphi$. Then f is a locally bounded above convex function on K. Hence $f^{-1}(<-\infty, \alpha>)$ is an open set in K. Since φ is a surjection, we get

$$\varphi(f^{-1}(<-\infty, \alpha>)) = g^{-1}(<-\infty, \alpha>)$$

and since, by assumption, the left hand side of this equation is open in Q, we are through. $\square$

Theorem 1.13 Let K be a non-empty convex set. Then the following four properties are equivalent.

(i) K is an excellent set.

(ii) The function $\Lambda(\cdot, x_0)$ is lower semi-continuous whenever $x_0 \in K$.

(iii) The function $\Lambda(\cdot, x_0)$ is lower semi-continuous at x_0 whenever $x_0 \in K$.

(iv) The homothetic image $h_\lambda(K, x_0)$ of K is a neighborhood of x_0 in K whenever $x_0 \in K$ and $\lambda \in < 0, 1 >$.

For the proof we need the following

Lemma 1.14 Let K_1 and K_2 be convex sets with a non-empty intersection $K_1 \cap K_2$. Let $x_0 \in K_1 \cap K_2$. Then

$$\Lambda_{K_1 \cap K_2}(x, x_0) = \min\{\Lambda_{K_1}(x, x_0), \Lambda_{K_2}(x, x_0)\}; \; x \in K_1 \cap K_2. \qquad (1.20)$$

Proof Let $x \in K_1 \cap K_2$. Since (1.20) is trivially true when $x = x_0$, we shall assume $x \neq x_0$. Let

$$I(x) = \{\lambda \in [0, 1 >: \; x - \lambda x_0 \in (1 - \lambda)(K_1 \cap K_2)\}$$

and

$$I_j(x) = \{\lambda \in [0, 1 >: \; x - \lambda x_0 \in (1 - \lambda)K_j\}; \quad j = 1, 2.$$

Hence

$$I(x) = I_1(x) \cap I_2(x). \qquad (1.21)$$

By definition

$$\Lambda_{K_1 \cap K_2}(x, x_0) = \sup\{\lambda : \lambda \in I(x)\}$$

and

$$\Lambda_{K_j}(x, x_0) = \sup\{\lambda : \lambda \in I_j(x)\}; \quad j = 1, 2.$$

Hence we get from (1.21)

$$0 \leq \Lambda_{K_1 \cap K_1}(x, x_0) \leq \min\{\Lambda_{K_1}(x, x_0), \Lambda_{K_2}(x, x_0)\}.$$

Assume that the right hand side is positive, and let λ be a positive number less than this minimum. By (1.21) we get

$$\lambda \in I_1(x) \cap I_2(x) = I(x),$$

and therefore $\lambda \leq \Lambda_{K_1 \cap K_2}(x, x_0)$. It follows that

$$\min\{\Lambda_{K_1}(x, x_0), \Lambda_{K_2}(x, x_0)\} \leq \Lambda_{K_1 \cap K_2}(x, x_0).$$

We have thus proved (1.20). □

Proof of Theorem 1.13. (i)$\Rightarrow$(ii). This is clear, since by Lemma 1.4, $\Lambda(\cdot, x_0)$ is a concave function taking values in $[0,1]$. (ii)$\Rightarrow$(iii). Obvious. (iv)$\Leftrightarrow$(iii). This follows from Lemma 1.9 and Comment 1.10. (iii)$\Rightarrow$(i). Let f be a locally bounded above convex function on K. Choose $x_0 \in K$, and let V be an open convex neighborhood of x_0 in E such that f is bounded above on $K \cap V$, say

$$f(x) \leq \delta < \infty; \quad x \in K \cap V. \tag{1.22}$$

Since

$$\limsup_{x \to x_0, x \in K \cap V} f(x) = \limsup_{x \to x_0, x \in K} f(x),$$

we have to prove that

$$\limsup_{x \to x_0, x \in K \cap V} f(x) \leq f(x_0) \tag{1.23}$$

Let $x \in K \cap V$. By Lemma 1.14

$$\Lambda_{K \cap V}(x, x_0) = \min\{\Lambda_K(x, x_0), \Lambda_V(x, x_0)\}. \tag{1.24}$$

Since V is open, $\Lambda_V(\cdot, x_0)$ is continuous. Furthermore, since we assume $\Lambda_K(\cdot, x_0)$ to be lower semi-continuous at x_0, we get from (1.19) and (1.24)

$$\lim_{x \to x_0, x \in K \cap V} \Lambda_{K \cap V}(x, x_0) = \Lambda_{K \cap V}(x_0, x_0) = 1. \tag{1.25}$$

Therefore, if $\lambda \in <0, 1>$, there is a neigborhood U of x_0 in E such that

$$\lambda < \Lambda_{K \cap V}(x, x_0); \quad x \in K \cap V \cap U.$$

This implies that if $x \in K \cap V \cap U$, then there is an $x^* \in K \cap V$ such that

$$x = \lambda x_0 + (1 - \lambda) x^*.$$

By applying (1.22), we therefore get

$$f(x) \leq \lambda f(x_0) + (1 - \lambda) f(x^*) \leq \lambda f(x_0) + (1 - \lambda)\delta.$$

Hence

$$\limsup_{x \to x_0, x \in K \cap V} f(x) \leq \lambda f(x_0) + (1 - \lambda)\delta.$$

Letting $\lambda \to 1$, we obtain (1.23). □

Corollary 1. A non-empty convex set K is excellent if and only if the projection map

$$\Pi(\cdot, x_0) : K^* \mapsto K$$

is open whenever $x_0 \in K$.

Proof. An immediate consequence of Proposition 1.7 and Theorem 1.13(ii). □

We denote with $\operatorname{ext} K$ the set of extreme points of K. In addition, we denote with $\operatorname{ext}(K, x_0)$ the set of all points $x \in K$ that are *extremal relative to* x_0, which means that x is not an interior point of any segment $[a, x_0] \subset K$. Thus

$$\operatorname{ext}(K, x_0) = \{x \in K : \mu x + (1-\mu)x_0 \notin K \text{ whenever } \mu > 1\}$$

We note that

$$\operatorname{ext}(K, x_0) = \{x \in K : \Lambda(x, x_0) = 0\} \tag{1.26}$$

Corollary 2. If K is an excellent set, then $\operatorname{ext} K$ is a subset of K without accumulation points, and $\operatorname{ext}(K, x_0)$ is closed relative to K whenever $x_0 \in K$.

Proof If $x_0 \in K$ is an accumulation point of $\operatorname{ext} K$, then there is a net $\{a_i\}_I$ on $(\operatorname{ext} K)\backslash\{x_0\}$ converging to x_0. Since $\Lambda(a_i, x_0) = 0$ whenever $i \in I$, we get

$$0 = \liminf_{x \to x_0} \Lambda(x, x_0) < 1 = \Lambda(x_0, x_0)$$

Thus, the property (ii) of Theorem 1.13 is contradicted. Furthermore, since for any $x_0 \in K$

$$\{x \in K : \Lambda(x, x_0) \leq 0\} = \{x \in K : \Lambda(x, x_0) = 0\},$$

it follows from (1.26) and Theorem 1.13(ii) that $\operatorname{ext}(K, x_0)$ is closed for any $x_0 \in K$. □

Corollary 3. If K is an excellent compact convex set, then K is a polytope.

Proof The set $\operatorname{ext} K$ has to be finite, and hence, by the Krein-Milman theorem, K is a polytope. □

The proof of the next lemma is very similar to the proof of Lemma 1.14, and is therefore omitted.

Lemma 1.15 Assume that K_1 and K_2 are non-empty convex sets. If $(a_1, a_2) \in K_1 \times K_2$ and if the center of $\Lambda_{K_1 \times K_2}$ is (a_1, a_2) and the center of Λ_{K_j} is a_j where $j = 1, 2$, then

$$\Lambda_{K_1 \times K_2}((x_1, x_2)) = \min\{\lambda_{K_1}(x_1), \Lambda_{K_2}(x_2)\}; \ (x_1, x_2) \in K_1 \times K_2 \qquad \square$$

Proposition 1.16 Let $K_1 \subset E$ and $K_2 \subset F$ be two excellent convex sets. Then the cartesian product $K_1 \times K_2$ is an excellent set. Furthermore, if $E = F$ and the intersection $K_1 \cap K_2$ is non-empty, then this set is an excellent set as well.

Proof This is an immediate consequence of Lemma 1.14, Lemma 1.15 and Theorem 1.13(ii). $\square$

Proposition 1.17 If $K \subset E$ is an excellent set, and if $P \subset K$ is an open (in relative topology) non-empty convex subset of K, then P is an excellent set.

Proof Let f be a locally bounded above convex function on P, and let $a \in P$. It will suffice to show that there is an open convex neighborhood V of a in E such that $P \cap V$ is an excellent set. Because, in that case

$$\limsup_{x \to a, x \in P} f(x) = \limsup_{x \to a, x \in P \cap V} f(x) \le f(a).$$

By assumption, there is an open O in E such that $P = O \cap K$. Hence we can find an open convex neighborhood V of a in E such that $V \subset O$. Consequently

$$V \cap K = V \cap O \cap K = V \cap P.$$

By Proposition 1.16, the set $V \cap K$ is excellent. Hence $V \cap P$ is excellent, as required. $\square$

Lemma 1.18 Let $K \neq \emptyset$ be convex and closed, let $a \in K$ and assume that $V \subset E$ is a convex set with $0 \in V$. If

$$(a + V) \cap \mathrm{ext}(K, a) = \emptyset,$$

then for any $\lambda \in <0, 1>$

$$(a + \lambda V) \cap K \subset h_\lambda(K, a) \tag{1.27}$$

Proof Choose $\lambda \in <0,1>$. Let $v \in V$ and assume that $x_0 = a + \lambda v \in K$. We have to find an $x \in K$ such that $a + \lambda v = x_0 = a + \lambda(x - a)$. This means that we have to prove that $a + v \in K$. Hence we can and shall assume $v \neq 0$. Consider the ray

$$r = r(x_0, a) = \{a + \mu(x_0 - a) : \mu \geq 0\} = \{a + \mu\lambda v : \mu \geq 0\}. \tag{1.28}$$

If $r \subset K$, we choose $\mu = \lambda^{-1}$, and get $a + \mu\lambda v = a + v \in K$. Assume therefore $r \not\subset K$. Hence $K \cap r$ has to be a closed line segment of the form

$$K \cap r = [a, b]. \tag{1.29}$$

Hence $b \in \text{ext}(K, a)$, and so, by assumption

$$b - a \notin V. \tag{1.30}$$

Since $x_0 \in K \cap r$, there is, by (1.29) an $\alpha \in <0,1]$ such that $a + \lambda v = x_0 = a + \alpha(b - a)$. Hence $b - a = \lambda\alpha^{-1}v$. Since V is convex and $0 \in V$, we must have $\lambda\alpha^{-1} > 1$, since otherwise $b - a \in V$, thereby contradicting (1.30). Put $x = a + \alpha\lambda^{-1}(b - a)$. It follows from (1.29) that $x \in K$, and since $x = a + v$, we are through. □

Theorem 1.19 Let $K \neq \emptyset$ be closed and convex. Then the following five properties are equivalent.

(i) K is an excellent set.

(ii) Every locally bounded above and lower semi-continuous convex function on K is continuous.

(iii) The function $\Lambda(\cdot, x_0)$ is continuous whenever $x_0 \in K$.

(iv) The set $\text{ext}(K, x_0)$ is closed whenever $x_0 \in K$.

(v) If $x_0 \in K$, then $x_0 \notin \overline{\text{ext}(K, x_0)}$.

Proof It is an immediate consequence of Lemma 1.5 and Theorem 1.13 that the first three properties are equivalent. Furthermore, applying Corollary 2 of Theorem 1.13, we get that (i) implies (iv). And since x_0 is not a member of $\text{ext}(K, x_0)$, it is trivial that (v) follows from (iv). Finally, applying Lemma 1.18, we get that (v) implies that $h_\lambda(K, x_0)$ is a neighborhood of x_0 in K whenever $x_0 \in K$, and hence, by Theorem 1.13, the property (i) is true. □

The following example shows that if K is not closed, then the property (iv) of Theorem 1.19 does not necessarily imply the property (i) of that theorem.

Example 1.20 Let K be the open unit disc in the plane and with the point $a = (1, 0)$ added. If $x_0 \in K \setminus \{a\}$, then $\text{ext}(K, x_0) = \{a\}$, and since $\text{ext}(K, a) = \emptyset$, the property (iv) of Theorem 1.19 is true. The set K is, however, not excellent. Consider for instance

$$h_{1/2}(K, a) = \frac{1}{2}K + \frac{1}{2}a\,.$$

This set is the open disc with radius $\frac{1}{2}$ and center $\frac{1}{2}a$ and added the point a. Hence this set is not a neighborhood of a in K. By Theorem 1.13, it follows that K is not an excellent set. □

We shall now exhibit a 3-dimensional example to show that if K is not closed, then the property (ii) of Theorem 1.19 does not necessarily imply that K is en excellent set.

Example 1.21 Let K consist of all the points (x, y, z) of the unit cube $[0, 1]^3$, except that the 'front face' $1 \times [0, 1]^2$ only contains the points in the closed disc with center $(1, 0, \frac{1}{2})$ and radius $\frac{1}{2}$, that is points of the form $(1, y, z)$ where

$$y = r\cos\varphi\,,\ z = \frac{1}{2} + \frac{1}{2}r\sin\varphi\,;\quad 0 \leq r \leq \frac{1}{2}\,,\quad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\,.$$

K is convex, and we note that any point of the form $(1, \frac{1}{2}\cos\varphi, \frac{1}{2} + \frac{1}{2}\sin\varphi)$ is an extreme point of K Hence these points are accumulation points of $\text{ext}\, K$, and therefore, by Corollary 2 of Theorem 1.13, K is not excellent. However, we shall show in the next section, as a consequence of a rather general result, that K is 'conic at non-interior points', and therefore, as we shall show, satisfies the property (ii) of Theorem 1.19. □

The next lemma will be of use in section 4.

Lemma 1.22 Let $K \subset E$ be convex and let $M \subset E$ be an affine manifold. If the set $\text{ext}(K \cap M)$ admits an accumulation point $x_0 \in K \cap M$, then $\Lambda(\cdot, x_0)$ is discontinuous at x_0.

Proof Let $a \in \text{ext}(K \cap M) \setminus \{x_0\}$. We are through if we can show that $\Lambda(a, x_0) = 0$. Let $\lambda \in [0, 1>$ and assume that $a - \lambda x_0 = (1 - \lambda)x$, where $x \in K$. Then $x \neq x_0$, and since $a, x_0 \in M$ it follows that $x \in K \cap M$. Therefore, $\lambda = 0$, and hence $\Lambda(a, x_0) = 0$. □

Proposition 1.23 The closed unit ball of c_0 is an excellent convex set.

Proof Let K be the closed unit ball of c_0. Hence

$$K = \{a = (\alpha_n) : \lim \alpha_n = 0 \ \text{ and } \ \|a\| \leq 1\}$$

where

$$\|a\| = \|(\alpha_n)\| = \sup\{|\alpha_n| : n \in \mathbf{N}\}.$$

According to Theorem 1.13, we have to prove that if $b = (\beta_n) \in K$ and $\lambda \in <0,1>$, then the homothetic image $h_\lambda(K, b)$ of K is a neighborhood of b in K. Hence we have to find an $\epsilon > 0$, such that

$$K \cap B(b, \epsilon) \subset h_\lambda(K, b), \tag{1.31}$$

where

$$B(b, \epsilon) = \{x \in c_0 : \|x - b\| \leq \epsilon\}$$

Note that if $a \in c_0$, then $a \in h_\lambda(K, b)$ if and only if $a = \lambda x + (1 - \lambda)b$, where $x \in K$. This means, however, that $\|a - b + \lambda b\| \leq \lambda$. Hence to prove (1.31) we have to find an $\epsilon > 0$ such that

$$\sup\{|\alpha_n - \beta_n + \lambda\beta_n|\} \leq \lambda \tag{1.32}$$

whenever $\sup\{|\alpha_n|\} \leq 1$ and $\sup\{|\alpha_n - \beta_n|\} \leq \epsilon$. At this point we observe that if $\beta = \pm 1$ and $0 < \epsilon \leq \lambda$, then

$$[-1, 1] \cap [\beta - \epsilon, \beta + \epsilon] \subset [\beta - \lambda\beta - \lambda, \beta - \lambda\beta + \lambda]. \tag{1.33}$$

We note, furthermore, that if $|\alpha| \leq 1$ and $|\beta| \leq \frac{1}{2}$ and $|\alpha - \beta| \leq \epsilon \leq \frac{1}{2}\lambda$, then

$$|\alpha - \beta + \lambda\beta| \leq |\alpha - \beta| + \frac{1}{2}\lambda \leq \lambda. \tag{1.34}$$

We are now ready to determine ϵ: There is an $n_0 \in \mathbf{N}$ such that $|\beta_n| \leq \frac{1}{2}$ whenever $n \geq n_0$. Put

$$\epsilon = \min\{\frac{\lambda}{2}, \lambda(1 - |\beta_n|) : n \leq n_0 \text{ and } |\beta_n| \neq 1\}$$

We claim that (1.31) is valid with this ϵ. In fact, let $\sup\{|\alpha_n|\} \leq 1$ and $\sup\{[\alpha_n - \beta_n|\} \leq \epsilon$. Choose $n \in \mathbf{N}$. If $n \geq n_0$, then it follows from (1.34) that

$$|\alpha_n - \beta_n + \lambda\beta_n| \leq \lambda,$$

and if $n \leq n_0$ and $|\beta_n| = 1$, then the same inequality follows from (1.33). Assume therefore that $n \leq n_0$ and $|\beta_n| \neq 1$. Then we get, by the definition of ϵ,

$$|\alpha_n - \beta_n + \lambda\beta_n| \leq \epsilon + \lambda \mid \beta_n| \leq \lambda(1 - \mid \beta_n|) + \lambda|\beta_n| = \lambda.$$

Hence (1.32) is valid and thus (1.31) is proved. □

Comment 1.24 It is well-known and easy to prove that if K is the unit ball of c_0, then $\text{ext}\, K = \emptyset$. In particular, the set $\text{ext}\, K$ is without accumulation points in K. That this condition is not sufficient to secure that the unit ball of a normed space is an excellent set, is shown by the next example.

Example 1.25 The unit ball of l_1 is not an excellent set.

Proof Let K be the unit ball of l_1. Hence

$$K = \{a = (\alpha_n) : \|a\|_1 = \sum |\alpha_n| \leq 1\}.$$

Choose $a = (\alpha_n) \in K$ with $\sum |\alpha_n| = 1$ and such that $\alpha_n \neq 0$ whenever $n \in \mathbf{N}$. By Corollary 2 of Theorem 1.13, we are through if we can prove that $\text{ext}(K, a)$ is not closed. Let, as usual,

$$e_n = (0, \dots, 0, \underset{n}{1}, 0, \dots),$$

and put

$$x_n = a - 2\alpha_n e_n.$$

Then $\|x_n\|_1 = \|a\|_1 = 1$. Furthermore, let $\mu > 1$. Then

$$\|\mu x_n + (1 - \mu)a\|_1 = \sum_{j \neq n} |\alpha_j| + (2\mu - 1)|\alpha_n| = 1 + 2|\alpha_n|(\mu - 1) > 1.$$

Hence $x_n \in \text{ext}(K, a)$. Since $x_n \to a$ and since $a \notin \text{ext}(K, a)$, we have proved that $\text{ext}(K, a)$ is not closed. □

At this point we remark that it follows from Proposition 1.16 and Corollary 3 of Theorem 1.13 that if E is a normed space such that the closed unit ball of E is an excellent set, then the closed unit ball of any finite dimensional subspace of E has to be a polytope.

We pose the converse of this statement as the following open

Problem If E is a normed space such that the closed unit ball of any finite dimensional subspace of E is a polytope, is it then true that the closed unit ball of E is an excellent set?

2 Convex sets that are conic at non-interior points

We introduce in the present section convex sets that are conic at non-interior points, and prove that any closed set of this kind is an excellent set. The main tool in proving this is the property stated in the Bourbaki exercise mentioned in the introduction. We state this property and, for the convenience of the reader, we supply a proof.

Lemma 2.1 ([Bo, Chap. II §2, Ex. 29]) Let $A \subset E$ be an affine manifold, let $x_0 \in A$ and assume that $C \subset A$ is an open (relative to A), punctured, convex cone with x_0 as vertex. Furthermore, let $V \subset A$ be an open (relative to A) convex neighborhood of x_0. If f is any bounded above convex function on $C \cap V$, then the limit

$$\lim_{x \to x_0, x \in C \cap V} f(x)$$

exists as a real number. Furthermore, if f admits a convex extension to $\{x_0\} \cup (C \cap V)$, then this limit is less or equal $f(x_0)$.

Proof By applying a translation, we can and shall assume that $x_0 = 0$. Hence A is a linear subspace of E. Define

$$\alpha = \liminf_{x \to 0, x \in C \cap V} f(x)\,, \quad \beta = \limsup_{x \to 0, x \in C \cap V} f(x).$$

Thus $\beta < \infty$, since f is bounded above. We have to show that $\alpha = \beta$. Assume that this is not the case. Let $\epsilon = \frac{1}{2}(\beta - \alpha)$ if $\alpha > -\infty$, otherwise let $\epsilon = 1$. We note that in the first case $\alpha + \epsilon = \beta - \epsilon$. Hence, by the definition of α, for any $U \in \mathcal{V}_A(0)$ there is a $y \in U \cap V \cap C$ such that

$$f(y) < \beta - \epsilon. \tag{2.1}$$

Claim: Given $\delta > 0$ there is an $a \in C \cap V$ such that

$$f(\lambda a) \geq \beta - \delta\,, \quad \lambda \in <0, 1] \tag{2.2}$$

In fact, there exists a convex $U \in \mathcal{V}_A(0)$ such that

$$\beta \leq \sup\{f(x) : x \in U \cap V \cap C\} < \beta + \frac{1}{3}\delta. \tag{2.3}$$

Define $U_0 = \frac{1}{2}(U \cap V)$. Hence $U_0 \subset U \cap V$. Thus we can find an $a \in U_0 \cap C$ such that

$$\beta - \frac{1}{3}\delta < f(a) \tag{2.4}$$

Since $2a \in U \cap V \cap C$, we get from (2.3)

$$f(2a) < \beta + \frac{1}{3}\delta. \tag{2.5}$$

Let $\mu \in [0, 1>$. Then $a = (1+\mu)^{-1}(1-\mu)a + (1+\mu)^{-1}\mu 2a$, and where $(1-\mu)a$ and $2a$ belong to $V \cap C$. Hence

$$f(a) \leq (1+\mu)^{-1} f((1-\mu)a) + \mu(1+\mu)^{-1} f(2a).$$

Applying (2.4) and (2.5) we get

$$(1+\mu)(\beta - \frac{1}{3}\delta) \leq f((1-\mu)a) + \mu(\beta + \frac{1}{3}\delta)$$

By a simple computation, we thus obtain

$$\beta - \delta \leq f((1-\mu)a),$$

thereby proving (2.2).

We now choose $\delta = \frac{\epsilon}{2}$ in the inequality (2.2). Since $a \in C \cap V$, there exists a symmetric and convex $U_1 \in \mathcal{V}_A(0)$ such that

$$a + U_1 \subset C \cap V. \tag{2.6}$$

Let $k > 1$ be given and define

$$U = \frac{1}{k-1} U_1 . \tag{2.7}$$

Choose $\frac{1}{k}a \neq y \in U \cap C \cap V$ according to (2.1), and let l be the line through the two points $\frac{1}{k}a$ and y. Hence

$$l(t) = \frac{t}{k}a + (1-t)y; \quad t \in \mathbf{R}.$$

In particular

$$l(k) = a + (1-k)y. \tag{2.8}$$

It follows from (2.6) and (2.7) that $l(k) \in C \cap V$. Now (2.8) can be written

$$\frac{1}{k}a = (1 - \frac{1}{k})y + \frac{1}{k}l(k)$$

Hence we get from (2.2) and (2.1)

$$\beta - \frac{\epsilon}{2} \le f(\frac{1}{k}a) \;\le\; (1-\frac{1}{k})f(y) + \frac{1}{k}f(l(k))$$
$$< \; (1-\frac{1}{k})(\beta-\epsilon) + \frac{1}{k}f(l(k)).$$

Accordingly, we obtain

$$\beta + \epsilon(\frac{k}{2} - 1) < f(l(k)).$$

Since $l(k) \in C \cap V$, and $k > 1$ can be chosen arbitrarily large, this inequality contradicts the boundedness from above of f on $C \cap V$. This proves the first statement in the lemma. As for the second one, we choose an element $b \in C \cap V$. By assumption, the restriction of f to $[0, b]$ is convex, and is therefore, as mentioned in the introduction, an upper semi-continuous function. Hence

$$\lim_{x \to 0, x \in C \cap V} f(x) = \limsup_{x \to 0, x \in <0, b]} f(x) \le f(0). \qquad \square$$

Lemma 2.2 Let K be a convex set contained in the affine manifold A. Assume that $\mathrm{int}_A K \neq \emptyset$. Let f be a lower semi-continuous convex function on K. Then, for any $x_0 \in K$,

$$\limsup_{x \to x_0, x \in K} f(x) = \limsup_{x \to x_0, x \in \mathrm{int}_A K} f(x). \tag{2.9}$$

Proof Since the left hand side of (2.9) is greater or equal the right hand side, we have to prove the opposite inequality. Let U be an open (relative to A) convex neighborhood of x_0, and let $a \in K \cap U$. Choose $b \in \mathrm{int}_A K$. Exactly the same proof as in [Bo, p. 54] shows that $<a, b] \subset \mathrm{int}_A K$. Since U is open and convex, we can find an element $c \in <a, b] \cap U$. Hence $<a, c] \subset U \cap \mathrm{int}_A K$. The restriction of f to $[a, c]$ is, as mentioned in the introduction, upper semi-continuous, and hence, by assumption, continuous. It follows that

$$f(a) = \lim_{x \to a, x \in <a, c]} f(x) \le \sup\{f(x) : x \in U \cap \mathrm{int}_A K\}.$$

Hence

$$\sup\{f(a) : a \in U \cap K\} \le \sup\{f(x) : x \in U \cap \mathrm{int}_A K\}.$$

Since the family of open convex neighborhoods of x_0 constitutes a base of $\mathcal{V}_A(x_0)$, this proves (2.9). $\square$

Comment 2.3 Without the assumption that f is lower semi-continuous, the above Lemma 2.2 is not necessarily true. A simple example is given by the function on $[0, 1]$ with the value one at the point 1 and zero otherwise.

Comment 2.4 It is easy to prove that if the convex set K is contained in the affine manifold A and $\text{int}_A K \neq \emptyset$, then A is in fact the affine manifold generated by K. This comment is relevant for the next definition.

Definition 2.5 Let A be the affine manifold generated by the convex set K. Assume that $\text{int}_A K \neq \emptyset$, and let $x_0 \in K \setminus \text{int}_A K$. We say that K is *conic at* x_0 if there are an open (relative to A) punctured convex cone $C \subset A$ with x_0 as vertex and an open (relative to A) convex neighborhood $V \in \mathcal{V}_A(x_0)$ such that

$$V \cap C = V \cap \text{int}_A K.$$

If K is conic at x_0 whenever $x_0 \in K \setminus \text{int}_A K$, then K is said to be *conic at every non-interior point.* □

Proposition 2.6 Assume that the convex set K is conic at every non-interior point. If f is a locally bounded above lower semi-continuous convex function on K, then f is continuous.

Proof We have to prove that f is upper semi-continuous at every point $x_0 \in K$. If $x_0 \in \text{int}_A K$, this follows from [Bo, Prop. 21, p. 60]. Assume therefore that $x_0 \in K \setminus \text{int}_A K$. Choose C and V according to Definition 2.5. Hence

$$V \cap C = V \cap \text{int}_A K. \tag{2.10}$$

By assumption, there is a convex open neighborhood U of x_0 such that f is bounded above on $U \cap K$. Applying Lemma 2.2 and Lemma 2.1 and (2.10) we get

$$\limsup_{x \to x_0, x \in K} f(x) = \limsup_{x \to x_0, x \in \text{int}_A K} f(x) = \limsup_{x \to x_0, x \in U \cap V \cap \text{int}_A K} f(x) =$$
$$= \limsup_{x \to x_0, x \in U \cap V \cap C} f(x) = \lim_{x \to x_0, x \in U \cap V \cap C} f(x) \leq f(x_0). \qquad \square$$

Corollary 2.7 If K is a closed convex set, and K is conic at every non-interior point, then K is an excellent set.

Proof This is an immediate consequence of Theorem 1.19 and Proposition 2.6. □

Proposition 2.8 Assume that K is conic at every non-interior point. Let $P \subset K$ be a non-empty open (relative to K) convex subset of K. Then P is conic at every non-interior point.

Proof Let A be the affine manifold generated by K. By a simple argument, we get

$$\mathrm{int}_A P = P \cap \mathrm{int}_A K. \tag{2.11}$$

This set is, however, non-empty. In fact, choose $x_0 \in P$ and $b \in \mathrm{int}_A K$, and let U be a convex neighborhood of x_0, open relative to A and such that

$$U \cap K \subset P. \tag{2.12}$$

As in the proof of Lemma 2.2, we have $<x_0, b] \subset \mathrm{int}_A K$. Since $<x_0, b] \cap U$ is non-empty, we get

$$\emptyset \neq <x_0, b] \cap U \subset U \cap \mathrm{int}_A K \subset P \cap \mathrm{int}_A K = \mathrm{int}_A P\,.$$

It follows, as remarked in Comment 2.4, that A is the affine manifold generated by P. Let $x_0 \in P \setminus \mathrm{int}_A P$, and choose U as above. Applying (2.11) we have $x_0 \in P \setminus \mathrm{int}_A K$. But K is conic at every non-interior point. So we can choose C and V as in Definition 2.5. Hence

$$V \cap U \cap C = V \cap U \cap \mathrm{int}_A K = V \cap U \cap P \cap \mathrm{int}_A K = V \cap U \cap \mathrm{int}_A P\,.$$

Since $V \cap U$ is an open neighborhood of x_0, we are through. □

Lemma 2.9 Let $P \subset K$ be a subset such that $\mathrm{int}_K P \neq \emptyset$. Then the affine manifold A generated by K equals the affine manifold M generated by P.

Proof Clearly $M \subset A$. To prove the converse, it suffices to prove $K \subset M$. Choose $a \in \mathrm{int}_K P$. Hence there is $V \in \mathcal{V}_E(0)$ such that $(a+V) \cap K \subset P$.

Let $x \in K$. We want to show that $x \in M$, and therefore we can assume $x \neq a$. There is a $\lambda_0 \in <0, 1>$ satisfying $\lambda_0(x-a) \in V$. Let

$$x_0 = a + \lambda_0(x-a) = (1-\lambda_0)a + \lambda_0 x.$$

Then

$$x_0 \in (a+V) \cap K \subset P.$$

Since $x \neq a$ and $0 < \lambda_0 < 1$, the point x belongs to the line through x_0 and a. Since this line is contained in M, we get $x \in M$. Therefore $K \subset M$.

□

We referred to the next proposition in the Example 1.21.

Proposition 2.10 Let K be a convex set with $\text{int}_A K \neq \emptyset$, and let P be a convex subset of K such that $\text{int}_A P = \text{int}_A K$. If K is conic at every non-interior point, then so is P.

Proof Applying Lemma 2.9, we get that A is the affine manifold generated by P. Let $x_0 \in P \setminus \text{int}_A P$. By assumption, $x_0 \in K \setminus \text{int}_A K$. Choose C and V as in Definition 2.4. Hence

$$V \cap C = V \cap \text{int}_A K = V \cap \text{int}_A P$$

This shows that P is conic at every non-interior point. □

The next proposition will be of use in section 4, in our study of locally compact excellent sets.

Proposition 2.11 If the convex set K is the union of a sequence $\{K_n\}$ of convex sets K_n such that every K_n is conic at non-interior points and satisfies

$$K_n \subset \text{int}_K K_{n+1}\,; \quad n \in \mathbf{N}, \tag{2.13}$$

then K is conic at non-interior points.

Proof Let A be the affine manifold generated by K. We first want to show

$$\text{int}_A K = \cup\{\text{int}_A K_n : n \in \mathbf{N}\}. \tag{2.14}$$

Obviously, the relation $\supset$ is true. To prove the opposite inclusion, let $x \in \text{int}_A K$. Choose $n \in \mathbf{N}$ such that $x \in K_n$. There exists an open set U in A with the property

$$\text{int}_K K_{n+1} = U \cap K \subset K_{n+1}. \tag{2.15}$$

Applying (2.13), it follows that

$$x \in U \cap \text{int}_A K \subset U \cap K \subset K_{n+1}.$$

Therefore

$$x \in U \cap \mathrm{int}_A K \subset \mathrm{int}_A K_{n+1}.$$

Thus (2.14) is proved. We now observe that it follows from (2.13) and Lemma 2.9 that the affine manifold generated by K_{n+1} equals A. Hence, by assumption, $\mathrm{int}_A K_{n+1} \neq \emptyset$. By (2.14), we conclude that $\mathrm{int}_A K \neq \emptyset$.

Let $x_0 \in K \setminus \mathrm{int}_A K$. Applying (2.14) once more, there is $n \in \mathbf{N}$ such that

$$x_0 \in K_n \setminus \mathrm{int}_A K_{n+1} \subset K_{n+1} \setminus \mathrm{int}_A K_{n+1}.$$

Since K_{n+1} is conic at x_0, there are an open convex cone $C \subset A$, punctured at x_0, and an open neighborhood V of x_0, with $V \subset A$, such that

$$V \cap C = V \cap \mathrm{int}_A K_{n+1}. \tag{2.16}$$

With U as in (2.15) we claim that

$$U \cap \mathrm{int}_A K_{n+1} = U \cap \mathrm{int}_A K. \tag{2.17}$$

Indeed,

$$U \cap \mathrm{int}_A K \subset U \cap K = \mathrm{int}_K K_{n+1} \subset K_{n+1}.$$

Since the set on the left hand side is open in A, it follows that

$$U \cap \mathrm{int}_A K \subset U \cap \mathrm{int}_A K_{n+1} \subset U \cap \mathrm{int}_A K,$$

as claimed. Now

$$x_0 \in K_n \subset \mathrm{int}_K K_{n+1} = U \cap K.$$

Hence $U \cap V$ is an open neighborhood of x_0. By (2.16) and (2.17) we get

$$U \cap V \cap C = U \cap V \cap \mathrm{int}_A K_{n+1} = U \cap V \cap \mathrm{int}_A K.$$

This proves that K is conic at x_0. □

3 Polyhedral convex sets

The main goal of the present section is to prove that a polyhedral convex set is conic at every non-interior point.

We shall first fix some notations. If n is a natural number, we put $\mathbf{N}(n) = \{j \in \mathbf{N} : 1 \leq j \leq n\}$, whereas $\mathbf{N}(0)$ denotes the empty set. A *closed half space* H (in the given topological vector space E) is a subset of the form $H = f^{-1}([\alpha, \infty >)$, where $f \neq 0$ is a continuous linear functional on E and $\alpha \in \mathbf{R}$.

In the finite dimensional case, a polyhedral convex set is defined to be the intersection of a finite number of closed half spaces (see for instance [Ro]). However, in the infinite dimensional case, such an intersection has to be of infinite dimension. Hence a *polytope*, which by definition is the convex hull of finitely many points, would not be a polyhedral convex set according to this definition. In order to remedy this, we have chosen the following

Definition 3.1 A convex set $K \subset E$ is called a polyhedral convex set if there are an affine manifold $A \subset E$ and a finite family of closed half spaces $\{H_j : j \in \mathbf{N}(n)\}$ with $n \geq 0$ such that

$$K = A \cap \bigcap \{H_j : j \in \mathbf{N}(n)\} \tag{3.1}$$

Note, that by choosing $n = 0$, we get in particular that every affine manifold is a polyhedral set.

Proposition 3.2 Let $K \subset E$ be a polytope and let $M \subset E$ be an affine manifold. Then $M + K$ is a polyhedral convex set.

Proof It is evident that the translate of a polyhedral convex set is a set of the same kind. Therefore, we can and shall assume that M is a linear subspace of E.

(i) *We first assume* $M = \{0\}$. Let A be the affine manifold generated by K. Since K is a polytope, A is finite dimensional. Choose $a \in A$ and let $L = A - a$. Then $K - a$ is a polytope in the finite dimensional linear space L. Referring for instance to [Ro], we can find finitely many non-zero linear functionals $\varphi_1, \ldots, \varphi_n$ on L and real numbers $\beta_1, \ldots, \beta_n$ such that

$$K - a = \bigcap \{\varphi_j^{-1}([\beta_j, \infty >) : j \in \mathbf{N}(n)\}.$$

Now, by the Hahn-Banach theorem, there exists a linear continuous extension f_j of φ_j to E. Put $\alpha_j = \beta_j + f_j(a)$. By an easy argument it follows that

$$K = A \cap \bigcap \{f_j^{-1}([\alpha_j, \infty >) : j \in \mathbf{N}(n)\}.$$

This proves that K is a polyhedral set.

(ii) *We now consider the genral case.* Consider the quotient map

$$\eta : E \mapsto E/M,$$

where E/M is equipped with the quotient topology. Hence E/M is a locally convex vector space. It is not hard to prove that $\eta(K)$ is also a polytope. From (i) we therefore get

$$\eta(K) = A \cap \bigcap\{H_j : j \in \mathbf{N}(n)\}$$

where A is an affine manifold and $H_1, \ldots, H_n$ are closed half spaces in E/M. Hence

$$\eta^{-1}(\eta(K)) = \eta^{-1}(A) \cap \bigcap\{\eta^{-1}(H_j) : j \in \mathbf{N}(n)\}$$

where $\eta^{-1}(A)$ is an affine manifold and $\eta^{-1}(H_1), \ldots, \eta^{-1}(H_n)$ are closed half spaces in E. Since

$$\eta^{-1}(\eta(K)) = M + K,$$

we conclude that $M + K$ is a polyhedral set. □

Lemma 3.3 Let $J \neq \emptyset$ be a finite set, let $\{A_j : j \in J\}$ be a family of affine manifolds in E. If K is convex and

$$K \subset \bigcup\{A_j : j \in J\},$$

then there exists a $k \in J$ such that $K \subset A_k$.

Proof Define for any $x \in K$

$$J(x) = \{j \in J : x \in A_j\}.$$

By assumption, $J(x) \neq \emptyset$. The proof will obviously be finished if we can prove that the intersection

$$\bigcap\{J(x) : x \in K\}$$

is non-empty. Equip J with the discrete topology. Then J is a compact Hausdorff space and every $J(x)$ is a closed subset of J. Hence it is sufficient to prove that the family $\{J(x) : x \in K\}$ has the finite intersection property. We thus have to prove that if $m \in \mathbf{N}$ and $x_1, \ldots, x_m \in K$, then

$$\bigcap\{J(x_i) : i \in \mathbf{N}(m)\} \neq \emptyset \tag{3.2}$$

To prove this, we use induction on m. Since $J(x) \neq \emptyset$ for any $x \in K$, the relation (3.2) is true when $m = 1$. Assume therefore that $m \in \mathbf{N}$ is given and that the relation (3.2) is true whenever $x_1, \ldots, x_m \in K$.

Let $x_1, \ldots, x_{m+1} \in K$ be given. By the induction hypothesis, we can and shall assume that these elements are all different. Choose $i \in \mathbf{N}(m)$, and let l_i be the line between x_i and x_{m+1}. Thus

$$l_i(t) = t x_{m+1} + (1-t) x_i \,; \quad t \in \mathbf{R}$$

Choose $\lambda \in <0, 1>$. Then

$$\{l_1(\lambda), \ldots, l_m(\lambda)\} \subset K\,.$$

Therefore, by the induction hypothesis, there exists an element

$$k(\lambda) \in \bigcap \{J(l_i(\lambda)) : i \in \mathbf{N}(m)\}$$

Since J is finite, we can find two different $\lambda, \lambda' \in <0, 1>$ such that $k(\lambda) = k(\lambda')$. We call this element k. Then $l_i(\lambda) \neq l_i(\lambda')$ and

$$l_i(\lambda), l_i(\lambda') \in A_k\,; \quad i \in \mathbf{N}(m)\,.$$

Hence the line l_i is contained in A_k. In particular

$$x_i, x_{m+1} \in A_k\,; \quad i \in \mathbf{N}(m).$$

But this means that

$$k \in \bigcap \{J(x_i) : i \in \mathbf{N}(m+1)\}.$$

The induction step is thus proven. □

Let K be a polyhedral convex set as given by (3.1). We want to determine $\mathrm{int}_A K$. If $\mathring{H}_j$ denotes the interior of H_j in E, one might believe that $\mathrm{int}_A K$ will be the set

$$A \cap \bigcap \{\mathring{H}_i \colon i \in \mathbf{N}(n)\}.$$

However, if we choose $n = 1$ and $A = H_1 \setminus \mathring{H}_1$, then $K = A$, whereas the set above is empty. Motivated by this example, we introduce the set

$$I = \{j \in \mathbf{N}(n) : A \subset H_j \setminus \mathring{H}_j\}. \tag{3.3}$$

We then have the following

Lemma 3.4 If the polyhedral convex set K is given by

$$K = A \cap \bigcap\{H_j : j \in \mathbf{N}(n)\},$$

then

$$\begin{cases} (i) & K = A \cap \bigcap\{H_j : j \in \mathbf{N}(n) \setminus I\} \\ (ii) & \mathrm{int}_A K = A \cap \bigcap\{\overset{\circ}{H}_j : j \in \mathbf{N}(n) \setminus I\}, \end{cases} \tag{3.4}$$

where I is given by (3.3).

Comment 3.5 In the formulas (3.4) and in what follows, we use the convention that the intersection of a family of subsets of E with an empty set of indices, is the set E itself.

Proof The first formula in (3.4) follows immediately from the definition of the set I. As for the second formula we have to prove that the left hand side of (3.4)(ii) is contained in the right hand side. Assume that this is not true. Hence there exist an $x_0 \in \mathrm{int}_A K$ and an index $j \in \mathbf{N}(n) \setminus I$ such that

$$x_0 \in H_j \setminus \overset{\circ}{H}_j \, .$$

We can find a continuous linear functional f_j and a real number α_j such that

$$H_j = f_j^{-1}([\alpha_j, \infty>), \tag{3.5}$$

Furthermore

$$\overset{\circ}{H}_j = f_j^{-1}(<\alpha_j, \infty>), \tag{3.6}$$

and hence

$$x_0 \in H_j \setminus \overset{\circ}{H}_j = f_j^{-1}(\alpha_j) \tag{3.7}$$

Let $U \subset A$ be an open (relative to A) convex neighborhood of x_0 such that $U \subset K$. We shall show that if we assume that f_j is constant on U, then we shall get

$$A \subset H_j \setminus \overset{\circ}{H}_j, \tag{3.8}$$

a contradiction since $j \notin I$. So let $a \in A$. Assume first that $a \neq x_0$. Let l be the line between x_0 and a. Then $U \cap l$ is an open interval containing x_0. Choose $x_1 \in U \cap l$ with $x_1 \neq x_0$. Any $x \in l$ can be written

$$x = (1-t)x_0 + t x_1 ; \quad t \in \mathbf{R}.$$

Since we assume f_j constant on U, we get

$$f_j(x) = (1-t)f_j(x_0) + tf_j(x_1) = f_j(x_0) = \alpha_j,$$

where we used (3.7) in the last equation. In particular, $f_j(a) = f_j(x_0) = \alpha_j$. By (3.7), we have thus proved the contradiction (3.8). It follows that there exists an $a \in U$ such that $f_j(a) \neq f_j(x_0)$. Let again l be the line between a and x_0. Since $U \cap l$ is an open intervall around x_0, there exists an $\epsilon > 0$ such that

$$x = (1-t)x_0 + ta \in U \cap l \subset K; \quad |t| < \epsilon. \tag{3.9}$$

Since $f_j(x_0) = \alpha_j$, we get

$$f_j(x) = \alpha_j + t(f_j(a) - f_j(x_0))$$

But we know that $f_j(a) \neq f_j(x_0)$. By (3.9) we therefore get that for some $x \in K$, $f_j(x) < \alpha_j$. This is the desired contradition, since

$$x \in H_j = f_j^{-1}([\alpha_j, \infty>). \qquad \square$$

We shall now study the case where $\text{int}_A K = \emptyset$. For that purpose the following lemma is useful.

Lemma 3.6 Let K be given as in Lemma 3.4. Then $\text{int}_A K = \emptyset$ if and only if the set

$$J \stackrel{d}{=} \{j \in \mathbf{N}(n) \setminus I : K \subset H_j \setminus \overset{\circ}{H}_j\} \tag{3.10}$$

is non-empty.

Proof Assume that $\text{int}_A K = \emptyset$. It follows from Lemma 3.4 that for any $x \in K$ there exists $j \in \mathbf{N}(n) \setminus I$ such that $x \in H_j \setminus \overset{\circ}{H}_j$. Hence

$$K \subset \bigcup \{H_j \setminus \overset{\circ}{H}_j : j \in \mathbf{N}(n) \setminus I\}$$

Since $H_j \setminus \overset{\circ}{H}_j$ is a hyperplane in E, we get from Lemma 3.3 the existence of $j \in \mathbf{N}(n) \setminus I$ such that $K \subset H_j \setminus \overset{\circ}{H}_j$. This means that $J \neq \emptyset$. Assume conversely that $j \in J$. Hence

$$K \subset (H_j \setminus \overset{\circ}{H}_j) \cap A.$$

Since $j \in \mathbf{N}(n) \setminus I$, we have

$$(H_j \setminus \overset{\circ}{H}_j) \cap A \subsetneqq A.$$

It follows that the affine manifold generated by K is a proper subset of A. By Comment 2.4, we conclude that $\text{int}_A K = \emptyset$. $\square$

Applying Lemma 3.6 we define the *reduction* $r(A)$ of A by the formula

$$r(A) = \begin{cases} \bigcap\{A \cap (H_j \setminus \overset{\circ}{H}_j) : j \in J\}; & \text{when } \text{int}_A K = \emptyset \\ A; & \text{when } \text{int}_A K \neq \emptyset. \end{cases} \tag{3.11}$$

We note that $r(A)$ is an affine manifold with $K \subset r(A) \subset A$. Furthermore, we get by a straightforward argument

$$K = r(A) \cap \bigcap\{H_j : j \in \mathbf{N}(n) \setminus (I \cup J)\}. \tag{3.12}$$

Lemma 3.7 With K as above, the following formula is valid

$$\text{int}_{r(A)} K = r(A) \cap \bigcap\{\overset{\circ}{H}_j : j \in \mathbf{N}(n) \setminus (I \cup J)\}. \tag{3.13}$$

Proof If $I \cup J = \mathbf{N}(n)$, then (3.12) shows that $K = r(A)$, and so (3.13) is valid. We can therefore assume $\emptyset \neq \mathbf{N}(n) \setminus (I \cup J)$. Let $j \in \mathbf{N}(n)$ and suppose

$$r(A) \subset H_j \setminus \overset{\circ}{H}_j .$$

Then $j \in I \cup J$. In fact, if $j \in \mathbf{N}(n) \setminus I$, the inclusion $K \subset r(A)$ implies

$$K \subset H_j \setminus \overset{\circ}{H}_j,$$

and hence $j \in J$. Consequently, if $j \in \mathbf{N}(n) \setminus (I \cup J)$, then

$$r(A) \not\subset H_j \setminus \overset{\circ}{H}_j$$

Therefore, if we use Lemma 3.4 with K represented as in (3.12), we get that the corresponding I-set is empty. Hence (3.13) follows from the formula (3.4)(ii). $\square$

Lemma 3.8 With $r(A)$ defined by (3.11) we always have

$$\text{int}_{r(A)} K \neq \emptyset$$

Proof Assume contrarily that $\mathrm{int}_{r(A)}K = \emptyset$. It follows from Lemma 3.7 that for any $x \in K$ there exists $j \in \mathbf{N}(n) \setminus (I \cup J)$ such that

$$x \in H_j \setminus \overset{\circ}{H}_j$$

Hence

$$K \subset \bigcup \{H_j \setminus \overset{\circ}{H}_j : j \in \mathbf{N}(n) \setminus (I \cup J)\}$$

Applying Lemma 3.3 we conclude that there exists an element $k \in \mathbf{N}(n) \setminus (I \cup J)$ such that

$$K \subset H_k \setminus \overset{\circ}{H}_k$$

Since $k \in \mathbf{N}(n) \setminus I$, this implies that $k \in J$, a contradiction. □

Proposition 3.9 If K is a polyhedral convex set, then K is conic at every non-interior point.

Proof By the formula (3.12), Lemma 3.7 and Lemma 3.8, we can assume, with a slight change of notation, that

$$K = A \cap \bigcap \{H_j : j \in \mathbf{N}(n)\}, \tag{3.14}$$

and

$$\mathrm{int}_A K = A \cap \bigcap \{\overset{\circ}{H}_j : j \in \mathbf{N}(n)\} \neq \emptyset. \tag{3.15}$$

It follows, as in Comment 2.4, that A has to be the affine manifold generated by K.

Let $x_0 \in K \setminus \mathrm{int}_A K$. Hence there exists $j \in \mathbf{N}(n)$ such that

$$x_0 \in H_j \setminus \overset{\circ}{H}_j \,.$$

Let

$$J(x_0) \overset{d}{=} \{j \in \mathbf{N}(n) : x_0 \in H_j \setminus \overset{\circ}{H}_j\} \neq \emptyset,$$

and define

$$C(x_0) = A \cap \bigcap \{\overset{\circ}{H}_j : j \in J(x_0)\} \tag{3.16}$$

Then $C(x_0)$ is a convex relatively open subset of A. Furthermore, since $\mathrm{int}_A K$ is contained in $C(x_0)$, we get $C(x_0) \neq \emptyset$. We claim that $C(x_0)$ is a punctured convex cone with x_0 as vertex. In fact, since $J(x_0) \neq \emptyset$, $x_0 \notin C(x_0)$. Furthermore, let $x \in C(x_0)$ and let l^+ be the open half-line through x_0 and x with start in x_0. Hence

$$l^+(t) = (1-t)x_0 + tx\,; \quad t > 0.$$

We have to show that $l^+ \subset C(x_0)$. Of course, $l^+ \subset A$. Therefore, let $j \in J(x_0)$. There exist an f_j with $f_j \neq 0$ and a real number α_j such that

$$\mathring{H}_j = f_j^{-1}(<\alpha_j, \infty>) \text{ and } H_j \setminus \mathring{H}_j = f_j^{-1}(\alpha_j).$$

Let $t > 0$. Then

$$\begin{aligned} f_j(l^+(t)) &= (1-t)f_j(x_0) + tf_j(x) = \\ &= \alpha_j + t(f_j(x) - \alpha_j) > \alpha_j \end{aligned}$$

Hence $l^+ \subset \mathring{H}_j$. This proves the claim. Define

$$V(x_0) = A \cap \bigcap\{\mathring{H}_j : j \in \mathbf{N}(n) \setminus J(x_0)\}$$

Thus $V(x_0)$ is a convex set, open relative to A, and $x_0 \in V(x_0)$. By (3.15) and the definitions of $C(x_0)$ and $V(x_0)$ we get

$$\begin{aligned} V(x_0) \cap C(x_0) &= A \cap \bigcap\{\mathring{H}_j : j \in \mathbf{N}(n)\} = \\ &= \mathrm{int}_A K = V(x_0) \cap \mathrm{int}_A K. \end{aligned}$$

This proves that K is conic at x_0. □

Combining Proposition 3.9 with Corollary 2.7, we get the following

Corollary 3.10 A closed polyhedral convex set is an excellent convex set.

□

In the next section we shall have need of the following extension of Proposition 3.9

Proposition 3.11 Let K be a convex set such that

$$K = \bigcup\{K_n : n \in \mathbf{N}\},$$

where $\{K_n\}$ is a sequence of polyhedral convex sets with the property

$$K_n \subset \mathrm{int}_K K_{n+1}\,; \quad n \in \mathbf{N}.$$

Then K is conic at non-interior points.

Proof Use Proposition 3.9 and Proposition 2.11. □

4 Locally compact excellent convex sets

We have proved in the preceding section that if the convex set K is the union of a strictly increasing family of polyhedral convex sets, as in Proposition 3.11, then K is conic at non-interior points. Furthermore, it was shown in section 2 that any closed convex set of the latter kind is an excellent set. The main objective of the present section is to prove that if K is closed and locally compact, then these three properties are indeed equivalent. A corollary of this characterization is that every closed locally compact excellent set is finite dimensional. This is an extension of the classical theorem that every locally compact topological vector space is finite dimensional, a theorem we shall make use of in the proof. Otherwise, our main analytical tool will be a theorem of V.L. Klee [Kl] stating that if K is a closed locally compact convex set containing no line, then there is a closed half space H such that $K \cap H$ is compact.

Recall that $\operatorname{ext} K$ denotes the set of extreme points of K.

Lemma 4.1 If $K \subset E$ is convex and if $f \neq 0$ is a continuous linear functional on E and $\alpha \in \mathbf{R}$, then

$$\operatorname{ext}[K \cap f^{-1}(<-\infty, \alpha])] \subset (\operatorname{ext} K) \cup \operatorname{ext}(K \cap f^{-1}(\alpha)).$$

Proof Let c be an extreme point of $K \cap f^{-1}(<-\infty, \alpha])$. If $f(c) = \alpha$, then c is an extreme point of $K \cap f^{-1}(\alpha)$. Assume therefore $f(c) < \alpha$. If c is not an extreme point of K, then $2c = a + b$, where $a, b \in K$ and $a \neq b$. Hence

$$f(a) + f(b) = 2f(c) < 2\alpha. \tag{4.1}$$

We must have $\min\{f(a), f(b)\} < \alpha$, say $f(a) < \alpha$. Since c is an extreme point of $K \cap f^{-1}(< -\infty, \alpha])$, it follows that $f(b) > \alpha$. Define $\lambda_0 = (f(b) - \alpha)(f(b) - f(a))^{-1}$. Then $\lambda_0 > 0$ and using (4.1) we get $f(a) - f(b) = f(a) + f(b) - 2f(b)) < 2(\alpha - f(b))$. Hence $0 < \lambda_0 < \frac{1}{2}$. Let $a_0 = \lambda_0 a + (1 - \lambda_0) b$. Then $f(a_0) = \alpha$ and c is an interior point of $[a, a_0]$. Since

$$a, a_0, \in K \cap f^{-1}(<-\infty, \alpha]) \tag{4.2}$$

we have got the desired contradiction. □

We now assume that the convex set K is closed. Let $a \in K$. We shall follow Bourbaki [Bo, Chap. II, §2, Ex. 14] and call

$$C_K = \bigcap \{\lambda(K - a) : \lambda > 0\}$$

the *asymptotic cone* of K. Then C_K is a closed convex cone with zero as vertex, and C_K is independent of the choice of a. Furthermore, $C_K + a$ is the union of all half lines with start in a and contained in K. Let

$$L = C_K \cap (-C_K) \subset E$$

Then L is a closed linear subspace of E, and it is easy to see that L is the union of $\{0\}$ and of all lines through zero contained in K. Let $N \subset E$ be a linear subspace supplementary to L in E. If we assume $0 \in K$, then as in [Kl] we have

$$K = L + N \cap K, \tag{4.3}$$

where $N \cap K$ contains no line. We now assume that K is locally compact as well. Hence L is a locally compact space, and is therefore finite dimensional. By a well-known result L admits a closed supplementary linear subspace N such that the projection

$$\tau : E = L + N \mapsto N : x = u + v \mapsto v \tag{4.4}$$

is continuous. If we choose this N in (4.3), we get in particular that $N \cap K$ is a closed locally compact convex set containing no line.

Lemma 4.2 Let L and N be supplementary linear subspaces in E such that

$$K = L + N \cap K. \tag{4.5}$$

Choose $x_0 \in N \cap K$. Then $\Lambda_{N \cap K}(\cdot, x_0)$ is equal to the restriction of $\Lambda(\cdot, x_0)$ to the set $N \cap K$.

Proof Let $a \in N \cap K$, and let as in section 1

$$I(a) = \{\lambda \in [0,1] : a - \lambda x_0 \in (1-\lambda)K\}.$$

We are finished if we can prove

$$I(a) = \{\lambda \in [0,1] : a - \lambda x_0 \in (1-\lambda) N \cap K\}. \tag{4.6}$$

Of course, the relation $\supset$ is valid. Therefore, let $\lambda \in I(a)$. If $\lambda = 1$, then $a = x_0$ and the right hand side of (4.6) equals $[0,1]$. We shall therefore assume $\lambda \neq 1$. There is $x \in K$ such that $a = \lambda x_0 + (1-\lambda)x$. By (4.5), $x = u + v$, where $u \in L$ and $v \in N \cap K$. Hence

$$(1-\lambda)u = a - \lambda x_0 - (1-\lambda)v \in L \cap N = \{0\}$$

Therefore, $u = 0$ and consequently $x = v \in N \cap K$. This proves the inclusion $\subset$ in (4.6). □

A *continuous affine function* f on E is by definition of the form $f = \alpha + g$, where $\alpha \in \mathbf{R}$ and where g is a continous linear functional on E.

Theorem 4.3 Let $K \subset E$ be a closed convex set. Consider the following five properties P(1),...,P(5).

P(1) There is a continuous affine function f on E such that

$$K \cap f^{-1}(<-\infty, n])$$

is a polyhedral convex set whenever $n \in \mathbf{N}$.

P(2) There are a finite dimensional affine manifold M and a sequence of polytopes $\{P_n\}$ such that

$$K = \bigcup \{M + P_n, n \in \mathbf{N}\}$$

and

$$M + P_n \subset \mathrm{int}_K(M + P_{n+1}); \quad n \in \mathbf{N}.$$

P(3) There is a sequence of polyhedral convex sets $\{K_n\}$ such that

$$K = \bigcup \{K_n, n \in \mathbf{N}\}$$

and

$$K_n \subset \mathrm{int}_K K_{n+1}; \quad n \in \mathbf{N}.$$

P(4) K is conic at non-interior points.

P(5) K is an excellent set.

Then P(1) $\Rightarrow$P(3), P(2) $\Rightarrow$P(3), P(3) $\Rightarrow$P(4) and P(4) $\Rightarrow$P(5). Furthermore, if K in addition is locally compact, then all five properties are equivalent.

Proof P(1) $\Rightarrow$P(3). Putting $K_n = K \cap f^{-1}(<-\infty, n])$, we get

$$K_n \subset K \cap f^{-1}(<-\infty, n+1>) \subset K_{n+1},$$

and since the middle term is an open subset of K, P(3) follows.

P(2) $\Rightarrow$P(3). Use Proposition 3.2

P(3) $\Rightarrow$P(4). Use Proposition 3.11.

P(4) $\Rightarrow$P(5). Use Corollary 2.7.

It remains to prove, that if K is locally compact, then P(5) implies both P(1) and P(2). Assume therefore that K is a closed locally compact excellent set. Hence, by Theorem 1.19, the function $\Lambda(\cdot, x_0)$ is continuous for all $x_0 \in K$. Let us first assume $0 \in K$. It then follows from (4.3) that

$$K = L + N\cap K \tag{4.7}$$

where we have remarked that L is finite dimensional and that $N\cap K$ is a closed locally compact convex set containg no line. It follows from [Kl, p. 236] that there is a continuous linear functional g on N such that if we define

$$P_n = N\cap K\cap g^{-1}(<-\infty, n]); \quad n\in\mathbf{N}, \tag{4.8}$$

then P_n is compact whenever $n\in\mathbf{N}$. We claim that P_n is a convex polytope. In fact, applying Lemma 4.2, we get that $\Lambda_{N\cap K}(\cdot, x_0)$ is continuous whenever $x_0\in N\cap K$. Now, by Lemma 4.1,

$$\operatorname{ext} P_n \subset [P_n\cap \operatorname{ext}(K\cap N)] \cup \operatorname{ext}(K\cap N\cap g^{-1}(n)). \tag{4.9}$$

Since P_n and $K\cap N\cap g^{-1}(n)$ are compact subsets of $K\cap N$, it follows from Lemma 1.22 that both of the sets on the righthand side of (4.9) are finite. Hence P_n is a polytope. Let τ be the continuous projection as given in (4.4). Let $f = go\tau$. Then f is a continuous linear functional on E. Without difficulty we get

$$L + P_n = K\cap f^{-1}(<-\infty, n]). \tag{4.10}$$

It follows from Proposition 3.2 that the left hand side of (4.10) is a polyhedral convex set. This proves P(1). Furthermore, as in the proof of P(1)$\Rightarrow$(P(3), we get

$$L + P_n \subset \operatorname{int}_K(L + P_{n+1}). \tag{4.11}$$

Finally, using (4.10), it follows that

$$K = \bigcup\{L + P_n : n\in\mathbf{N}\}.$$

This proves P(2). In the general case, choose $a_0\in K$ and let $K' = K - a_0$. Thus $0\in K'$, and K' is a closed locally compact excellent convex set. By the first part of the proof there are a finite dimensional linear space L, a continuous linear functional f_0 on E and a sequence $\{P_n\}$ of polytopes such that

$$L + P_n = (K - a_0)\cap f_0^{-1}(<-\infty, n]); \quad n\in\mathbf{N}.$$

Hence

$$L + a_0 + P_n = K\cap (f_0^{-1}(<-\infty, n]) + a_0) \tag{4.12}$$

Let

$$f = f_0 - f_0(a_0), \quad M = L + a_0.$$

Then f is an affine continuous function, and M is a finite dimensional affine manifold. Furthermore, from (4.12) we get without difficulty

$$M + P_n = K \cap f^{-1}(<-\infty, n]); \quad n \in \mathbf{N}.$$

As in the preceding part of the proof, it follows that P(1) and P(2) are satisfied. □

Corollary 4.4 If K is a closed, locally compact excellent convex set, then K is finite dimensional.

Proof By P(2) and Lemma 2.9, the affine manifold generated by K is equal to the affine manifold generated by $M + P_2$. Since M is finite dimensional and P_2 is a polytope, this manifold is finite dimensional. □

Proposition 4.5 If K is a closed locally compact excellent convex set, then any convex function on K is upper semi-continuous.

Proof We know, by Corollary 4.4, that $K \subset E$, where E is a finite dimensional linear space. Hence we can and shall equip E with a norm $\|\cdot\|$ such that the closed ball $B_r = \{x : \|x\| \leq r\}$ is a polytope whenever $r > 0$. We now make use of the property P(3) of Theorem 4.3. Hence

$$K = \bigcup\{K_n : n \in \mathbf{N}\}, \tag{4.13}$$

where every K_n is a polyhedral convex set such that

$$K_n \subset \text{int}_K K_{n+1}. \tag{4.14}$$

Let f be a convex function on K. We have to show that f is locally upper bounded on K. Assume first that K itself is a polyhedral convex set. Let $r > 0$ be given. Then $K \cap B_r$ is a bounded polyhedral convex set. It follows, for instance by [Ro, pp. 170–171], that $K \cap B_r$ itself is a polytope. Let $e_1, \ldots, e_m$ be the extreme points of this set. Hence any x in $K \cap B_r$ can be written as a convex combination $x = \sum \lambda_j e_j$. Therefore

$$f(x) \leq \sum \lambda_j f(e_j) \leq \max\{f(e_J) : 1 \leq j \leq m\}.$$

This shows that f is bounded above on $K \cap B_r$. In particular, if $x_0 \in K$, and $r = \|x_0\| + 1$, then $K \cap B_r$ is a neighborhood of x_0 on which f is bounded above. We now consider the general case. Let $x_0 \in K$. According to (4.13), there is an n so that $x_0 \in K_n$. Choose $r = \|x_0\| + 1$. By the first part of the proof, the function f is bounded above on $B_r \cap K_{n+1}$, and therefore on the set $B_r \cap \text{int}_K K_{n+1}$. But this set is, by (4.14), a neighborhood of x_0 relative to K. □

5 Openness of affine maps between excellent convex sets

We study in the present section the problem when a continuous affine surjection $\varphi : K \mapsto Q$ between excellent convex sets K and Q will be open. As is well known, the map φ is open if and only if the correspondence $\varphi^{-1} : Q \mapsto 2^K$ is lower semi-continuous. Now a theorem in Kuratowski [Ku, vol. 2] asserts that if K and Q are contained in metric spaces, then a correspondence $\phi : Q \mapsto 2^K$ is lower semi-continuous if and only if the function

$$\delta : K \times Q \mapsto [0, \infty>: \delta(x, q) = \mathrm{dist}(x, \phi(q))$$

is upper semi-continuous. We prove that if K is contained in a normed space and the correspondence ϕ is *convex*, then δ is a convex function on $K \times Q$. Since we have proved in section 1 that $K \times Q$ is excellent if K and Q are excellent, the function δ is upper semi-continuous if and only if it is locally bounded above. We thus obtain a general criterion for ϕ to be lower semi-continuous. As corollaries we get that if K is bounded, then any $\varphi : K \mapsto Q$ is open, and that the same conclusion is valid if K and Q are locally compact closed excellent sets. A third consequence of this criterion is that if P is any convex set, and $\varphi : K \mapsto P$ is a *closed* continuous affine surjection, then φ is open. Finally, we prove that any excellent convex set K contained in a normed space is a *stable* convex [Pa], which means that the middle-point map $(a, b) \mapsto \frac{1}{2}(a + b)$ is open.

Throughout this section we assume that the convex set K is contained in a normed linear space $(E, \|\cdot\|)$. Furthermore, we shall assume that Q is a convex set contained in a metrizable locally convex vector space F.

We recall that a correspondence

$$\phi : Y \mapsto 2^X,$$

where X and Y are topological spaces, is called *lower semi-continuous* provided the set

$$\{y \in Y : \phi(y) \cap U \neq \emptyset\}$$

is open in Y whenever U is open in X.

The next lemma can be found in [Ku, v. 2, p. 63, Th. 3]. Actually, it is assumed in this theorem that the metric spaces are compact, but the proof works without this assumption.

Lemma 5.1 Let (X, ρ) and (Y, σ) be metric spaces, let

$$\phi : Y \mapsto 2^X$$

be a correspondence (by definition we require that every $\phi(y) \neq \emptyset$). Then ϕ is lower semi-continuous if and only if the function

$$\delta : X \times Y \mapsto [0, \infty\rangle : \delta(x, y) = \text{dist}(x, \phi(y)) \tag{5.1}$$

is upper semi-continuous. □

If X and Y are convex sets, then a correspondence

$$\phi : Y \mapsto 2^X$$

is said to be *convex* provided

$$\lambda_1 \phi(y_1) + \lambda_2 \phi(y_2) \subset \phi(\lambda_1 y_1 + \lambda_2 y_2) \tag{5.2}$$

whenever $y_1, y_2 \in Y$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$.

Lemma 5.2 Let

$$\phi : Q \mapsto 2^K$$

be a convex correspondence. Then the function

$$\delta : K \times Q \mapsto [0, \infty\rangle : \delta(x, q) = \text{dist}(x, \phi(q)) \tag{5.3}$$

is convex.

Proof Let $j = 1, 2$ and let $a_j \in K$, $q_j \in Q$, $x_j \in \phi(q_j)$ and $\lambda_j \geq 0$ with $\lambda_1 + \lambda_2 = 1$. By convexity of ϕ,

$$\lambda_1 x_1 + \lambda_2 x_2 \in \phi(\lambda_1 q_1 + \lambda_2 q_2)$$

Hence

$$\delta(\lambda_1(a_1, q_1) + \lambda_2(a_2, q_2)) \leq \lambda_1 \|a_1 - x_1\| + \lambda_2 \|a_2 - x_2\|.$$

From this inequality we immediately get

$$\delta(\lambda_1(a_1, q_1) + \lambda_2(a_2, q_2)) \leq \lambda_1 \delta((a_1, q_1)) + \lambda_2 \delta((a_2, q_2)).$$ □

Lemma 5.3 Let (X, ρ), (Y, σ) and δ be as in Lemma 5.1. Then the function δ is locally bounded above if and only if whenever $\{y_n\}$ is a convergent sequence on Y there is a bounded sequence $\{x_n\}$ on X such that $x_n \in \phi(y_n)$ for any $n \in \mathbf{N}$.

Proof We first note that it follows by a straightforward argument that δ is locally bounded above if and only if

$$\sup_{n \in \mathbf{N}} \delta(x_n, y_n) < \infty, \tag{5.4}$$

whenever $\{(x_n, y_n)\}$ is a convergent sequence on $X \times Y$.

Assume that δ is locally bounded above. Let $y_n \to y_0$ on Y. Choose $x_0 \in \phi(y_0)$. There are a neighborhood V of (x_0, y_0) and an $R < \infty$ such that $\delta(x, y) < R$ whenever $(x, y) \in V$. Furthermore, there is an $n_0 \in \mathbf{N}$ such that if $n \geq n_0$, then $(x_0, y_n) \in V$. Hence $\delta(x_0, y_n) < R$ whenever $n \geq n_0$. This implies, by the definition of δ, that there is $x_n \in \phi(y_n)$ such that $\rho(x_0, x_n) < R$ whenever $n \geq n_0$. It follows that if we choose an arbitrary $x_n \in \phi(y_n)$ when $n < n_0$, then the sequence $\{x_n\}$ is bounded and $x_n \in \phi(y_n)$ for any $n \in \mathbf{N}$.

To prove the converse implication, let $(x_n, y_n) \to (x_0, y_0)$ on $X \times Y$. Since $y_n \to y_0$, there is, by assumption, a bounded sequence $\{x'_n\}$ on X with $x'_n \in \phi(y_n)$ whenever $n \in \mathbf{N}$. Hence

$$\delta(x_n, y_n) \leq \rho(x_n, x'_n) \leq \rho(x_n, x_0) + \rho(x_0, x'_n).$$

Since $x_n \to x_0$, and the sequence $\{x'_n\}$ is bounded, it follows that (5.4) is valid. □

Theorem 5.4 Assume that K and Q are excellent convex sets, and that

$$\phi : Q \mapsto 2^K$$

is a convex correspondence. Then ϕ is lower semi-continuous if and only if the following condition (#) is satisfied

Whenever $\{q_n\}$ is a convergent sequence on Q, there is a bounded sequence $\{x_n\}$ on K such that $x_n \in \phi(q_n)$ for any $n \in \mathbf{N}$. (#)

Proof The correspondence ϕ is, by Lemma 5.1, lower semi-continuous if and only if the function δ is upper semi-continuous. Applying Lemma 5.2, we get that δ is a convex function on $K \times Q$. Since, by Proposition 1.16, the product $K \times Q$ is excellent, it follows that δ is upper semi-continuous if and only if δ is locally bounded above. However, by Lemma 5.3, this occurs if and only if the condition (#) is satisfied. □

Comment It is known that in the situation of Lemma 5.1, the correspondence ϕ is lower semi-continuous if and only if whenever $y_n \to y_0$ on Y and $x_0 \in \phi(y_0)$, there is $x_n \in \phi(y_n)$ such that $x_n \to x_0$. The condition (#) of Theorem 5.4 is thus a considerable weakening of this general criterion.

Corollary 5.5 Let $\varphi : K \mapsto Q$ be a continuous affine surjection. Then φ is open if and only if the following condition (*) is fulfilled.

Whenever $\{q_n\}$ is a convergent sequence on Q, there is a bounded sequence $\{x_n\}$ on K such that $\varphi(x_n) = q_n$ for any $n \in \mathbf{N}$. (*)

Proof By general topology, the map φ is open if and only if the correspondence

$$\varphi^{-1} : Q \mapsto 2^K$$

is lower semi-continuous. Since it is immediate that this correspondence is convex, the conclusion follows. □

Corollary 5.6 If K is a bounded excellent convex set, then any continuous affine surjection

$$\varphi : K \mapsto Q$$

is open.

Proof The condition (*) is in this case fulfilled. □

Corollary 5.7 If K and Q are closed locally compact excellent sets, then any continuous affine surjection

$$\varphi : K \mapsto Q$$

is open.

Proof As was noted in the proof of the theorem, the product $K \times Q$ is an excellent convex set. Since this set is also closed and locally compact, it follows from Proposition 4.5 that the function δ in Lemma 5.2 is locally bounded above. This means, by Lemma 5.3, that the condition (*) is fulfilled. □

Comment 5.8 Since any normed linear space is an excellent convex set, the Corollary 5.5 above gives in particular a necessary and sufficient condition for a continuous and linear surjection between two normed spaces to be open. Let us look at the following example:

Define

$$T : C[0,1] \mapsto C[0,1] : Tf(s) = \int_0^s f(t)dt; \quad s \in [0,1].$$

Then T is linear, continuous and injective. Hence it follows, by general functional analysis, that T is not open onto its image. Hence the condition (*) cannot be true. It is easy to show this directly. In fact, define for any $n \in \mathbf{N}$ the continuous function f_n on $[0,1]$ by requiring that f_n is zero on $[0, 1 - \frac{1}{n^2}]$, and that f_n is linear on $[1 - \frac{1}{n^2}, 1]$ with $f_n(1) = n$. Then $\|f_n\|_\infty = n$, whereas $\|Tf_n\|_\infty = \frac{1}{2n}$. Hence the condition (*) is violated.

□

Proposition 5.9 Let P be any non-empty convex set in a locally convex Hausdorff topological vector space, let K be an excellent convex set, and assume that the continuous affine surjection

$$\varphi : K \mapsto P$$

is *closed* (that is $\varphi(A)$ is closed in P whenever A is closed in K). Then φ is open, and hence P itself is an excellent convex set.

Proof It follows by general topology that φ is open if and only if the correspondence

$$\phi : K \mapsto 2^K : \phi(x) = \varphi^{-1}(\varphi(x)); \quad x \in K$$

is lower semi-continuous. It is easy to see that ϕ is convex. Furthermore, since $x \in \phi(x)$ whenever $x \in K$, it follows immediately that the condition (#) of Theorem 5.4 is fulfilled. Finally, applying Proposition 1.12, we get that P is an excellent convex set. □

Proposition 5.10 Any excellent convex set K contained in a normed vector space is a stable convex set.

Proof We have to prove that the middle point map

$$m : K \times K \mapsto K : (a,b) \mapsto \frac{a+b}{2}$$

is open. Since m is a continuous affine surjection, and $K \times K$ is excellent, it suffices to show that the condition (*) of Corollary 5.5 is satisfied. But this is easy. In fact, let $x_n \to x_0$ on K. Hence $(x_n, x_n) \to (x_0, x_0)$, and therefore the sequence $\{(x_n, x_n)\}$ is bounded. Since $m(x_n, x_n) = x_n$, we are through.

□

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