Characterizations of White Noise Test Function Space (S)

by

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1 Introduction

Let $S(R)$ be the Schwartz space of real valued rapidly decreasing functions on R. Denote it's dual space by $S'(R)$. Let $(S'(R), \mu)$ be the white noise space, i.e. μ is the standard Gaussian measure on $S'(R)$ with following characteristic function

$$\int_{S'(R)} e^{i\langle x,\xi\rangle} \mu(dx) = e^{-\frac{1}{2}|\xi|_2^2}, \qquad \xi \in S(R) \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the pairing of $S'(R)$ and $S(R)$, and $|\cdot|_2$ is the norm $L^2(R)$.

Let H be the following operator on $L^2(R)$

$$H = -\Big(\frac{d}{dx}\Big)^2 + x^2 + 1 \tag{1.2}$$

Let ξ_n be the Hermite function of order n, $n \geq 1$, i.e.

$$\xi_n(x) = \big(\sqrt{\pi}\, 2^{n-1}(n-1)!\big)^{\frac{1}{2}} H_{n-1}(x) e^{-\frac{x^2}{2}} \tag{1.3}$$

Here $H_n(x) = (-1)^n e^{x^2} \big(\frac{d}{dx}\big)^n e^{-x^2}$ is the Hermite polynomial of order n, $n \geq 0$.

It is well-known that the set $\{\xi_n, n \geq 1\}$ is contained in $S(R)$ and forms an orthonormal basis for $L^2(R)$, ξ_n is an eigenfunction of H with eigenvalue $2n$. By using the operator H, one constructs the white noise test function and generalized functional as follows: Let $\mathcal{P}$ be the algebra of polynomials, i.e. generated by $\langle x, \xi \rangle$, $\xi \in S(R)$. Define the following norm on $\mathcal{P}$ by

$$\|\varphi\|_{2,p} = |\Gamma(H^p)\varphi|_2, \qquad \varphi \in \mathcal{P}. \tag{1.4}$$

where $\Gamma(H^p)$ stands for the second quantization of H^p which satisfies

$$T(H^p) : \langle \cdot, f_1 \rangle \dots \langle \cdot, f_n \rangle := \langle \cdot, H^p f_1 \rangle \dots \langle \cdot, H^p f_n \rangle : \tag{1.5}$$

for $f_1 \dots f_n \in S(R)$. (See [1] for details.)

Let $(S)_p$ denote the completion of $\mathcal{P}$ with respect to norm $\|\cdot\|_{2,p}$. The white noise test function space (S) is defined as the intersection of $\{(S)_p, p\geq 0\}$ equipped with the projective limit topology. The dual of (S), denoted by $(S)^*$, is called the space of generalized Brownian functional. So we have following Gel'fand triple

$$(S) \subset L^2(S'(R), \mu) \subset (S)^* . \tag{1.6}$$

We refer reader to [1] for the motivations of the study of generalized Brownian functional. Now we introduce the S-transform of generalized functional, which plays a essential role in the study of white noise analysis. For $\Phi \in (S)^*$, the S-transform of Φ is a functional defined on $S(R)$ by

$$S\Phi(\xi) = \langle\langle \Phi : e^{\langle \cdot,\xi\rangle} :\rangle\rangle \qquad \xi \in S(R) . \tag{1.7}$$

Here $:e^{\langle \cdot,\xi\rangle}: = e^{\langle \cdot,\xi\rangle - \frac{1}{2}|\xi|^2_{L^2}}$.

We now present another transform for function Φ in $L^2(S'(R), \mu)$ introduced in [3], called H-transforms, which is quite useful sometimes. If $\alpha = (\alpha_1 \dots \alpha_m)$ is a multi-index of non-negative integers, we define $h_\alpha(u_1 \dots u_m) := h_{\alpha_1}(u_1)h_{\alpha_2}(u_2)\dots h_{\alpha_m}(u_m)$. Fix the orthonormal base in (1.3) for $L^2(R)$. Define

$$H_\alpha(x) = h_\alpha(\theta_1(x) \dots \theta_m(x)), \qquad x \in S'(R) \tag{1.8}$$

where $\theta_k(x) = \int_R \xi_k(t) dB_t(x) = \langle \xi_k, x\rangle$, $x \in S'(R)$. Then any $\Phi \in L^2(\mu)$ has unique expansion.

$$\Phi(x) = \sum_\alpha C_\alpha H_\alpha(x). \tag{1.9}$$

where $C_\alpha \in R$ for each multi-index α and $\|\Phi\|^2_{L^2} = \sum_\alpha \alpha! C^2_\alpha$, $\alpha! = \prod_{i=1}^{m} \alpha_m!$.

The H-transform $H(\Phi)$ of Φ is the formal power series in infinitely many complex variables $z_1, z_2, \dots$ defined by

$$H(\Phi) = \widetilde{\Phi}(z) = \sum_\alpha C_\alpha z^\alpha . \tag{1.10}$$

Many results about H-transform have been proved in [3]. For instance, it is shown that

$$\Phi = \int \widetilde{\Phi}(\theta + i\eta)\lambda(d\eta) \tag{1.11}$$

where $\theta = (\theta_1, \theta_2, \dots)$ with $\theta_k = \langle \xi_k, x\rangle$. as before. $\lambda(d\eta)$ is the measure on $R^{\mathbf{N}}$ defined by $\lambda(d\eta) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\eta_k^2/2} d\eta_k$.

For $X = \sum C_\alpha H_\alpha(x)$, $Y = \sum a_\beta H_\beta(x)$, the Wick product $X \diamond Y$ is defined in [3] by:

$$X \diamond Y = \sum_{\alpha,\beta} C_\alpha a_\beta H_{\alpha+\beta}(x) \tag{1.12}$$

if the right side makes sense in $L^2(\mu)$.

In this paper we give a characterization of white noise test function (S) in terms of the coefficients of its Hermite transform. It turns out that it is a quite good analogue of the characterization of Schwartz space in finite dimensional case. Thus one can think that the white noise test function is rapidly decreasing in some sense. We also obtained some sufficient conditions under which the generalized functional is in $L^2(\mu)$ and presented an example of positive generalized functional which is not absolutely continuous with respect to Gaussian measure.

2 Main Results

For any multi-index $\alpha = (\alpha_1 \dots \alpha_m)$, define

$$(2\mathrm{N})^\alpha = \prod_{k=1}^{m} (2k)^{\alpha_k} \tag{2.1}$$

We have following

Theorem 2.1: $\Phi \in (S)$ if and only if

$$\sup_\alpha C_\alpha^2 \alpha! (2\mathrm{N})^{\alpha P} < +\infty \qquad \forall P \geq 1 \tag{2.2}$$

where C_α are the coefficients of the H-transform of Φ.

Proof. Suppose $\Phi \in (S)$. By the definition of H-transform, we have that

$$\begin{aligned} \Phi &= \sum_\alpha C_\alpha H_\alpha(x) = \sum_\alpha C_\alpha \int \xi_1^{\otimes\alpha_1} \hat{\otimes} \xi_2^{\otimes\alpha_2} \hat{\otimes} \cdots \hat{\otimes} \xi_m^{\otimes\alpha_m} dB^{\otimes|\alpha|} \\ &= \sum_\alpha C_\alpha : \underbrace{\langle x, \xi_1\rangle \langle x, \xi_1\rangle \cdots \langle x, \xi_1\rangle}_{\alpha_1} \cdots \underbrace{\langle x, \xi_m\rangle \cdots \langle x, \xi_m\rangle}_{\alpha_m} : \end{aligned}$$

Hence, it holds that:

$$\begin{aligned} \Gamma(H^P)\Phi &= \sum_\alpha C_\alpha \Gamma(H^P) : \langle x, \xi_1\rangle \cdots \langle x, \xi_1\rangle \cdots \langle x, \xi_m\rangle \cdots \langle x, \xi_m\rangle : \\ &= \sum_\alpha C_\alpha : \langle x, H^P\xi_1\rangle \cdots \langle x, H^P\xi_1\rangle \cdots \langle x, H^P\xi_m\rangle \cdots \langle x, H^P\xi_m\rangle : \\ &= \sum_\alpha C_\alpha \prod_{k=1}^{m} (2k)^{\alpha_k P} : \langle x, \xi_1\rangle \cdots \langle x, \xi_1\rangle \cdots \langle x, \xi_m\rangle \cdots \langle x, \xi_m\rangle : \\ &= \sum_\alpha C_\alpha (2\mathrm{N})^{\alpha P} H_\alpha(x). \end{aligned}$$

Our assumption implies that $\Gamma(H^P)\Phi \in L^2(\mu)$, $\forall P \geq 1$. This is equivalent to

$$\sum_\alpha C_\alpha^2 \alpha! (2\mathrm{N})^{2\alpha P} < +\infty \qquad \forall P \geq 1 \tag{2.3}$$

which indicates

$$\sup_\alpha C_\alpha^2 \alpha! (2\mathrm{N})^{\alpha P} < +\infty \qquad \forall P \geq 1 . \tag{2.4}$$

Conversely, suppose (2.2) holds for any $P \geq 1$. We prove (2.3), i.e. $\Phi \in (S)$. In fact, for any $P \geq 1$

$$\begin{aligned}
\sum_\alpha C_\alpha^2 \alpha! (2\mathrm{N})^{\alpha 2P} &\leq \sum_\alpha C_\alpha^2 \alpha! (2\mathrm{N})^{\alpha 2(P+2)} (2\mathrm{N})^{-4\alpha} \\
&\leq \sum_\alpha \sup_\alpha C_\alpha^2 \alpha! (2\mathrm{N})^{\alpha 2(P+2)} \left((2\mathrm{N})^\alpha\right)^{-4} \\
&\leq \sup_\alpha C_\alpha^2 \alpha! (2\mathrm{N})^{\alpha 2(P+2)} \sum_\alpha \left((2\mathrm{N})^\alpha\right)^{-4} .
\end{aligned}$$

So in order to complete the proof, it suffices to show that for any positive integer $P \geq 2$

$$\sum_\alpha \left((2\mathrm{N})^\alpha\right)^{-P} < +\infty \tag{2.5}$$

For this end, we introduce following notation for any multi-index α

$$\text{Index}\,\alpha = \max\{m, \alpha_m \neq 0\} .$$

Then $\text{Index}\,\alpha < +\infty$ for any α, and

$$\sum_\alpha \left((2\mathrm{N})^\alpha\right)^{-P} = \sum_{N=0}^{\infty} \sum_{\text{Index}\,\alpha = N} \left((2\mathrm{N})^\alpha\right)^{-P}$$

Let

$$\begin{aligned}
a_N &\triangleq \sum_{\text{Index}\,\alpha = N} \left((2\mathrm{N})^\alpha\right)^{-P} \\
&= \sum_{\substack{(\alpha_1 \cdots \alpha_N) \\ \alpha_N \neq 0}} \left((2\mathrm{N})^\alpha\right)^{-P} \\
&= \sum_{\substack{\alpha_1 \cdots \alpha_{N-1} = 0 \\ \alpha_N = 1}}^{\infty} \prod_{k=1}^{N} (2k)^{-P\alpha_k} \\
&= \prod_{k=1}^{N-1} \left(\sum_{\alpha_k = 0}^{\infty} \left((2k)^{-P}\right)^{\alpha_K} \right) \left(\sum_{\alpha_N = 1}^{\infty} \left((2N)^{-P}\right)^{\alpha_N} \right) \\
&= \prod_{k=1}^{N-1} \left(\frac{(2k)^P}{(2k)^P - 1} \right) \left(\frac{1}{(2N)^P - 1} \right)
\end{aligned}$$

Now we calculate

$$\begin{aligned}\frac{a_N}{a_{N+1}} - 1 &= \frac{(2N+2)^P - 1}{(2N)^P} - 1 \\ &= \frac{\sum_{k=0}^{P} C_P^k (2N)^{P-k} 2^k - 1 - (2N)^P}{(2N)^P} \\ &= 2P\frac{1}{2N} + \sum_{k=2}^{P} C_P^k 2^k \frac{1}{(2N)^k} - \frac{1}{(2N)^P}\end{aligned}$$

Therefore

$$\lim_{N\to\infty} N\left(\frac{a_N}{a_{N+1}} - 1\right) = \lim_{N\to\infty} 2P\frac{N}{2N} = P > 1$$

From the Albel criteria for the convergence of series, we conclude that

$$\sum_{N=0}^{\infty} a_N < +\infty$$

This proves (2.5).

Corollary 2.2: $X, Y \in (\varphi) \Rightarrow X \diamond Y \in (\varphi)$.

Proof: We suppose the H-transforms of X and Y are given as following

$$\widetilde{X} = \sum_{\alpha} C_\alpha Z^\alpha, \qquad \tilde{Y} = \sum_{\beta} \bar{C}_\beta Z^\beta .$$

By definition the H-transform $\widetilde{X \diamond Y}$ of $X \diamond Y$ is

$$\begin{aligned}\widetilde{X \diamond Y} = \tilde{X} \cdot \tilde{Y} &= \sum_{\alpha,\beta} C_\alpha \bar{C}_\beta Z^\alpha \cdot Z^\beta \\ &= \sum_K C_K^* Z^K\end{aligned}$$

Now for any $P \geq 1$

$$\begin{aligned}&\sup_K \sqrt{K!}\, |C_K^*| \left((2N)^K\right)^P \\ &\leq \sup_K \sum_{\alpha+\beta=K} \sqrt{K!}\, |C_\alpha||\bar{C}_\beta| (2\mathrm{N})^{PK} \\ &\leq \sup_K \sum_{\alpha+\beta=K} \sqrt{\alpha!}\, |C_\alpha| \sqrt{\beta!}\, |\bar{C}_\beta| (2\mathrm{N})^{PK} \sqrt{\frac{K!}{\alpha!\beta!}}\end{aligned}$$

$$\leq \sup_K \sum_{\alpha+\beta=K} \sqrt{\alpha!}\,|C_\alpha|\sqrt{\beta!}\,|\bar{C}_\beta|(2\mathrm{N})^{PK}\sqrt{\prod_i \frac{K_i!}{\alpha_i!\beta_i!}}$$

$$(K_i = \alpha_i + \beta_i)$$

$$\leq \sup_K \sum_{\alpha+\beta=K} \sqrt{\alpha!}\,|C_\alpha|\sqrt{\beta!}\,|\bar{C}_\beta|(2\mathrm{N})^{PK}\sqrt{\prod_i 2^{K_i}}$$

$$\leq \sup_K \sum_{\alpha+\beta=K} \sqrt{\alpha!}\,|C_\alpha|(2N)^{\alpha(P+1)}\sqrt{\beta!}\,|\bar{C}_\beta|(2\mathrm{N})^{\beta(P+1)}$$

$$\leq \left(\sum_\alpha \sqrt{\alpha!}\,|C_\alpha|(2\mathrm{N})^{\alpha(P+1)}\right)\left(\sum_\beta \sqrt{\beta!}\,|\bar{C}_\beta|(2\mathrm{N})^{\beta(P+1)}\right) < +\infty$$

This implies from Theorem 2.1 that $X \diamond Y \in (\varphi)$.

In the rest of this section, we give some sufficient conditions under which the generalized functional Φ is in $L^2(\mu)$. First we recall that for any $\Phi \in (S)^*$, $\Phi \sim (F^n, n \geq 0)$ and there exists some $P \geq 0$

$$F^n \in S_{-P}(R^n), \quad \forall n \geq 0, \quad \sum_{n=0}^{\infty} n!|F^n|_{2,-P}^2 < +\infty .$$

The S-transform of Φ is given:

$$S\Phi(\xi) = \sum_{n=0}^{\infty} \langle F^n, \xi^{\otimes n}\rangle . \tag{2.6}$$

We claim that for any $m \geq 0$, $\eta_1 \ldots \eta_m \in S(R)$, the mapping $S\Phi(\lambda_1\eta_1 + \cdots + \lambda_m\eta_m) : R^m \to R$ has an entire analytic extension to $Z \in \mathbf{C}^m$. This follows by following estimates

$$\sum_{n=0}^{\infty} |\langle F^n, (\lambda_1\eta_1 + \cdots + \lambda_m\eta_m)^{\otimes n}\rangle|$$

$$\leq \sum_{n=0}^{\infty} |F^n|_{2,-P}\,|\lambda_1\eta_1 + \cdots + \lambda_m\eta_m|_{2,P}^n$$

$$\leq \left(\sum_{n=0}^{\infty} n!|F^n|_{2,-P}^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \frac{1}{n!}(|\lambda_1||\eta_1|_{2,P} + \cdots + |\lambda_m||\eta_m|_{2,P})^{2n}\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{n=0}^{\infty} n!|F^n|_{2,-P}^2\right)^{\frac{1}{2}} \prod_{K=1}^{m} \left(\sum_{n=0}^{\infty} \frac{|\lambda_K|^2|\eta_K|_{2,P}^2}{n!}\right)^{\frac{m}{2}} < +\infty$$

We still use $S\Phi(z_1\eta_1 + \cdots + z_m\eta_m)$ to denote the extension of $S\Phi(\lambda_1 + \cdots + \lambda_m\eta_m)$. Define a measurable map $F : C_0^{\mathrm{N}} \to C$ as

$$F(z) = S\Phi(z_1\xi_1 + z_2\xi_2 + \cdots + z_n\xi_n + \cdots) \tag{2.7}$$

Here ξ_i is the Hermite function.

Let $\lambda(dx)$ be the measure on R^{N} defined as in section 1. Then we state following

Theorem 2.3: If $\iint |F(x+iy)|^2 \lambda(dx)\lambda(dy) < +\infty$, then $\Phi \in L^2(\mu)$.

Proof: We note first that $F(z)$ is analytic in each $z_K \in C$ and, for any $z_1 \ldots z_m \in C^m$

$$\begin{aligned}\overline{F(z)} &= \sum_{n=0}^{\infty} \overline{\langle F^n, (z_1\xi_1 + \cdots + z_m\xi_m)^{\otimes n}\rangle} \\ &= \sum_{n=0}^{\infty} \langle F^n, (\bar{z}_1\xi_i + \cdots + \bar{z}_m\xi_m)^{\otimes n}\rangle \\ &= F(\bar{z})\end{aligned}$$

Thus from (L.Ø.U [4], page 11) and our assumption, it follows that $F(z)$ is a H-transform of a random variable $X \in L^2(\mu)$. But the Theorem 5.7 in [3] says that

$$F(z) = SX(z_1\xi_1 + z_2\xi_2 + \cdots) \qquad \text{for } z = (z_1, z_2, \ldots) \in C_0^{\mathbf{N}}$$

Consequently we get that

$$SX(z_1\xi_1 + z_2\xi_2 + \cdots) = S\Phi(z_1\xi_1 + z_2\xi_2 + \cdots), \qquad z \in C_0^{\mathbf{N}}. \tag{2.8}$$

Fix any $\xi \in S(R)$, we have that $\sum_{K=1}^{N} \langle \xi, \xi_K\rangle \xi_K \to \xi$ in $S(R)$. Hence the following holds

$$\begin{aligned}SX(\xi) &= \lim_{N\to\infty} SX\left(\sum_{K=1}^{N} \langle \xi, \xi_K\rangle \xi_K\right) = \lim_{N\to\infty} S\Phi\left(\sum_{K=1}^{N} \langle \xi, \xi_K\rangle \xi_K\right) \\ &= S\Phi(\xi)\end{aligned} \tag{2.9}$$

Since the S-transform uniquely determines the generalized functional, it is obtained by (2.9) that $\Phi = X \in (L^2)$. This proves the Theorem.

For the application of this theorem, one can see that the S-transform $S\Phi(\xi) = \int_0^1 dt\, e^{\int_0^t f(u)\xi(u)du}$ represents an element Φ in $L^2(\mu)$ if $f \in L^2(0,1)$, since

$$\begin{aligned}&\lim_{N\to\infty} \iint |S\Phi(z_1\xi_1 + \cdots + z_N\xi_N)|^2 \lambda(dx)\lambda(dy) \\ &= \lim_{N\to\infty} \iint \Big| \int_0^1 dt\, e^{\sum_{i=1}^N (x_i+iy_i)\int_0^t f(u)\xi_i(u)du}\Big|^2 \lambda(dx)\lambda(dy) \\ &= \lim_{N\to\infty} \iint \Big| \int_0^1 dt\, e^{\sum_{i=1}^N (x_i+iy_i)\int_0^t f(u)\xi_i(u)du}\Big|^2 \lambda(dx)\lambda(dy) \\ &\leq \lim_{N\to\infty} \iint \int_0^1 dt\, e^{2\sum_{i=1}^N x_i \int_0^t f(u)\xi_i(u)du} \lambda(dx)\lambda(dy) \\ &= \lim_{N\to\infty} \int_0^1 dt\, e^{2\sum_{i=1}^N (\int_0^t f(u)\xi_i(u)du)^2} \\ &= \int_0^1 dt e^{2\int_0^t f^2(u)du} < +\infty.\end{aligned}$$

Finally, we present one class of examples of positive generalized functionals which is generally not absolutely continuous with respect to Gaussian measure.

Assume $\sigma(w,s)$ is a $\mathcal{F}_t=\sigma(B_s, s\leq t)$-adapted process such that $|\sigma(w,s)|\leq M$, (M is a constant). We define process X_t as

$$X_t = \begin{cases} \int_0^t \sigma(w,s)dB_s & t\geq 0 \\ 0 & t\leq 0 \end{cases} \tag{2.10}$$

Then X_t is a continuous martingale. Due to the fact $\int_{-\infty}^{\infty}\frac{|X_t|}{1+t^2}\,dt<+\infty$ a.s. (This is from $E\int_{-\infty}^{\infty}\frac{|X_t|}{1+t^2}\,dt<+\infty$), we regard $X.(w)$ as a mapping from (Ω,P) into $S'(R)$.

Denote the distributional derivative of X by $\dot{X}$. Let ν be the distribution of $\dot{X}$ on $S'(R)$. Then we have

Theorem 2.4: $\nu\in(S)^*_+$.

Proof: According to the characterization given by J. Potthoff and L. Streit, we only need to prove that the characteristic function $F(\xi)$ of ν is a U-functional. (See [1] for the definition of U-functional.) By definition, for $\xi\in S(R)$

$$\begin{aligned} F(\xi) &= \int e^{i\langle x,\xi\rangle}\nu(dx)=\int e^{i\langle \dot{X},\xi\rangle}dP \\ &= \int e^{-i\int X_s\dot{\xi}(s)ds}dP=\int e^{i\int \xi(s)dX_s}dP \\ &= \int e^{i\int_0^\infty \xi(s)\sigma(s)dB_s}dP \end{aligned}$$

In order to show that F is ray entire, it suffices to prove $F(z\xi)=\int e^{iz\int_0^\infty \xi(s)\sigma(s)dB_s}dP$ is analytic for $\xi\in S(R)$. Fix $z_0\in\mathbf{C}$, since $e^{iz\int_0^\infty \xi(s)\sigma(s)dB_s}\to e^{iz_0\int_0^\infty \xi(s)\sigma(s)dB_s}$, as $z\to z_0$ and

$$\begin{aligned} &\sup_{|z-z_0|\leq 1}\int \left|e^{iz\int_0^\infty \xi(s)\sigma(s)dB_s}\right|^2 dP \\ &\leq \sup_{|z-z_0|\leq 1}\int e^{-2\operatorname{Im} z\int_0^\infty \xi(s)\sigma(s)dB_s}dP \\ &\leq \sup_{|z-z_0|\leq 1}\left(\int e^{-2\operatorname{Im} z\int_0^\infty \xi(s)\sigma(s)dB_s-2(\operatorname{Im} z)^2\int_0^\infty \xi^2(s)\sigma^2(s)ds}dP\right)e^{2(\operatorname{Im} z)^2M^2\int_0^\infty \xi^2(s)ds} \\ &= \sup_{|z-z_0|\leq 1} e^{2(\operatorname{Im} z)^2M^2\int_0^\infty \xi^2(s)ds}<+\infty \end{aligned}$$

We conclude that $F(z\xi)\to F(z_o\xi)$ as $z\to z_0$, which means that $F(z\xi)$ is continuous in z. On the other hand, for any closed curve D in complex plane, it holds that

$$\begin{aligned} \int_D F(z\xi)dz &= \int_D dz\int e^{iz\int_0^\infty \xi(s)\sigma(s)dB_s}dP \\ &= \int dP\int_D e^{iz\int_0^\infty \xi(s)\sigma(s)dB_s}dz=0 \end{aligned} \tag{2.11}$$

this is because $e^{iz\int_0^\infty \xi(s)\sigma(s)dB_s}$ is analytic and $e^{iz\int_0^\infty \xi(s)\sigma(s)dB_s}$ is absolutely integrable. Thus we deduce from Morena's Theorem that $F(z\xi)$ is analytic.

Furthermore, for any $R>0$

$$\begin{aligned}
\sup_{|z|=R} |F(z\xi)| &= \sup_{|z|=R} \left| \int e^{iz\int_0^\infty \xi(s)\sigma(s)dB_s} dP \right| \\
&\leq \sup_{|z|=R} \int e^{-\operatorname{Im} z\int_0^\infty \xi(s)\sigma(s)dB_s} dP \\
&\leq \sup_{|z|=R} \left(\int e^{-\operatorname{Im} z\int_0^\infty \xi(s)\sigma(s)dB_s - \frac{1}{2}(\operatorname{Im} z)^2 \int_0^\infty \xi^2(s)\sigma^2(s)ds} dP \right) e^{\frac{1}{2}M^2R^2\int_0^\infty \xi^2(s)ds} \\
&\leq e^{\frac{1}{2}M^2R^2|\xi|_{2,0}^2} .
\end{aligned} \tag{2.12}$$

The combination of (2.11) and the ray entire property shows that F is a U-functional. This ends the proof.

Remark 2.1: Theorem 2.3 can be easily extended to the case which X_t is a Ito process, that is,

$$X_t = \int_0^t \sigma(w,s)dB_s + \int_0^t b(w,s)ds .$$

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