The Shuffle Filtration of Hochschild Cohomology

Arne B. Sletsjøe

In [Q] Quillen introduced a decomposition of Hochschild cohomology in the case where the ground field contains the rational numbers. Later this Hodge-type decomposition has been studied from many different points of view. Gerstenhaber and Schack define in [G-S] a decomposition in terms of eigenvectors of certain operators on the Hochschild complex. Burghelea and Vigue-Poirrier ([B-V]) use powers of the differentials of a minimal model of the algebra. Finally, in [L] Loday defines the γ-filtration of the Hochschild complex, also proving that in the case $\mathbf{Q} \subset k$ this gives a decomposition of Hochschild homology whcich coincides with the one defined in [G-S] and [B-V]. In [R] Ronco encloses the circle, proving that all decompositions are the same as defined in [Q].

The Hochschild complex has a structure as an associative algebra via the shuffle-product. By considering shuffle-powers of the augmentation ideal we obtain a filtration of the Hochschild cochain complex. The purpose of this noter is to show that this filtration coincides with the γ-filtration in [L] and thus in characteristic zero gives another interpretation of the decomposition of Hochschild cohomology.

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Let A be a commutative k-algebra and M a symmetric A-bimodule (i.e. with commuting left and right action). We define the "symmetrized" bar complex

$$B_n = A \otimes A^{\otimes n}$$

viewed as a symmetric A-bimodule through multiplication on the left A factor. A general element $a \otimes a_1 \otimes \ldots \otimes a_n$ is denoted $a[a_1, \ldots, a_n]$ and the differential

$$\partial = \partial_n : B_n \to B_{n-1}$$

is given by the action on the element $a[a_1, \ldots, a_n]$;

$$\begin{aligned}\partial(a[a_1,\ldots,a_n]) =& aa_1[a_2,\ldots,a_n] + \sum_{i=1}^{n-1}(-1)^i a[a_1,\ldots,a_ia_{i+1},\ldots,a_n] \\ &+ (-1)^n a_n a[a_1,\ldots,a_{n-1}]\end{aligned}$$

In particular $\partial_1 = 0$.

Definition 1.

With the notation as above we define Hochschild homology of A with coefficients in M as

$$H_\bullet(A,M) = H(B_\bullet \otimes_A M)$$

Hochschild cohomology of A with values in M is defined as

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It is easily seen that Hochschild cohomology can be computed as cohomology of the complex $Hom_k(A^{\otimes\bullet}, M)$ with differential

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Hochschild cohomology is the correct cohomology in the category of (non-commutative) k-algebras in the sense that all cohomology groups of order ≥ 2 vanish for a free k-algebra. When working in the category of commutative k-algebras, where the free objects are polynomial rings, we need a cohomology theory where the higher cohomology groups vanish on these objects. Hochschild cohomology does not satisfy this condition, and we need some modifications of the complex.

Definition 2.

A permutation $\pi \in S_n$ is called a shuffling if $\exists \quad 1 \leq i \leq n$ such that

$$\pi(j) < \pi(k) \qquad \text{whenever} \quad 1 \leq j < k \leq i \quad \text{or} \quad i+1 \leq j < k \leq n$$

We name the shufflings by the i; $(i, n-i)$-shufflings.

Notice that the i in the definition is not unique; a shuffling π is a $(i, n-i)$-shuffling for more than one i.

There is a 1-dimensional representation of the group-ring $\mathbf{Q}[S_n]$ given by the signature of a permutation

$$\begin{aligned} sgn:\quad \mathbf{Q}[S_n] &\longrightarrow \mathbf{Q}\\ \pi &\longmapsto sgn\pi\end{aligned}$$

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We also define $s_n = \sum_{i=0}^n s_{i,n-i}$. Notice that $sgn(s_{i,n-i}) = \binom{n}{i}$ and consequently $sgn(s_n) = 2^n - 2$.

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We would like to call $s_{i,n-i}$ a "shuffle-product", and the following will justyfy the name. The tensor algebra $T = T_k(A) = \oplus_{n\geq 0} A^{\otimes n}$ is obviously an algebra under the tensor-product. But it is also an algebra under the shuffle-product

$$[a_1,\ldots,a_i]\star[a_{i+1},\ldots,a_n] = s_{i,n-i}[a_1,\ldots,a_n]$$

The $\star$-product is associative and graded-commutative and we have the relation ([B])

$$\partial([a_1,\ldots,a_i]\star[a_{i+1},\ldots,a_n]) = \partial[a_1,\ldots,a_i]\star[a_{i+1},\ldots,a_n] + (-1)^i[a_1,\ldots,a_i]\star\partial[a_{i+1},\ldots,a_n]$$

This makes the algebra $(T,\star)$ into a differential graded-commutative algebra and the shuffle-product is a "real" product. We denote by Λ the algebra $\oplus_{n\geq 0} A^{\otimes n}$ with the shuffle-product $\star$ (to distinguish it from the ordinary tensoralgebra T with the $\otimes$-product) and put $\Lambda_+ = \oplus_{n>0} A^{\otimes n}$; the augmentation ideal. Thus we obtain a descending sequence of ideals of Λ;

$$\Lambda_+ \supset \Lambda_+\star\Lambda_+ \supset \Lambda_+\star\Lambda_+\star\Lambda_+ \supset\ldots$$

Notice that the algebra Λ is not generated in degree 1. In fact it is not even finitely generated.

The sequence of inclusions of ideals induces a sequence of surjections

$$\ldots\to\Lambda/_{\Lambda_+\star\Lambda_+\star\Lambda_+}\to\Lambda/_{\Lambda_+\star\Lambda_+}\to 0 \qquad (1)$$

The $\star$-product is homogenous and ∂ is a differential with respect to $\star$. The quotients $\Lambda/_{\Lambda_+^{\star n}}$ are thus associative, differential graded-commutative algebras. The sequence itself is not stabilized, but if we focus on each degree (i.e. $\otimes$-degree) it will stabilize. It is easily seen that in $\otimes$-degree n there are no $\star$-products of degree $\geq n+1$, and in that case the quotient is no longer a quotient, but the whole algebra Λ.

The quotients $\Lambda_+/_{\Lambda_+^{\star n}}$ are equipped with a symmetric A-bimodule structure by tensoring by A from left. Thus we obtain a sequence of symmetric A-bimodules

$$\ldots\to\Lambda_4\to\Lambda_3\to\Lambda_2\to 0$$

where $\Lambda_j = A\otimes_k\Lambda_+/_{\Lambda_+^{\star n}}$. If M is another symmetric A-bimodule we get, as before, the two complexes

$$\Lambda_j\otimes_A M \qquad \text{and} \qquad Hom_A(\Lambda_j, M) \qquad j\geq 2$$

Let us consider the complexes $Hom_A(\Lambda_j, M)$. Since the sequence (1) is a sequence of surjections there is a filtration of complexes

$$0\subset Hom_A(\Lambda_2,M)\subset Hom_A(\Lambda_3,M)\subset\ldots$$

where we as well could write $Hom_A(\Lambda_j,M) = Hom_k(\Lambda_+/_{\Lambda_+^{\star n}},M)$ with the differential given in the beginning of this section. We put $F^jC^\bullet = Hom_A(\Lambda_j,M)$ and consider the short-exact sequences of complexes

$$0\to F^nC^\bullet\to F^{n+1}C^\bullet\to F^{n+1}C^\bullet/_{F^nC^\bullet}\to 0$$

Summing up over n we obtain another exact sequence of complexes

$$0 \to \oplus_n F^n C^\bullet \to \oplus_n F^{n+1} C^\bullet \to \oplus_n F^{n+1} C^\bullet /_{F^n C^\bullet} \to 0$$

and therefore an exact couple

$$\begin{array}{ccc} H(\oplus_n F^n C^\bullet) & \longrightarrow & H(\oplus_n F^n C^\bullet) \\ \nwarrow & & \swarrow \\ & H(\oplus_n F^{n+1} C^\bullet /_{F^n C^\bullet}) & \end{array}$$

Remembering the observation that the sequence (1) is stabilized to the Hochschild complex in each degree we have proved the following

Theorem 3. (Shuffle-filtration spectral sequence)
There is a 2. quadrant spectral sequence

$$E_1^{p,q} = H^{p+q}(F^{-p+1} C^\bullet /_{F^{-p} C^\bullet})$$

converging to the Hochschild cohomology $H^\bullet(A, M)$.

◇

Notice that $E_1^{-1,n+1} = Harr^n(A, M)$ is the Harrison cohomology of A.

Definition 4.
Let $I = (i_1, \ldots, i_m)$ be anordered m-tuple of positive integers such that $i_1 + \ldots + i_m = n$. An I-shuffling is a permutation $\pi \in S_n$ with the property

$$\pi(j) < \pi(k) \qquad \text{whenever} \quad 1 \leq j < k \leq i_1 \quad \text{or} \quad \alpha_l + 1 \leq j < k \leq \alpha_{l+1}$$

for some $1 \leq l \leq n-1$, where $\alpha_l = \alpha_l(I) = i_1 + \ldots + i_l$ for $l \geq 1$ and $\alpha_0 = 0$.

For I defined as above we let

$$s_I = \sum (sgn\pi)\pi$$

where the sum is taken over all I-shufflings, the "multi-shuffle-products".

Remember that $s_n = \sum_{i=1}^{n-1} s_{i,n-i}$. A better name would have been $s_n^{(2)}$, since it contains all squares. For the same reason we put

$$s_n^{(m)} = \sum s_I$$

where the sum is taken over all m-tuples as defined in Definition 4.

Put $I(j) = I - (0, \ldots, 0, 1, 0, \ldots, 0)$, subtraction as m-tuples by 1 in the j-th place.

Lemma 5.

$$\partial s_I[r_1,\dots,r_n] = \sum_{j=0}^{n-1}(-1)^{\alpha_j} s_{I(j)}[r_1,\dots,r_{\alpha_j}]\otimes\partial[r_{\alpha_j+1},\dots,r_{\alpha_{j+1}}]\otimes[r_{\alpha_{j+1}+1},\dots,r_n]$$

Proof. Repeated use of formula (1.)

◇

Lemma 6.

$$\begin{aligned}\partial[r_1,\dots,r_n] =& \partial[r_1,\dots,r_{\alpha_1+1}]\otimes[r_{\alpha_1+2},\dots,r_n]\\ &+\sum_{j=1}^{m-2}(-1)^{\alpha_j}[r_1,\dots,r_{\alpha_j}]\otimes\partial[r_{\alpha_j+1},\dots,r_{\alpha_{j+1}+1}]\otimes[r_{\alpha_{j+1}+2},\dots,r_n]\\ &+[r_1,\dots,r_{\alpha_{m-1}}]\otimes\partial[r_{\alpha_{m-1}+1},\dots,r_n]\end{aligned}$$

Proof. Repeated use of Proposition 2.3 of [B].

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Proposition 7.
Fix $m \geq 1$. The family $\{s_n^{(m)}\}$ commutes with the differential ∂, i.e.

$$\partial s_n^{(m)} = s_{n-1}^{(m)}\partial$$

Proof. An easy consequence of Lemma 5 and 6.

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The element $s_n^{(m)}$ is the sum of all m-multi-shuffles and plays an important role in this theory. Nevertheless it is lacking some good properties. We have to introduce a related element, $e_n^{(m)}$, defined in the next lemma, which essentially is due to Barr ([B]). He stated it for $m = 2$ only, but the proof workes also for $m \geq 3$.

Lemma 8.
Given $s_n^{(k)}$ as above, there exists another element in $\mathbf{Q}[S_n]$, denoted $e_n^{(k)}$, with the following properties;

i) $e_n^{(k)}$ is a polynomial in $s_n^{(k)}$ without constant term
ii) $sgn(e_n^{(k)}) = 1$

iii) $\partial e_n^{(k)} = e_{n-1}^{(k)}\partial$
iv) $(e_n^{(k)})^2 = e_n^{(k)}$
v) $e_n^{(k)} \cdot s_I = s_I$ for all k − shuffleproducts s_I;
$I = (p_1, \ldots, p_k)$ and $p_1 + p_2 + \ldots + p_k = n$

Proof. We have $sgn\, s_n^{(k)} \neq 0$, in fact [L] gives $sgn\, s_n^{(k)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} i^n$. Put $e_k^{(k)} = \frac{1}{k!} s_k^{(k)} = \epsilon_k$. Suppose we have found $e_k^{(k)}, e_{k+1}^{(k)}, \ldots, e_{n-1}^{(k)}$ satisfying the given conditions. Suppose $e_{n-1}^{(k)} = p(s_{n-1}^{(k)})$. We define

$$e_n^{(k)} = p(s_n^{(k)}) + (1 - p(s_n^{(k)})) \cdot \frac{s_n^{(k)}}{sgn\, s_n^{(k)}}$$

We start by proving the lemma for $e_k^{(k)}$. By construction it satisfies i) and ii). Furthermore $\partial \epsilon_k = 0 = e_{k-1}^{k} \partial$. $\epsilon_k^2 = \epsilon_k$ and the only k-shuffling in s_k is multiplication by ϵ_k. Hence ϵ_k satisfies i)-v).

Consider $e_n^{(k)}$. Once more; by construction it satisfies i) and ii). In [L] Loday proves that $\partial s_n^{(k)} = s_{n-1}^{(k)} \partial$ and therefore

$$\begin{aligned}
\partial e_n^{(k)} &= p(s_{n-1}^{(k)})\partial + \frac{1}{sgn\, s_n^{(k)}}(1 - p(s_{n-1}^{(k)})) \cdot s_{n-1}^{(k)} \partial \\
&= e_{n-1}^{(k)} \partial + + \frac{1}{sgn\, s_n^{(k)}}(1 - e_{n-1}^{(k)}) \cdot s_{n-1}^{(k)} \partial \\
&= e_{n-1}^{(k)} \partial
\end{aligned}$$

since $s_{n-1}^{(k)} = \sum_I s_I$ and $s_{n-1}^{(k)} - e_{n-1}^{(k)} s_{n-1}^{(k)} = 0$. Furthermore, $\partial (e_n^{(k)})^2 = (e_{n-1}^{(k)})^2 \partial = e_{n-1}^{(k)} \partial = \partial e_n^{(k)}$. Hence $\partial ((e_n^{(k)})^2 - e_n^{(k)}) = 0$ and therefore $(e_n^{(k)})^2 = e_n^{(k)}$. The equalities

$$\begin{aligned}
&\partial e_n^{(k)} s_I[r_1, \ldots, r_n] \\
&\quad = e_{n-1}^{(k)} \partial s_I[r_1, \ldots, r_n] \\
&\quad = \sum_{j=0}^{n-1} (-1)^{\alpha_j} e_{n-1}^{(k)} s_{I(j)}[r_1, \ldots, r_{\alpha_j}] \otimes \partial [r_{\alpha_j+1}, \ldots, r_{\alpha_{j+1}}] \otimes [r_{\alpha_{j+1}+1}, \ldots, r_n] \\
&\quad = \sum_{j=0}^{n-1} (-1)^{\alpha_j} s_{I(j)}[r_1, \ldots, r_{\alpha_j}] \otimes \partial [r_{\alpha_j+1}, \ldots, r_{\alpha_{j+1}}] \otimes [r_{\alpha_{j+1}+1}, \ldots, r_n] \\
&\quad = \partial s_I[r_1, \ldots, r_n]
\end{aligned}$$

implies that $\partial(e_n^{(k)} s_I - s_I) = 0$ hence $e_n^{(k)} s_I - s_I = sgn(e_n^{(k)} s_I - s_I)\epsilon_n = 0$. Thus we have also proved v), which completes the proof.

◇

Corollary 9.

The ideal in $\mathbf{Q}[S_n]$ generated by all $(i, n-i)$-shufflings equals the principal ideal generated by $e_n^{(k)}$, equals the principal ideal generated by $s_n^{(k)}$.

Proof. An immediate consequence of Lemma 8.

◇

Theorem 10. ([B-V], [G-S], [Q], [L])

If $\mathbf{Q} \subset k$ the decomposition of Hochschild cohomology

$$H^n(A, M) = \oplus_i H^n_{(i)}(A, M)$$

is obtained by putting

$$H^n_{(i)}(A, M) = E_1^{-i,n+i}$$

where the term $E_1^{-i,n+i}$ referres to the spectral sequence of Th.3.

Proof.Loday has shown [L] that there is a γ-filtration $F_i^\gamma B_n$ of B_n which in the case $\mathbf{Q} \subset k$ gives the same decomposition of Hochschild homology as studied in [Q], [B-V] and [G-S]. Following Loday it is easy to show that F_m^γ is generated by $s_n^{(m-1)}, s_n^{(m)}, \ldots, s_n^{(n)}$. But this is exactly the ideal generated by the $(m-1)$-multishuffles, and the γ-filtration and the shuffle-filtration coincide. Hence Th.3.7. of [L] gives the desired result.

◇

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