ISBN 82–553–0755–9
Pure Mathematics

No.19
November 1991

On the spectral sequence for the equivariant cohomology of a circle action.

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PREPRINT SERIES – Matematisk institutt, Universitetet i Oslo

When a circle group S^1 is acting continuously on a paracompact topological space X, an important invariant of the group action is the equivariant cohomology ring $H^\star_{S^1}(X;k)$ where k is a field of arbitrary characteristic. This cohomology ring is the cohomology of the space X_{S^1} which is the total space of the Borel fibering ([1,3])

$$X \to X_{S^1} \to B_{S^1}.$$

The spectral sequence E_r, $1 \leq r \leq \infty$, of this fibering is such that E_∞ is the sum of subquotients

$$F^q/F^{q-1} \simeq E^{\star q}_\infty \quad ,q \geq 0,$$

where $F^{q-1} \subset F^q \subset H^\star_{S^1}(X;k)$ is a filtration of the module $H^\star_{S^1}(X;k)$ over $k[t] = H^\star(B_{S^1};k)$ where t is a generator of $H^2(B_{S^1};k)$.

We now state the result of this paper. We assume that

$$dim_k H^q(X;k) < \infty \text{ for } q \geq 0.$$

Theorem.

As graded modules over the polynomial ring $k[t]$ the cohomology module $H^\star_{S^1}(X;k)$ is isomorphic to the module E_∞ of the spectral sequence.

When $Y \subseteq X$ is a closed invariant subspace, the corresponding statement on $H^\star_{S^1}(X,Y;k)$ is equally valid.

The case of $H^\star_{S^1}(X,Y;k)$ is similar to the case of $H^\star_{S^1}(X;k)$ and we focus on the latter.

The localization theorem for equivariant cohomology will not be used in this paper. Hence the field k may be of any characeristic.

We will define a mapping of sets

$$E : H^\star_{S^1}(X;k) \to E_\infty$$

which is *not* a module homomorphism. We define $E(0) = 0$ and if

$$x \in F^q, x \notin F^{q-1}, q \geq 0,$$

then $E(x)$ is the image of x by the module homomorphism

$$F^q \to F^q/F^{q-1} \xrightarrow{\simeq} E^{\star q}_\infty$$

associated to the spectral sequence. Each E^{pq}_∞ lies in the image of E and $E(x) \neq 0$ for $x \neq 0$, but E is not injective. The mapping E has the following four properties where x_j are homogeneous elements of $H^\star_{S^1}(X;k)$.

(1) If $E(x_1)E(x_2) \neq 0$, then $E(x_1 x_2) = E(x_1)E(x_2)$

(2) If $t^aE(x_1) \neq 0$, then $E(t^ax_1) = t^aE(x_1), a \geq 1$.
(3) If $E(x_1) \in E_\infty^{\star q}$ with $q \geq 0$, then $E(t^ax_1) \in E_\infty^{\star s}$ with $s \leq q$ for $a \geq 1$.
(4) If $x_1 \neq 0$ and $t^aE(x_1) = 0$ and $E(x_1) \in E_\infty^{\star q}, q \geq 0$, then $E(t^ax_1) \in E_\infty^{\star s}$ with $s < q$.

We shall use the following lemma of T.Chang and the author.

Lemma. ([2])

The $k[t]$-module $E_r^{\star q}, 2 \leq r \leq \infty$, is generated as a module by the linear subspace E_r^{oq}.

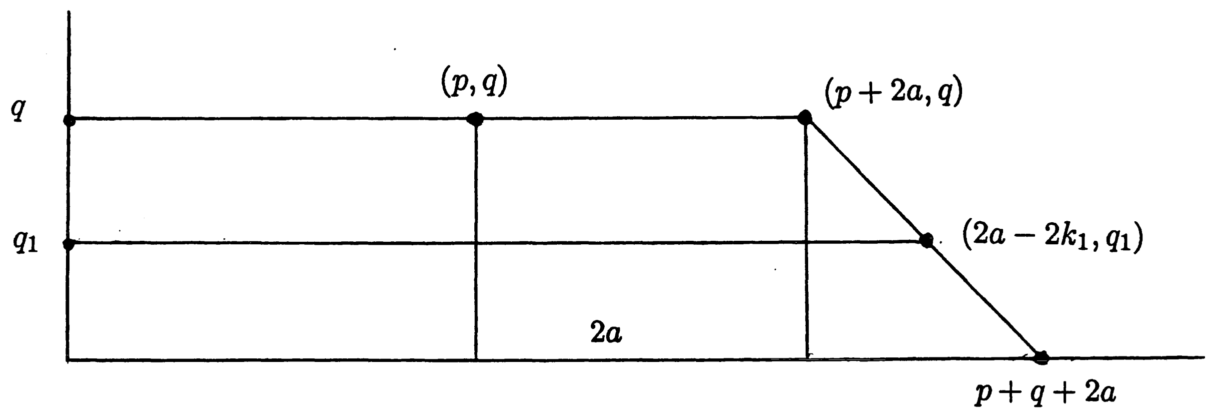
We first prove a key lemma.

Lemma.

Let $x \in E_\infty^{pq}$ be such that $t^ax = 0$ for some $a \geq 1$. Then there is an $u \in H_{S^1}^{p+q}(X;k)$ with $E(u) = x$ and $t^au = 0$.

Proof.
If $q = 0$ so that $x \in E_\infty^{po} \subset F^o \subseteq H_{S^1}^\star(X;k)$, this is evident. Thus we may assume that $q > 0$. Choose $v \in H_{S^1}^{p+q}(X;k)$ such that $E(v) = x$. As $t^aE(v) = t^ax = 0$, whereas $t^av \neq 0$ in general, we have $t^av \in E_\infty^{\star q_1}$ for some $q_1 < q$, by property (4).
As $E_\infty^{\star q_1}$ is generated over $k[t]$ by $E_\infty^{oq_1}$, there is some $v_1 \in H_{S^1}^{q_1}(X;k)$ with $E(v_1) \in E_\infty^{oq_1}$ and $t^{a+k_1}E(v_1) = E(t^av) \neq 0$, (in general),where $k_1 > 0$.

It is convenient to draw a picture of E_∞,



As $E(t^av) - E(t^{a+k_1}v_1) = 0$, it follows that $E(t^av - t^{a+k_1}v_1) \in E_\infty^{\star q_2}$ with $q_2 < q_1$. Thus there is some $v_2 \in H_{S^1}^{q_2}(X;k)$ with $E(v_2) \in E_\infty^{oq_2}$ and, with $k_2 > k_1, t^{a+k_2}E(v_2) = E(t^av - t^{a+k_1}v_1)$. We then have

$$E(t^av - t^{a+k_1}v_1 - t^{a+k_2}v_2) \in E_\infty^{\star q_3}$$

with $q_3 < q_2 < q_1 < q$.

We go on in this manner until we get $q_j \leq 0$. We then get

$$E(t^a v - (t^{a+k_1} v_1 + t^{a+k_2} v_2 + \cdots + t^{a+k_j} v_j)) = 0,$$

where $0 < k_1 < k_2 \cdots < k_j$, and hence,

$$t^a v = t^{a+k_1} v_1 + t^{a+k_2} v_2 + \cdots + t^{a+k_j} v_j.$$

We now define $u \in H_{S^1}^{p+q}(X; k)$ by the equation

$$v = t^{k_1} v_1 + t^{k_2} v_2 + \cdots + t^{k_j} v_j + u.$$

We then have $t^a u = 0$ and as $v_1, v_2, \cdots v_j \in F^{q_1} \subseteq F^{q-1}$ and $v \notin F^{q-1}$, we obtain $x = E(v) = E(u)$ where $t^a u = 0$.

We now prove the theorem together with the following lemma.

Lemma.

For each $q \geq 0$ the exact sequence

$$0 \to F^{q-1} \hookrightarrow F^q \to E_\infty^{\star q} \to 0$$

is a split exact sequence of graded $k[t]$ modules.

Proof.
Choose elements

$$\alpha_1, \ldots., \alpha_a, \beta_1, \ldots, \beta_b \in E_\infty^{\sigma q}$$

such that the cyclic $k[t]$-modules generated by α_j are torsion modules of dimension $d_j \geq 1$ over k, and the submodules generated by the β_j are free modules, and such that $E_\infty^{\star q}$ is the direct sum of those $a + b$ submodules.

Let $\alpha_j' \in H_{S^1}^q(X; k)$ be such that $t^{d_j} \alpha_j = 0$ and $E(\alpha_j') = \alpha_j$, and let $\beta_j' \in H_{S^1}^q(X; k)$ be such that $E(\beta_j') = \beta_j$. Then the $a + b$ cyclic submodules of $H_{S^1}^\star(X; k)$ generated by the α_j' and the β_j' form a direct sum in $F^q \subseteq H_{S^1}^\star(X; k)$, and this sum maps isomorphically onto $E_\infty^{\star q}$ under the homomorphism $F^q \to E_\infty^{\star q}$.

The proof of the theorem follows by using the split sequences of this lemma for all $q \geq 0$.

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