Carleman Approximation on Totally Real Subsets of Class C^k

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Introduction.

Let X be a complex manifold and $S \subset X$ a totally real submanifold of class C^k. In [10] we showed that there is a Stein neighborhood Ω of S in X such that $\mathcal{O}(\Omega)$ is dense in $C^k(S)$ in the Whitney C^k-topology on $C^k(S)$ (or equivalently, that Carleman approximation of class C^k is possible). In this paper we extend these results to the case where $S \subset X$ is a totally real subset of class C^k.

This type of approximation was first introduced by Carleman in [2]. Papers which deal with Carleman approximation in several complex variables are [1], [4], [10], [11], [13], and [14].

The present paper will be part of the author's doctoral thesis, written under the direction of Nils Øvrelid. I would like to thank Nils Øvrelid for his advice and support, which has been of great help to me.

Notation.

We will use standard multiindex notation

$$\nu = (\nu_1, \dots, \nu_n) \in \mathbf{N}_0^n,$$
$$|\nu| = \nu_1 + \cdots + \nu_n.$$

Differentiation in $\mathbf{R}^n$ is denoted by

$$D^\nu = D_t^\nu = \frac{\partial^{|\nu|}}{\partial t^\nu} = \frac{\partial^{|\nu|}}{\partial t_1^{\nu_1} \dots \partial t_n^{\nu_n}}.$$

All manifolds are assumed to be second countable. We will refer to both the "usual" C^k-topology on a manifold, and a stronger topology which we will call the Whitney C^k-topology. If Whitney's name is not mentioned explicitly, we always mean the "usual" C^k-topology.

Jet Bundles and Whitney Functions.

We give a description of the jet bundle and the various subbundles that will be used. For more details and proofs we refer to [5] and [9]. Let X and Y be smooth manifolds. If $f : X \to Y$ is a map of class C^k, then $df : TX \to TY$ is a map of class C^{k-1}. We say that two maps $f, g : X \to Y$ have 0-th order contact at $p \in X$ if $f(p) = g(p)$. Inductively, we say that two maps $f, g : X \to Y$ of class C^k have k-th order contact at p if $df, dg : TX \to TY$ have $(k-1)$-th order contact at every point of T_pX. The notion of k-th order contact at p is an equivalence relation on $C^k(X, Y)$, and the equivalence classes are called k-jets at p. The set of such equivalence classes will be denoted by $J_p^k(X, Y)$. The disjoint union

$$J^k(X, Y) = \cup_{p \in X} J_p^k(X, Y)$$

is called the bundle of k-jets (or simply the jet bundle), it is a fiber bundle over X in a natural manner. We will only be concerned with the case where $Y = \mathbf{C}$, in this case $J^k(X, \mathbf{C})$ is a complex vector bundle over X.

Let $\Gamma_k(X)$ be the set of continuous sections of the jet bundle $J^k(X, \mathbf{C})$. We introduce a topology on $\Gamma_k(X)$ in the following manner. If $E \subset J^k(X, \mathbf{C})$ is an open subset, then let $M(E) = \{\sigma \in \Gamma_k(X) : \sigma(p) \in E \text{ for all } p \in X\}$. If $\sigma_0 \in \Gamma_k(X)$, then a neighborhood system at σ_0 is given by $\{M(E)\}$ where E runs over all open sets in $J^k(X, \mathbf{C})$ which contain $\sigma_0(X)$. The topology on $\Gamma_k(X)$ defined in this manner is called the Whitney C^k-topology.

We give a convenient alternative description of the Whitney C^k-topology on $\Gamma_k(X)$. Choose a norm $||\cdot||_p$ on each $J^k_p(X, \mathbf{C})$ which varies continuously with respect to p. Let $\mathcal{A} \subset \Gamma_k(X)$ be a set of sections of the jet bundle, and let $\phi \in \Gamma_k(X)$ be given. Then ϕ lies in the closure of $\mathcal{A}$ in the Whitney C^k-topology on $\Gamma_k(X)$ iff for each positive continuous function $\epsilon : X \to \mathbf{R}$ there exists $\phi_\epsilon \in \mathcal{A}$ such that

$$||\phi_\epsilon(p) - \phi(p)||_p < \epsilon(p)$$

for all $p \in X$.

Let $S \subset X$ be a closed subset. A continuous section over S of the jet bundle $J^k(X, \mathbf{C})$ is a continuous map $\phi : S \to J^k(X, \mathbf{C})$ such that $\phi(p) \in J^k_p(X, \mathbf{C})$ for all $p \in S$. The set of all continuous sections over S is denoted by $\Gamma_k(S)$. The map $\Theta : \Gamma_k(X) \to \Gamma_k(S)$ given by restricting the domain of a section is surjective, hence we can define the Whitney C^k-topology on $\Gamma_k(S)$ by letting $U \subset \Gamma_k(S)$ be open iff $\Theta^{-1}(U)$ is open in $\Gamma_k(X)$.

Any function $f : X \to \mathbf{C}$ of class C^k induces a continuous section $j_k(f)$ in the jet bundle $J^k(X, \mathbf{C})$. The question of which sections are induced by functions is answered by Whitney's extension theorem (see [15]). Let $S \subset X$ be a closed subset, and let ϕ be a continuous section over S. For each $p \in S$, let f_p be a representative for $\phi(p)$. Let (x, U) be some choice of local coordinates on X, and let $K \subset x(S \cap U)$ be a compact set. Then ϕ induces a family of functions on K (i.e. a jet in the sense of [9]) by

$$F(t) = \{D^\nu(f_{x^{-1}(t)} \circ x^{-1})(t)\}_{|\nu| \le k} = \{g_\nu(t)\}_{|\nu| \le k},$$

where $t \in K$. Whitney's condition is that

$$g_\nu(t) - D_t^\nu \sum_{|\alpha| \le k} \frac{g_\alpha(s)}{\alpha!}(t-s)^\alpha = o(|t-s|^{k-|\nu|})$$

uniformly for $s, t \in K$ and for all ν with $|\nu| \le k$. If Whitney's condition is satisfied for all choices (x, U) of local coordinates on X and for all compacts $K \subset x(S \cap U)$, then there is some $f \in C^k(X)$ such that the restriction of $j_k(f)$ to S is equal to ϕ. In that case we will call ϕ a Whitney function of class C^k. The set of Whitney functions of class C^k on S will be denoted by $W^k(S)$. We give $W^k(S)$ the induced topology from $\Gamma_k(S)$. Clearly, $W^k(S)$ is closed in $\Gamma_k(S)$.

From now on, if f is a function of class C^k in a neighborhood of $S \subset X$, then $j_k(f)$ will denote the section over S induced by f.

Let X be a complex n-dimensional manifold. Let $S \subset X$ be a closed subset, and let $\phi \in W^k(S)$ be given. Choose some function $f \in C^k(X)$ such that $j_k(f) = \phi$, and let (z, U) be some choice of holomorphic coordinates such that $S \cap U \neq \emptyset$. Consider the condition

$$(*) \qquad \bar{\partial}\frac{\partial^{|\nu|} f}{\partial z^\nu} = 0 \qquad \text{on } z(S \cap U)$$

for all multiindices $\nu = (\nu_1, \ldots, \nu_n)$ of order $\leq k-1$. This condition is independent of the choice of representative f. If (ζ, V) is another choice of local coordinates with $S \cap U \cap V \neq \emptyset$, then $(*)$ implies that

$$\bar{\partial}\frac{\partial^{|\nu|} f}{\partial \zeta^\nu} = 0 \qquad \text{on } \zeta(S \cap U \cap V)$$

for all multiindices of order $\leq k-1$. Hence we can define the closed subspace $H^k(S) \subset W^k(S)$ by $\phi \in H^k(S)$ iff $(*)$ is satisfied for all choices of representatives f for ϕ and all choices of local coordinates (z, U) with $S \cap U \neq \emptyset$. We give $H^k(S)$ the induced topology from $W^k(S)$. We will interpret $H^k(S)$ as those Whitney functions of class C^k which satisfy the Cauchy-Riemann equations up to order k on S.

We observe that if f is holomorphic in a neighborhood of S, then necessarily $j_k(f) \in H^k(S)$.

Totally Real Subsets.

Let X be a complex n-dimensional manifold. We say that a closed subset $S \subset X$ is a totally real subset of class C^k ($k \geq 1$) if there exists a non-negative function $\rho \in C^{k+1}(X)$ which is strictly plurisubharmonic on a neighborhood of S and such that $S = \rho^{-1}(0)$. It is shown in [7] that if $S \subset X$ satisfies the condition above then for each $p \in S$ there are a neighborhood U of p and a totally real submanifold $M \subset U$ of class C^k such that $S \cap U \subset M$. (In [7] only the case $k = 1$ is considered, but the same proof works without change for all positive integers k.) In [8] it is shown that a totally real submanifold of class C^1 is also a totally real subset of class C^1, and this is generalized in [12] to totally real submanifolds and subsets of class C^k, $k \geq 1$. The argument given in the Note added in proof of [6] shows that any closed subset of a totally real submanifold of class C^k is a totally real subset of class C^k. Hence $S \subset X$ is a totally real subset of class C^k iff S can locally be embedded as a closed subset of a totally real submanifold of class C^k. We note that in [3] an example is given of a totally real subset which cannot be globally embedded in any totally real submanifold.

Let $M \subset X$ be a totally real submanifold of class C^k and real dimension n (i.e. the maximal possible). Let $S \subset M$ be a closed subset, and let $\phi \in H^k(S)$. Let $\tilde{f} \in C^k(X)$ be a function such that $j_k(\tilde{f}) = \phi$, and let f be the restriction of $\tilde{f}$ to M. Then it is possible to recover ϕ from f, since the partial derivatives of $\tilde{f}$ in the non-tangential directions are determined by the partial derivatives in the tangential directions together with the Cauchy-Riemann equations.

Again, let $M \subset X$ be as in the preceeding paragraph, and let $f \in C^k(M)$ be given. In [8, Lemma 4.3] it is shown that there exists an extension $\tilde{f}$ of f which is C^k on a neighborhood of M and which satisfies the Cauchy-Riemann equations up to order k on M. Hence f

determines an element of $H^k(M)$, and since $\dim_{\mathbf{R}} M = n$ we see that this is a one-to-one correspondence between $C^k(M)$ and $H^k(M)$.

We can now state the theorem that we will prove in this paper.

THEOREM. *Let X be a complex n-dimensional manifold and let $S \subset X$ be a totally real subset of class C^k. Then there is a Stein neighborhood Ω of S in X such that the set $\{j_k(h) : h \in \mathcal{O}(\Omega)\}$ is dense in $H^k(S)$ in the Whitney C^k-topology.*

Approximation.

Proposition 1 and Proposition 2 below are both taken from [10].

PROPOSITION 1. *Let X be a complex manifold and let $M \subset X$ be a totally real submanifold of class C^k, $k \geq 1$. For each $p \in M$ there are neighborhoods*

$$U' \subset\subset U'' \subset\subset U \subset\subset X$$

around p and a neighborhood $W \subset U$ around $M \cap \partial U''$ such that if $f \in C^k(M)$ has compact support contained in $M \cap U'$, then there are holomorphic functions $h_t \in \mathcal{O}(U)$, $t > 0$, such that $h_t \to f$ in the C^k-topology on $M \cap U$ and $h_t \to 0$ in the C^k-topology on W as $t \to 0$. It is possible to choose U such that if $V \subset\subset U$ is an open subset, then U' and U'' can be chosen such that $V \subset\subset U'$.

The last assertion of Proposition 1 is not stated explicitly in [10], but it follows immediately from the proof, since U', U'', and U are images of polydiscs which may be chosen arbitrarily close to each other.

Let $\{U_j\}$ be a locally finite cover of S by open sets $U_j \subset\subset X$ with the following properties:

(1) For each j there is a totally real submanifold $M_j \subset U_j$ of class C^k and of real dimension n such that $S \cap U_j \subset M_j$.

(2) For each j there is an open set $V_j \subset\subset U_j$, and $\{V_j\}$ is also a locally finite cover of S.

(3) For each j there are open sets $U'_j \subset\subset U''_j \subset\subset U_j$ and $W_j \subset U_j$ such that $V_j \subset\subset U'_j$ and the conditions in Proposition 1 are satisfied for these sets.

Since S has a fundamental system of Stein neighborhoods (see [6]), we can choose a Stein neighborhood Ω of S such that $\Omega \cap \partial U''_j \subset W_j$ for all j. For each j, choose $\eta_j \in C^k(X)$ such that $0 \leq \eta \leq 1$, $\eta_j \equiv 1$ on V_j, $\operatorname{supp} \eta_j \subset U'_j$, and the k-jet induced by η_j lies in $H^k(M_j)$.

PROPOSITION 2. *Under the assumptions above, if $f \in C^k(M_j)$ has compact support contained in $M_j \cap U'_j$, then there are functions $h_t \in \mathcal{O}(\Omega)$ such that $h_t \to f$ in the C^k-topology on $\Omega \cap M_j \cap U_j$ and $h_t \to 0$ in the C^k-topology on $\Omega \setminus U''_j$ as $t \to 0$.*

Let $\phi \in H^k(S)$ be given. For each $p \in S$, choose a norm $||\cdot||_p$ on $J^k_p(X, \mathbf{C})$ such that $||\cdot||$ varies continuously with respect to $p \in S$. Let $A : S \to \mathbf{R}$ be a continuous function such that if g_1, g_2 are C^k-functions then

$$||j_k(g_1 g_2)(p)||_p \leq A(p)||j_k(g_1)(p)||_p||j_k(g_2)(p)||_p$$

for all $p \in S$. Let $\epsilon : S \to \mathbf{R}$ be a positive, continuous function. We will show that there is $h \in \mathcal{O}(\Omega)$ such that $||j_k(h)(p) - \phi(p)||_p < \epsilon(p)$ for all $p \in S$. Let $\tilde{f} \in C^k(X)$ be such that $j_k(\tilde{f}) = \phi$ at all points of S. Let $\tilde{f}_j = \eta_j \tilde{f}$, and let f_j be the restriction of $\tilde{f}_j$ to M_j. By Proposition 1, there are $\tilde{h}_j^{(t)} \in \mathcal{O}(\Omega \cap U_j)$ such that $\tilde{h}_j^{(t)} \to f_j$ in the C^k-topology on $\Omega \cap M_j \cap U_j$ and $\tilde{h}_j^{(t)} \to 0$ in the C^k-topology on $\Omega \cap W_j$ as $t \to 0$. By Proposition 2, there are $h_j^{(t)} \in \mathcal{O}(\Omega)$ such that $h_j^{(t)} \to f_j$ in the C^k-topology on $\Omega \cap M_j \cap U_j$ and $h_j^{(t)} \to 0$ in the C^k-topology on $\Omega \setminus U_j''$.

Let $\{K_m\}$ be a sequence of compact sets in Ω with $K_m \subset K_{m+1}$ such that

$$\Omega = \bigcup_{m=1}^{\infty} K_m,$$

$$\bigcup_{j=1}^{m} S \cap U_j \subset S \cap K_m$$

for all positive integers m. Let

$$k(m) = \max\{j : K_m \cap U_j \neq \emptyset\},$$
$$\alpha_m = \max\{A(p)\, \|j_k(1-\eta_j)(p)\|_p : p \in K_m, j \leq k(m)\},$$

and let $\{C_m\}$ be an increasing sequence such that $C_m \geq \alpha_{m+1} \dots \alpha_{k(m)}$ for all m. By Proposition 2, we can choose $h_1 \in \mathcal{O}(\Omega)$ such that

$$\|j_k(h_1 - \eta_1 \tilde{f})\|_{S \cap \overline{U}_1} < \inf\left\{\frac{\epsilon(p)}{2C_1} : p \in S \cap \overline{U}_1\right\},$$
$$\|j_k(h_1)\|_{K_1 \setminus U_1} < \inf\left\{\frac{\epsilon(p)}{2C_1} : p \in S \cap K_1\right\}.$$

Inductively, choose $h_m \in \mathcal{O}(\Omega)$ such that

$$(*) \qquad \left\| j_k\left(h_m - \eta_m\left(\tilde{f} - \sum^{m-1} h_j\right)\right)\right\|_{S \cap \overline{U}_m} < \inf\left\{\frac{\epsilon(p)}{2^m C_m} : p \in S \cap \overline{U}_m\right\},$$

$$(**) \qquad \|j_k(h_m)\|_{K_m \setminus U_m} < \inf\left\{\frac{\epsilon(p)}{2^m C_m} : p \in S \cap K_m\right\}.$$

Let $h = \sum h_m$. From (**) we easily get that the series converges uniformly on compacts in Ω, and hence that $h \in \mathcal{O}(\Omega)$. We claim that $\|j_k(h)(p) - \phi(p)\|_p < \epsilon(p)$ for all $p \in S$. So let $p \in S$ be given and let $m_0 = \max\{j : p \in V_j\}$, $m_1 = \max\{j : p \in U_j\}$. The norms below are all the norm $\|\cdot\|_p$ on $J_p^k(X, \mathbf{C})$. From (**) we get that

$$\left\| \sum_{j > m_1} j_k(h_j)(p) \right\| < \frac{\epsilon(p)}{2^{m_1} C_{m_1+1}},$$

and from $(*)$ we get that

$$\left\|\sum^{m_0} j_k(h_j)(p) - \phi(p)\right\| < \frac{\epsilon(p)}{2^{m_0}C_{m_0}},$$

Let $m_0 < m \leq m_1$, then $p \in K_m$. If $p \notin U_m$ then

$$\begin{aligned}\left\|\sum^{m} j_k(h_j)(p) - \phi(p)\right\| &\leq \|j_k(h_m)(p)\| + \left\|\sum^{m-1} j_k(h_j)(p) - \phi(p)\right\| \\ &\leq \frac{\epsilon(p)}{2^m C_m} + \left\|\sum^{m-1} j_k(h_j)(p) - \phi(p)\right\|.\end{aligned}$$

If $p \in U_m$ then

$$\begin{aligned}\left\|\sum^{m} j_k(h_j)(p) - \phi(p)\right\| &= \left\|j_k\left(h_m - (\eta_m + 1 - \eta_m)\left(\tilde{f} - \sum^{m-1} h_j\right)\right)(p)\right\| \\ &\leq \left\|j_k\left(h_m - \eta_m\left(\tilde{f} - \sum^{m-1} h_j\right)\right)(p)\right\| \\ &\quad + A(p)\|j_k(1-\eta_m)(p)\|\left\|\sum^{m-1} j_k(h_j)(p) - \phi(p)\right\| \\ &\leq \frac{\epsilon(p)}{2^m C_m} + \alpha_m\left\|\sum^{m-1} j_k(h_j)(p) - \phi(p)\right\|.\end{aligned}$$

Putting these results together, we get that

$$\begin{aligned}\|j_k(h)(p) - \phi(p)\| &\leq \left\|\sum^{m_1} j_k(h_j)(p) - \phi(p)\right\| + \left\|\sum_{j>m_1} j_k(h_j)(p)\right\| \\ &\leq \sum_{j=m_0+1}^{m_1} \alpha_j \ldots \alpha_{m_1} \frac{\epsilon(p)}{2^{j-1}C_{j-1}} + \frac{\epsilon(p)}{2^{m_1}C_{m_1}} + \frac{\epsilon(p)}{2^{m_1}C_{m_1+1}} \\ &< \epsilon(p).\end{aligned}$$

This ends the proof of the Theorem.

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