The Crossed Product of a UHF algebra by a Shift

by

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Abstract

We prove that the crossed product of the CAR algebra M_{2^∞} by the shift is an inductive limit of homogeneous algebras over the circle with fibres full matrix algebras. As a consequence the crossed product has real rank zero, and $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes M_{2^\infty}$ where $\mathcal{O}_2$ is the Cuntz algebra of order 2.

1 Introduction

Let M_{2^∞} be the CAR algebra, i.e. the UHF algebra of Glimm type 2^∞, [BR2], [Gli]. Then $M_{2^\infty} = \bigotimes_{-\infty}^{\infty} M_2$. It is well-known that if β is an automorphism of M_{2^∞} of product type, then the crossed product $M_{2^\infty} \times_\beta \mathbf{Z}$ is an inductive limit of algebras of the form $M_{2^n} \otimes C(\mathbf{T})$, where $\mathbf{T}$ is the circle, [Bra2, Theorem 2.1]. In this paper we will prove a similar result in the much more nontrivial situation that α is an automorphism of M_{2^∞} with strong ergodicity properties. Let α be the (Bernoulli) shift of M_{2^∞} obtained by translating each tensor factor by one to the right [BR1, Example 4.3.26]. The crossed product $\mathcal{B} = M_{2^\infty} \times_\alpha \mathbf{Z}$ is then a simple unital C^*-algebra, [Kis].

The algebra $\mathcal{B}$ has a canonical trace state obtained by extending the unique trace state τ on M_{2^∞} to $\mathcal{B}$ by

$$\tau\Big(\sum_n a_n u^n\Big) = \tau(a_0)$$

where $a_n \in M_{2^\infty}$ and u is the canonical unitary in $\mathcal{B}$ implementing α. This is the only trace state on $\mathcal{B}$ by the following reasoning: By [Bed] it suffices to show that the extension of the shift to the weak closure of M_{2^∞} in the cyclic trace representation $(\pi_\tau, \mathcal{H}_\tau, \Omega_\tau)$ is outer. But

$$\lim_{n\to\infty} \tau(a\alpha^n(b)) = \tau(a)\tau(b)$$

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for all $a, b \in M_{2^\infty}$ and hence the projection E_ω onto the $u_\tau(\alpha)$-invaviant vectors in $\mathcal{H}_\tau$ is one-dimensional, where $u_\tau(\alpha)$ is the canonical unitary operator implementing α in the trace representation, [BR1, Theorem 4.3.22]. It follows that the extension of α to $\pi_\tau(\mathcal{A})''$ is ergodic, [BR1, Theorem 4.3.20], and thus this extension is outer.

We conclude that $\mathcal{B}$ has a unique trace state.

The main result of this paper is

Theorem 1.1 There is an increasing sequence $\mathcal{B}_n$ of C^*-subalgebras of $\mathcal{B}$ such that $\cup_n \mathcal{B}_n$ is dense in $\mathcal{B}$, and each $\mathcal{B}_n$ has the form

$$\mathcal{B}_n \cong \bigoplus_{k=1}^{m_n} M_{[n,k]} \otimes C(\mathbf{T})$$

where $[n,k] \in \mathbf{N}$, and m_n is finite. In particular $\mathcal{B}$ has real rank zero.

Recall from [BP] that $\mathcal{B}$ is said to have real rank zero if for any $x = x^* \in \mathcal{B}$ and any $\varepsilon > 0$ there exists a $y = y^* \in \mathcal{B}$ such that y has finite spectrum and $\|x - y\| < \varepsilon$. As soon as we have established that $\mathcal{B}$ is an inductive limit of finite direct sum of circle algebras, it follows from the uniqueness of the trace state that the projections in $\mathcal{B}$ trivially separate the trace states, and hence $\mathcal{B}$ has real rank zero by [BBEK, Theorem 1.3] or [BDR, Theorem 2]. Thus the last statement in Theorem 1.1 is a consequence of the first. We will prove the first statement in Section 5.

As corollaries of Proposition 4.1, established in the course of the proof, we also deduce

Corollary 1.2: $\mathcal{B} \cong \mathcal{B} \otimes M_{2^\infty}$

Corollary 1.3: $\mathcal{O}_2 \cong \mathcal{O}_2 \otimes M_{2^\infty}$, where $\mathcal{O}_2$ is the Cuntz algebra of order 2, [Cun].

Remark 1.4 : Once Theorem 1.1 is established, one may use Elliott's classification in [Ell], [Su] to say more about the increasing sequence $\mathcal{B}_n$. One may for example take

$$\mathcal{B}_n \cong M_{4^n} \otimes C(\mathbf{T})$$

and the embedding $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$ to be 2 copies of the twice around embedding, [Bla2]. This is seen as follows:

We first compute the K-theory of $\mathcal{B}$. By the Pimsner-Voiculescu exact 6-term sequence, the K-groups of $\mathcal{B}$ are given by

$$K_0(\mathcal{B}) \cong \mathbf{Z}\left[\frac{1}{2}\right] \cong K_1(\mathcal{B}),$$

[Bla 1, 10.2.1]. Representatives in the K_0-and K_1-classes corresponding to a dyadic rational can be described explicitely as follows:

If p is a projection in the CAR algebra $\mathcal{A}$ with trace $\tau(p) \in \mathbf{Z}\left[\frac{1}{2}\right] \cap [0,1]$, then p is contained in the K_0 class $\tau(p)$ in $K_0(\mathcal{B})$. Since $1-p$ and $1-upu^*$ are equivalent in the CAR algebra, there exists a partial isometry $v \in \mathcal{A}$ which implements this equivalence. Now $up+v$ is an element in the K_1-class $\tau(p)$. For dyadic rationals outside [0,1] one may construct representatives in the K-classes by using matrix algebras over $\mathcal{A}$ or by considereing unitaries of the form $u^n p + v_n$, where $n \in \mathbf{Z}$.

Next note that $K_0(\mathcal{B})$ as an ordered group with order unit is the dyadic rationals with positive cone equal to the non-negative dyadic numbers and order unit equal to 1. This is because the positive cone in $\mathcal{A}$ is contained in the positive cone in $\mathcal{B}$. Assume that x belongs to the positive cone in $K_0(\mathcal{B})$, and that $x \neq 0$. As $\mathcal{B}$ is stably finite at most one of the elements x and $-x$ will belong to the positive cone in $K_0(\mathcal{B})$, [Bla1]. We know also (from $K_1(\mathcal{A}) = 0$ and Pimsner-Voiculescu's exact sequence) that at least one of the elements x and $-x$ belongs to the image of the positive cone in $K_0(\mathcal{A})$ in $K_0(\mathcal{B})$. This shows that the positive cone in $K_0(\mathcal{B})$ is exactly the image of the positive cone in $K_0(\mathcal{A})$, which is identified with the non-negative dyadic numbers.

The K-theory of the inductive limit described in the beginning of this remark is given by:

$$\begin{array}{ccc} & K_0 & K_1 \\ \mathcal{B}_1 & \mathbf{Z}, \mathbf{Z}_+ & \mathbf{Z} \\ \downarrow & \downarrow & \downarrow \\ \mathcal{B}_2 & \mathbf{Z}, \mathbf{Z}_+ & \mathbf{Z} \\ \downarrow & \downarrow & \downarrow \end{array}$$

where all the vertical arrows on the K-groups are given by multiplication by 2. Thus, if $\mathcal{C}$ is the inductive limit,

$$K_0(\mathcal{C}) \cong \mathbf{Z}\left[\frac{1}{2}\right] \cong K_1(\mathcal{C}),$$

and the positive cone in $K_0(\mathcal{C})$ is the non-negative dyadic numbers. It follows that the ordered K-theory of $\mathcal{C}$ is isomorphic to that of $\mathcal{B}$, and hence $\mathcal{B} \cong \mathcal{C}$ by [Ell] or [Su].

2 Voiculescu's almost inductive limit automorphisms

If $\mathcal{A}$ is a unital C^*-algebra, $\mathcal{F}(\mathcal{A})$ denotes the set of finite dimensional *-subalgebras of $\mathcal{A}$ containing the unit of $\mathcal{A}$. If $\mathcal{B}, \mathcal{C}$ are subalgebras of $\mathcal{A}$ we will follow [Voi] in using the notation $\mathcal{B} \subset^{\varepsilon} \mathcal{C}$ if

$$\sup\left\{\inf\{\|x-y\| \big| y \in \mathcal{C}, \|y\| \leq 1\} \big| x \in \mathcal{B}, \|x\| \leq 1\right\} < \varepsilon,$$

and the distance $d(\mathcal{B},\mathcal{C})$ between $\mathcal{B}$ and $\mathcal{C}$ is defined by

$$d(\mathcal{B},\mathcal{C}) = \inf\{\varepsilon > 0 | \mathcal{B} \subset^{\varepsilon} \mathcal{C} \text{ and } \mathcal{C} \subset^{\varepsilon} \mathcal{B}\}.$$

Since any automorphism β of the CAR algebra $\mathcal{A}$ is approximately inner it follows from [Voi, Lemma 3.1] that for any $\mathcal{D} \in \mathcal{F}(\mathcal{A})$ and any positive integer m, there are $\mathcal{B}_j \in \mathcal{F}(\mathcal{A})$, $j = 0, 1, \cdots, m$ with $\mathcal{B}_0 = \mathcal{B}_m$ such that

$$d(\beta(\mathcal{B}_j), \mathcal{B}_{j+1}) < \frac{5\pi}{m}$$

for $0 \leq j < m$, and

$$\mathcal{D} \subseteq \mathcal{B}_j$$

for $j = 0, 1, \cdots, m$. Thus, if e_{ij}, $i, j = 0, 1, \cdots, m-1$ is a complete set of matrix units for M_m, and σ_m is the cyclic shift of M_m, defined through

$$\sigma_m(e_{ij}) = e_{i+1,j+1}$$

(where the addition is modulo m), and $\mathcal{E} \in \mathcal{F}(\mathcal{A} \otimes M_m)$ is defined through

$$\mathcal{E} = \sum_{j=0}^{m-1} \mathcal{B}_j \otimes e_{jj},$$

then

$$\mathcal{D} \otimes 1 \subset \mathcal{E}$$

and

$$d((\beta \otimes \sigma_m)(\mathcal{E}), \mathcal{E}) < \frac{5\pi}{m}.$$

Now suppose β has the property that for any positive integer m_0 there is an integer $m > m_0$ such that for every positive integer N and every $\delta > 0$ there is a subalgebra $\mathcal{C} \subset^{\delta} \bigotimes_{k=N+1}^{\infty} M_2 \subseteq M_{2^\infty} = \mathcal{A}$ such that $\mathcal{C}$ contains the unit of $\mathcal{A}$, $\mathcal{C} \cong M_m$, $\mathcal{C}$ has a cyclic shift σ_m, and $\|\beta(x) - \sigma_m(x)\| \leq \delta\|x\|$ for $x \in \mathcal{C}$. Then β is an almost inductive limit automorphism of $\mathcal{A}$, i.e. for every $\mathcal{D} \in \mathcal{F}(\mathcal{A})$ and every $\varepsilon > 0$ there is a $\mathcal{E} \in \mathcal{F}(\mathcal{A})$ such that $\mathcal{D} \subset^{\varepsilon} \mathcal{E}$ and $d(\beta(\mathcal{E}), \mathcal{E}) < \varepsilon$.

This is seen as follows: By modifying the $\mathcal{D}$ and $\mathcal{B}_i$ above by a small amount we may assume $\mathcal{B}_i \subseteq \bigotimes_{-N}^{N} M_2$ for some N, but then, integrating over the unitary group of $\bigotimes_{-N}^{N} M_2$ and using the techniques of [Gli] and [Bra1], we may assume $\mathcal{C} \subseteq \Big(\bigotimes_{k=-\infty}^{-N-1} \otimes \bigotimes_{k=N+1}^{\infty}\Big) M_2$. Then redefine the $\mathcal{E}$ above as $\mathcal{E} = \sum_{j=0}^{m-1} \mathcal{B}_j e_{jj}$. If $x = \sum_{j=0}^{m-1} b_j e_{jj} \in \mathcal{E}$ one computes

$$\begin{aligned} \| \beta(x) - {}''\beta \otimes \sigma''_m(x) \| &= \| \sum_{j=0}^{m-1} \beta(b_j)(\beta - \sigma_m)(e_{jj}) \| \\ &\leq \sum_{j=0}^{m-1} \| b_j \| \delta , \end{aligned}$$

where the notation "$\beta \otimes \sigma_m$" is self explanatory. As $\| x \| = \sup_j \| b_j \|$, it follows that

$$\| \beta(x) - {}''\beta \otimes \sigma''(x) \| \leq m\delta \| x \|$$

Since we may choose this δ after m, we may make δm as small as we want, and as

$$d(({}''\beta \otimes \sigma'')(\mathcal{E}), \mathcal{E}) < \frac{5\pi}{m}$$

we may make $d(\beta(\mathcal{E}), \mathcal{E})$ as small as desired.

Thus, it follows from [Voi, Proposition 2.3] that for any $\varepsilon > 0$ there is a unitary $u \in \mathcal{A}$ such that $\|u - 1\| < \varepsilon$, and $\gamma = Adu \circ \beta$ is an inductive limit automorphism, i.e. there exists an increasing sequence $\mathcal{A}_n \in \mathcal{F}(\mathcal{A})$ such that $\cup_n \mathcal{A}_n$ is dense in $\mathcal{A}$ and $\gamma(\mathcal{A}_n) = \mathcal{A}_n$. But then $\mathcal{A} \times_\gamma \mathbf{Z}$ is the inductive limit of $\mathcal{A}_n \times_\gamma \mathbf{Z}$, and each of these latter algebras has the form $\mathcal{D}_n \otimes C(\hat{\mathbf{Z}}) = \mathcal{D}_n \otimes C(\mathbf{T})$ where $\mathcal{D}_n$ is the finite dimensional algebra obtained from $\mathcal{A}_n$ by merging the factors over the central orbits of γ into one factor of dimension equal to the product of the dimensions of each factor by the order of the orbit [Bra 2]. Note also that $\mathcal{A} \times_\gamma \mathbf{Z}$ is isomorphic to $\mathcal{A} \times_\beta \mathbf{Z}$, [Tak]. In conclusion, we have the following known lemma:

Lemma 2.1 [Voi, Lemma 3.3] Let β be an automorphism of $\mathcal{A} = M_{2^\infty}$ with the property that for any positive integer m_0 there is an integer $m > m_0$ such that for every positive integer N and every $\varepsilon > 0$ there is a subalgebra $\mathcal{C} \in \mathcal{F}(\mathcal{A})$ such that $\mathcal{C} \subset^\varepsilon \bigotimes_{k=N+1}^{\infty} M_2 \subseteq \mathcal{A}$, $\mathcal{C} \cong M_m$ and there exists a cyclic shift σ_m of order m on $\mathcal{C}$ such that

$$\|\beta(x) - \sigma_m(x)\| \leq \varepsilon \|x\|$$

for all $x \in \mathcal{C}$.

It follows that β is an almost inductive limit automorphism, and hence $\mathcal{A} \times_\beta \mathbf{Z}$ contains an increasing sequence $\mathcal{B}_n$ of C^*-subalgebras such that $\cup_n \mathcal{B}_n$ is dense, and each $\mathcal{B}_n$ has the form

$$\mathcal{B}_n \cong \bigoplus_{k=1}^{m_n} M_{[n,k]} \otimes C(\mathrm{T})$$

where $[n,k] \in \mathbf{N}$ and m_n is finite.

3 Quasifree automorphisms and the shift

If $\mathcal{H}$ is a separable infinite-dimensional Hilbert space with inner product $(\,,\,)$, the algebra $\mathcal{A} = M_{2^\infty}$ can be described as the universal C^*-algebra generated by operators $a(f), f \in \mathcal{H}$, satisfying

$$\begin{aligned}
&f \to a(f) \quad \text{is antilinear,}\\
&a(f)a(g) + a(g)a(f) = 0 \,,\\
&a(f)a(g)^* + a(g)^*a(f) = (f,g)\mathbf{1} \,,
\end{aligned}$$

see e.g. [BR2, Theorem 5.2.5]. If $(f_k)_{k=1}^\infty$ is an orthonormal basis for $\mathcal{H}$, the matrix units $\left(e_{ij}^{(k)}\right)_{i,j=1}^2$ for the k'th tensor factor of $\mathcal{A} = \bigotimes_{k=1}^\infty M_2$ can be given by

$$\begin{aligned}
e_{11}^{(k)} &= a(f_k)a(f_k)^* & e_{12}^{(k)} &= V_{k-1}a(f_k)\\
e_{21}^{(k)} &= V_{k-1}a(f_k)^* & e_{22}^{(k)} &= a(f_k)^*a(f_k) \,,
\end{aligned}$$

where

$$\begin{aligned}
V_k &= \prod_{i=1}^k (1 - 2a(f_i)^*a(f_i))\\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 \otimes 1 \otimes \cdots \,,
\end{aligned}$$

and there are k factors $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Conversely

$$\begin{aligned}
a(f_k) &= \left(\prod_{l=1}^{k-1} (e_{11}^{(l)} - e_{22}^{(l)})\right) e_{12}^{(k)}\\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1 \otimes 1 \otimes \cdots \,,
\end{aligned}$$

where there are $k-1$ factors $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If V is any isometry on $\mathcal{H}$, V defines a *-morphism of $\mathcal{A}$ by

$$a(f) \to a(Vf)$$

(and this is a *-automorphism if V is unitary, called a quasi-free or Bogoliubov automorphism). In particular, let β be the morphism defined by the one-sided shift:

$$Vf_k = f_{k+1}, \qquad k = 1, 2, \dots .$$

We call β the one-sided quasi-free shift. On the other hand, let α be the usual one-sided shift on $\mathcal{A}$:

$$\alpha(e_{ij}^{(k)}) = e_{ij}^{(k+1)}, \qquad k = 1, 2, \dots ,$$

i.e.

$$\alpha(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes x .$$

Let γ be the quasi-free automorphism defined by

$$\gamma(a(f)) = -a(f) ,$$

i.e.

$$\gamma(e_{ij}^{(k)}) = (-1)^{i-j} e_{ij}^{(k)} .$$

Define an element $x \in \mathcal{A}$ to be even if $\gamma(x) = x$ and odd if $\gamma(x) = -x$. Thus the *-algebra $\mathcal{A}^e$ of even elements is the closure of the set of polynomials in $a(f), a(f)^*$ with an even number of creators or annihilators in each constituent monomial. Using the expression of $a(f_k)$ in terms of the $e_{ij}^{(k)}$'s, one now easily computes

$$\beta(x) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes x & \text{if } x \in \mathcal{A}^e , \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes x & \text{if } x \text{ is odd} . \end{cases}$$

In particular we deduce

Lemma 3.1 The one-sided shift α and the one-sided quasifree shift β have the same restrictions to the even algebra $\mathcal{A}^e$.

4 Almost shift-invariant matrix sub-algebras of M_{2^∞}.

Proposition 4.1 Let $\mathcal{A} = M_{2^\infty} = \bigotimes_{k=1}^{\infty} M_2$ and let α be the one-sided shift on $\mathcal{A}$. If $\varepsilon > 0$ and $\rho_1, \rho_2, \cdots, \rho_n \in \mathbf{T} \subseteq \mathbf{C}$ then there exist a subalgebra $\mathcal{B} \in \mathcal{F}(\mathcal{A})$ such that $\mathcal{B} \cong M_{2^n}$, and an automorphism β of $\mathcal{B}$ of the form

$$\beta = Ad\left(\begin{pmatrix} 1 & 0 \\ 0 & \rho_1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & \rho_2 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & \rho_n \end{pmatrix}\right)$$

relatively to some tensor product decomposition of $\mathcal{B}$, such that

$$\|\alpha(x) - \beta(x)\| \leq \varepsilon \|x\|$$

for all $x \in \mathcal{B}$.

Remark 4.2 Once this is true for the one-sided shift α it is also trivially true for the two-sided shift.

Remark 4.3 In particular, putting

$$\rho_k = e^{2\pi i 2^{-k}}, \qquad k = 1, \cdots, n ,$$

the spectrum of the unitary operator implementing β becomes the set of all 2^n-roots of 1, and hence the proposition applies to the cyclic shift of order 2^n on M_{2^n}.

Proof of Proposition 4.1 First note that the two-sided shift V on $L^2(\mathbf{Z})$ is a unitary operator with spectrum $\mathbf{T}$, and hence by spectral theory there exist for any $\delta > 0$ mutually orthogonal unit vectors $\xi_0, \xi_1, \cdots, \xi_n$ in $L^2(\mathbf{Z})$ such that

$$\|V\xi_0 - \xi_0\| < \delta,$$

$$\|V\xi_k - \rho_k \xi_k\| < \delta, \qquad k = 1, \cdots, n.$$

Furthermore, shifting the ξ_k's sufficiently far to the right and changing each ξ_k by a small amount we may assume

$$\xi_k \in L^2(\mathrm{N}), \qquad k = 0, 1, \cdots, n.$$

Now, one checks that the operators a_k defined by

$$a_k = a(\xi_k)(a(\xi_0) + a(\xi_0)^*), \qquad k = 1, \cdots, n$$

satisfy the anti-commutation relations

$$\begin{aligned} a_k a_l + a_l a_k &= 0 \\ a_k a_l^* + a_l^* a_k &= \delta_{k,l} 1 \, , \end{aligned}$$

so the *-algebra B generated by the a_k's is isomorphic to M_{2^n} by [BR2, Theorem 5.2.5]. Furthermore, the automorphism β of $\mathcal{B}$ determined by

$$\beta(a_k) = \overline{\rho}_k a_k \, , \qquad k = 1, \cdots, n \, ,$$

has the required form

$$\beta = Ad\left(\begin{pmatrix} 1 & 0 \\ 0 & \rho_1 \end{pmatrix} \otimes \cdots \otimes \left(\begin{pmatrix} 1 & 0 \\ 0 & \rho_n \end{pmatrix} \right)\right)$$

relatively to the matrix units defined by $a_1, \cdots, a_n$.

If σ is the quasifree one-sided shift defined by $V|_{L^2(\mathbf{N})}$, then

$$\begin{aligned} &\sigma(a_k) - \overline{\rho}_k a_k \\ &\quad = a(V\xi_k)(a(V\xi_0) + a(V\xi_0)^*) - a(\rho_k \xi_k)(a(\xi_0) + a(\xi_0)^*) \, . \end{aligned}$$

Thus, using $\|a(\xi)\| = \|\xi\|$ we deduce

$$\begin{aligned} &\|\sigma(a_k) - \overline{\rho}_k a_k\| \\ &\quad \leq 2\|V\xi_k - \rho_k \xi_k\| + 2\|V\xi_0 - \xi_0\| \\ &\quad < 4\delta \, . \end{aligned}$$

Since any element in $\mathcal{B}$ is a polynomial in the a_k, a_k^* of degree at most n, it follows by choosing δ small enough that

$$\|\sigma(x) - \beta(x)\| \leq \varepsilon \|x\|$$

for all $x \in \mathcal{B}$. Finally, as the a_k are polynomials of homogeneous degree 2 in creators and annihilators, $\mathcal{B} \subseteq \mathcal{A}^e$ and it follows from Lemma 3.1 that $\alpha|_{\mathcal{B}} = \sigma|_{\mathcal{B}}$. Thus Proposition 4.1 follows.

5 The shift is an almost inductive limit automorphism

The proof of Theorem 1.1 is now immediate: By Remark 4.2 there exist for any $\varepsilon > 0$ and any n a subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing the unit of $\mathcal{A}$ such that $\mathcal{B} \cong M_{2^n}$ and an automorphism β of $\mathcal{B}$ isomorphic to the cyclic shift of order 2^n such that

$$\|\alpha(x) - \beta(x)\| \leq \varepsilon\|x\|$$

for all $x \in \mathcal{B}$. Since the latter estimate is not changed by the replacement $\mathcal{B} \to \alpha^m(\mathcal{B})$, $\beta \to \alpha^m \beta \alpha^{-m}$, we may also assume that $\mathcal{B} \subset^{\varepsilon} \bigotimes_{k=N+1}^{\infty} M_2$ for any given N. Theorem 1.1 now follows from Lemma 2.1.

Remark 5.1 By pushing the ideas from [Voi] and using the techniques in [BKR, Proposition 2.12] one could prove Theorem 1.1 if one could establish that for any $\varepsilon > 0$ there is a unitary $u \in \mathcal{A}$ such that $\|u - 1\| < \varepsilon$, and the C^*-dynamical system $(\mathcal{A}, Adu \circ \alpha)$ is isomorphic to $(\mathcal{A} \otimes \mathcal{A}, \beta \otimes \sigma)$, where $\sigma = \bigotimes_{n=1}^{\infty} \sigma_{2^n}$ on $\mathcal{A} \cong \bigotimes_{n=1}^{\infty} M_{2^n}$, and σ_{2^n} is the cyclic shift of order 2^n on M_{2^n}, and β some automorphism of $\mathcal{A}$. One may then appeal directly to [Voi, Lemma 3.3] to prove that $Adu \circ \alpha$ is an almost inductive limit automorphism.

6 Divisibility of $M_{2^\infty} \times_\alpha \mathbf{Z}$

In this section we will prove Corollary 1.2, i.e.

$$\mathcal{B} = M_{2^\infty} \times_\alpha \mathbf{Z} \cong (M_{2^\infty} \times_\alpha \mathbf{Z}) \otimes M_{2^\infty}.$$

To this end we will combine Proposition 4.1 with a special case of a result in [BKR]:

Lemma 6.1 [BKR, Proposition 2.12] Let $\mathcal{A}$ be a unital separable C^*-algebra with the property that for any finite set $\{x_1, \cdots, x_n\} \in \mathcal{A}$ and any $\varepsilon > 0$ there exists a $\mathcal{B} \in \mathcal{F}(\mathcal{A})$ such that $\mathcal{B} \cong M_2$ and $\|[x_i, y]\| \leq \varepsilon\|y\|$ for $i = 1, 2, \cdots, n$ and all $y \in \mathcal{B}$. It follows that

$$\mathcal{A} \cong \mathcal{A} \otimes M_{2^\infty}\ .$$

Proof: By [BKR, Proposition 2.12 and its proof], $\mathcal{A} \cong \mathcal{C} \otimes M_{2^\infty}$ for a suitable C^*-algebra $\mathcal{C}$. But as $M_{2^\infty} \otimes M_{2^\infty} \cong M_{2^\infty}$ it follows that

$$\begin{aligned} \mathcal{A} &\cong \mathcal{C} \otimes (M_{2^\infty} \otimes M_{2^\infty}) \\ &\cong (\mathcal{C} \otimes M_{2^\infty}) \otimes M_{2^\infty} \\ &\cong \mathcal{A} \otimes M_{2^\infty}. \end{aligned}$$

To prove Corollary 1.2, note that, by Proposition 4.1, for any $\varepsilon > 0$ there is a $\mathcal{C} \in \mathcal{F}(\mathcal{A})$ such that $\mathcal{C} \cong M_2$ and $\|\alpha(x) - x\| \leq \varepsilon \|x\|$ for all $x \in \mathcal{C}$. Replacing $\mathcal{C}$ by $\alpha^m(\mathcal{C})$ we may assume that $\mathcal{C}$ approximately commutes with any finite subset of M_{2^∞}, and as this replacement does not affect the estimate $\|\alpha(x) - x\| \leq \varepsilon \|x\|$ the new $\mathcal{C}$ (as well as the old) approximately commutes with the canonical unitary u in the crossed product $\mathcal{B} = M_{2^\infty} \times_\alpha \mathbf{Z}$.

Corollary 1.2 now follows from Lemma 6.1.

7 The Cuntz algebra $\mathcal{O}_2$

Recall from [Cun] that $\mathcal{O}_2$ is the universal C^*-algebra generated by two operators S_1, S_2 satisfying the relations

$$1 = S_1^* S_1 = S_2^* S_2 = S_1 S_1^* + S_2 S_2^*.$$

We will prove Corollary 1.3, i.e.

$$\mathcal{O}_2 \cong \mathcal{O}_2 \otimes M_{2^\infty} .$$

Recall from [Cun] that $\mathcal{O}_2$ contains M_{2^∞} canonically as a unital sub-algebra as follows: If μ is a multiindex of length n with values in $\{1, 2\}$, i.e. $\mu = (\mu_1, \cdots, \mu_n)$ with $\mu_j \in \{1, 2\}$, define $S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}$. Then the set of

$$S_\mu S_\nu^* ,$$

where μ, ν run over the multiindices of length n, constitute a complete set of $2^n \times 2^n$ matrix units. Letting $n \to \infty$, one establishes that the fixed point subalgebra $\mathcal{A}$ of $\mathcal{O}_2$ under the gauge group $\rho \in \mathbf{T} \to \sigma_\rho$, where $\sigma_\rho(S_1) = \rho S_1, \sigma_\rho(S_2) = \rho S_2$, is isomorphic to M_{2^∞}.

Now, define a morphism ϕ of $\mathcal{O}_2$ by

$$\phi(x) = S_1 x S_1^* + S_2 x S_2^*$$

Then ϕ commutes with the gauge action, so $\phi(\mathcal{A}) \subseteq \mathcal{A}$, and by applying ϕ to the matrix units $S_\mu S_\nu^*$, one sees that the restriction of ϕ to $\mathcal{A}$ is the one-sided shift. It follows from

Proposition 4.1 that for any $\varepsilon > 0$ there is a * subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing the unit of $\mathcal{A}$ such that $\mathcal{B} \cong M_2$ and such that

$$\|\phi(x) - x\| \leq \varepsilon \|x\|$$

for all $x \in \mathcal{B}$. But

$$\begin{aligned}(\phi(x) - x)S_1 &= S_1 x S_1^* S_1 + S_2 x S_2^* S_1 - x S_1 \\ &= S_1 x - x S_1 = [S_1, x] ,\end{aligned}$$

and correspondingly

$$S_1^*(\phi(x) - x) = [x, S_1^*] ,$$

$$(\phi(x) - x)S_2 = [S_1, x] ,$$

$$S_2(\phi(x) - x) = [x, S_2^*] ,$$

so the commutator of $x \in \mathcal{B}$ with any monomial in S_i, S_i^* of order 1 has norm less than or equal to $\varepsilon\|x\|$. Since the polynomials in S_i, S_i^* are dense in $\mathcal{O}_2$ it follows that $\mathcal{O}_2$ is approximately divisible in the sense that for all finite sequences $x_1, \cdots, x_n \in \mathcal{O}_2$ and all $\varepsilon > 0$, there exists a *-subalgebra $\mathcal{B} \subseteq \mathcal{O}_2$ containing the unit of $\mathcal{O}_2$ such that

$$\|[x_i, x]\| \leq \varepsilon \|x\|$$

for all $x \in \mathcal{B}$, and $\mathcal{B} \cong M_2$. It therefore follows from Lemma 6.1 that

$$\mathcal{O}_2 \cong \mathcal{O}_2 \otimes M_{2^\infty} .$$

8 Complements on almost shift-invariant projections

Let α be the shift on M_{2^∞}. It follows from Proposition 4.1 that for any $\varepsilon > 0$ we may find an m and a nontrivial projection

$$E \in \bigotimes_1^m M_2 \subseteq \bigotimes_{-\infty}^{\infty} M_2$$

such that $\|\alpha(E) - E\| < \varepsilon$. The following proposition says that $m \geq \pi/2 \arcsin \varepsilon$.

Proposition 8.1 If $E \in \bigotimes_1^m M_2$ is a nontrivial projection, then

$$\|E \otimes 1_2 - 1_2 \otimes E\| \geq \sin(\pi/2m) .$$

When $m = 2$, this is the best possible estimate. (Here $E \otimes 1_2$ and $1_2 \otimes E$ are both viewed as projections in $\bigotimes_1^{m+1} M_2$).

Proof: View $\bigotimes_1^m M_2$ as a subalgebra of $\bigotimes_{-\infty}^{\infty} M_2 = M_{2^\infty}$, let τ be the trace state on M_{2^∞}, and let $\mathcal{H}$ be the Hilbert space of the cyclic representation associated with τ. Since τ is invariant under the shift α, there is a unitary operator U on $\mathcal{H}$ such that

$$\alpha(x) = UxU^*$$

for all $x \in M_{2^\infty}$. We have the identification

$$1_2 \otimes E = \alpha(E \otimes 1_2) = \alpha(E) = UEU^* ,$$

so we must show

$$\|E - UEU^*\| \geq \sin(\pi/2m) .$$

But

$$U^m E U^{-m} = \alpha_m(E) \in \bigotimes_{m+1}^{2m} M_2,$$

thus E and $U^m E U^{-m}$ are distinct commuting projections, and there exists a nonzero vector $\xi \in \mathcal{H}$ such that

$$E\xi = \xi \quad , U^m E U^{-m} \xi = 0,$$

i.e. $\xi \in E\mathcal{H}$ and $U^{-m}\xi \perp E\mathcal{H}$. Since the angle between $U^{-m}\xi$ and $E\mathcal{H}$ is $\pi/2$ it follows by weak compactness of the unit ball in $\mathcal{H}$ that there is another unit vector $\xi \in E\mathcal{H}$ such that the angle between $U^{-1}\xi$ and $E\mathcal{H}$ is at least $\pi/2m$ (since the maximum such angle between $U^{-k}E\mathcal{H}$ and $U^{-1}\xi$ over all $\xi \in U^{-k}E\mathcal{H}$ is the same for $k = 0, 1, \cdots, m-1$, and the sum of these angles is at least $\pi/2$). But this means that

$$\|EU^{-1}\xi\| \leq \cos(\pi/2m) ,$$

and then

$$\|EU^*\xi - U^*\xi\| \geq \sin(\pi/2m) ,$$

i.e.

$$\|UEU^*\xi - \xi\| \geq \sin(\pi/2m) ,$$

and thus

$$\|\alpha(E) - E\| \geq \sin(\pi/2m).$$

Let us now look at the case $m = 2$, where the lower bound becomes

$$\|E \otimes 1_2 - 1_2 \otimes E\| \geq 1/\sqrt{2}.$$

This lower bound is actually achieved by the projection

$$E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

as one may verify by direct computation, or by noting that E is a solution of the braid relation:

$$(E \otimes 1_2)(1_2 \otimes E)(E \otimes 1_2) = \frac{1}{2}\big(E \otimes 1_2\big).$$

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