

## MINIMAL HYPERSPHERES IN TWO-POINT HOMOGENEOUS SPACES.

### Introduction.

In Riemannian geometry the study of minimal submanifolds has given the most important, higher-dimensional generalizations of geodesics. Especially significant from a global point of view are closed minimal submanifolds (generalizing closed geodesics); these raise many hard problems. In this paper we study existence and uniqueness questions in the case of the simplest topological type; i.e. minimal hyperspheres. We restrict ourselves to study such questions for the compact two-point homogeneous spaces; these spaces constitute the most natural generalization of classical (three-point homogeneous) spherical geometry. They can be characterized equivalently as (i) compact two-point homogeneous spaces, (ii) compact rank 1 symmetric spaces, or (iii) irreducible compact positively curved symmetric spaces. Since the standard spheres have been investigated in great detail in connection with the "Spherical Bernstein Problem", ([H-1,2], [HS], [HT], [T-1]), we only consider the complex projective spaces  $CP(n)$ , the quaternionic projective spaces  $HP(n)$ , and the Cayley projective plane  $Ca(2)$  here.

The compact two-point homogeneous spaces are also precisely the symmetric spaces which have a homogeneous, minimal hypersphere, unique up to congruence, and determined as the principal isotropy group orbit of maximal volume. We call it the "equator" of the space. Hence this is the natural class of spaces to study uniqueness questions of "spherical Bernstein type", i.e. the variety of non-equatorial minimal hyperspheres. The most effective method has been the "equivariant differential geometry" initiated by Wu-Yi Hsiang and B. Lawson, see [HL], [F-K], [H-T]). In the case of a cohomogeneity two subgroup  $G$  of the isometry group the reduced minimal equation in the orbit space is the geodesic equation for a suitably chosen metric; and the problem of finding  $G$ -invariant closed minimal submanifolds is reduced to finding geodesics that may "close up" by satisfying a delicate intersection requirement with the singular boundary. The technical difficulties in proving the existence of such "closed geodesics" may be expected to depend strongly on the coefficients of the reduced minimal equation; this is born out very clearly in the increasing difficulties in proving the following results:

**Theorem 1.** There exist infinitely many congruence classes of imbedded, minimal hyperspheres in  $CP(n)$ ;  $n \geq 2$ .

**Theorem 2.** There exist infinitely many congruence classes of immersed, minimal hyperspheres in  $HP(n)$ ,  $n \geq 1$ .

**Theorem 3.** There exist infinitely many congruence classes of immersed, minimal hyperspheres in the Cayley projective plane.

In fact, we will obtain a very explicit expression for how the oscillatory behaviour of a second order linearized variational equation for the reduced minimal equation changes with the dimension of the scalar field ( $\mathbf{C}$ ,  $\mathbf{H}$  or  $Ca$ ). Thus, although these spaces all have rich two-point homogeneous symmetry, the subtle changes in those symmetry groups related to change of scalars strongly impact existence proofs for non-equatorial minimal hyperspheres.

Theorem 1 was proved in [HHT-2], announced in [HHT-1]. In that case the proof is much simplified by lifting in the Hopf fibration  $S^1 \rightarrow S^{2n+1} \rightarrow CP(n)$ ; the orbital invariants are then easier to compute. This belongs to the simplest class of equivariant geometric structures ("corner singularity of focal type"), treated in [HHT-2], which enables one to prove Theorem 1 without too much analytical complication. The quaternionic projective space can also be treated by lifting in the Hopf fibration  $S^3 \rightarrow S^{4n+3} \rightarrow HP(n)$ , the resulting dynamical system is not of the above simple class and requires more complicated analysis in the orbit space ("corner singularity of nodal type", see [T-1]). It should, however, be feasible to compute the orbital invariants directly in the projective spaces; although harder, this more general approach should be of interest for other symmetric spaces. It becomes imperative in the case of the Cayley projective plane, since it is well known that there cannot exist any corresponding "Hopf fibration"  $S^7 \rightarrow S^{23} \rightarrow Ca(2)$ .

We therefore choose to give a unified treatment for direct computation of the orbital invariants for all these symmetric spaces. This is done in explicit detail for  $Ca(2)$ , which is the more complicated case, involving aspects (triviality) of exceptional Lie groups.

## 1 Preliminary results.

We first observe that only the above cases are relevant for the study of closed, minimal hyperspheres in two-point homogeneous spaces:

**Proposition 1.** Let  $M$  be a Riemannian two-point homogeneous space and let  $N$  be a minimal hypersphere in  $M$ . Then  $M$  is either a sphere, a complex or quaternionic projective space, or the Cayley projective plane.

**Proof:** As is well-known, a simply connected Riemannian manifold of non-positive curvature is diffeomorphic to Euclidean space and has no compact, minimal submanifolds. Since any two-point homogeneous space of non-positive sectional curvature is simply connected, we only need to consider those of positive curvature. By Frankel's theorem ([F]) it follows that for any closed minimal hypersurface  $N$  in a Riemannian manifold  $M$  of positive Ricci curvature, the induced map between fundamental groups is surjective.

This excludes the real projective spaces, and the result follows from the classification of two-point homogeneous spaces.

q.e.d.

**Proposition 2.** Let  $G$  be a cohomogeneity two Lie group of isometries acting on the Riemannian manifold  $M$  and let  $\pi : M \rightarrow M/G = X$  be the projection onto the orbit space. Let  $M^*$  be the union of principal orbits in  $M$ , and let  $X^* = \pi(M^*)$ , with its orbital distance metric. If  $\gamma$  is a curve in  $X^*$ , the mean curvature of  $\pi^{-1}(\gamma)$  in  $M^*$  is given by:  $H(z) = k(\pi(z)) - \frac{d}{dn} \ln V(\pi(z))$ , where  $k$  is the geodesic curvature of  $\gamma$ ,  $n$  is the unit normal of  $\gamma$ , and  $V$  is the volume functional registering the volume of the orbits.

This standard result makes it clear that the only needed data from the orbital geometry are the orbital distance metric of  $X^*$  and the volume functional  $V$ .

Note: We always allow normalization of  $V$  by a multiplicative constant; this does not change the equation.

Now, let  $M = \mathbf{CP}(N), \mathbf{HP}(n),$  or  $Ca(2)$ , and let  $p \in M$ . Let  $G$  be the isometry-isotropy group of  $p$ ; by two-point homogeneity the principal orbits of  $G$  are hyperspheres. The cut locus  $C(p)$  is a  $\mathbf{CP}(n-1), \mathbf{HP}(n-1),$  or  $Ca(1) = S^8$  respectively. We normalize the Riemannian metric such that the injectivity radius is  $\frac{\pi}{2}$ , all geodesics have length  $\pi$ , and the range of the sectional curvatures is  $[1,4]$ . The orbit space  $M/G$  is then parameterized by  $r \in [0, \frac{\pi}{2}]$ , the volume of the orbits is given by a function  $V(r)$  which vanishes at  $r = 0, \frac{\pi}{2}$ , and the (unique) orbit of maximal volume is automatically minimally imbedded in  $M/G$ , i.e. our equator. Let  $q \in C(p)$  and let  $H = G_q$  be the isotropy group of the  $G$ -action at  $q$ . The action of  $H$  on  $M$  then defines the cohomogeneity two transformation group which we will apply to prove Theorems 1-3.

## 2 The orbital invariants of the isotropy action.

Let  $M = \mathbf{CP}(n), \mathbf{HP}(n),$  or  $Ca(2)$  and  $p \in M$ , let  $G$  be the isometry-isotropy subgroup at  $p$ . Let  $X$  be a unit vector in  $T_p M$ , then the geodesic segment  $\exp rX$ ,  $r \in [0, \frac{\pi}{2}]$  parameterizes the orbit space  $M/G$  and selects a representative triple of subgroups  $K \subset H \subset G$  for the almost homogeneous  $G$ -space  $M$ , with  $K = G_{\exp rX}$ ,  $r \in (0, \frac{\pi}{2})$  and  $H = G_q$ ,  $q = \exp \frac{\pi}{2} X \in C(p)$ . We summarize the orbital geometry of the  $G$ -action and give explicit details for the case  $M = Ca(2)$ .

Let  $M = F/G$  as a symmetric space and  $\underline{F} = \underline{G} + \underline{P}$  be the Cartan decomposition of the corresponding Lie algebras. For  $Ca(2)$  we have:  $F = F_4$ ,  $G = \text{Spin } 9$ ,  $\underline{P}$  is the spin representation  $\Delta_9$  as a  $G$ -module,

$K = \text{Spin}^{-7}$ ,  $H = \text{Spin } 8$ , with corresponding orbit types:

$r = 0 : \{p\}$

$r \in (0, \frac{\pi}{2}) : G/K = S_r$ , the geodesic sphere of radius  $r$ .

$r = \frac{\pi}{2} : G/H = C(p)$

Furthermore:  $\Delta_9|H = \Delta_8^+ \oplus \Delta_8^-$ , the direct sum of the two half-spin representations, where  $X \in \Delta_8^+$ , and  $\Delta_8^+|K = \rho_7 \oplus \theta$ ,  $\Delta_8^-|K = \Delta_7$  with  $\Delta_7$  the spin representation,  $\rho_7$  the standard orthogonal representation of  $K = \text{Spin}^{-7}$ , and  $\theta_X$  the trivial one-dimensional representation along  $X$ . (Note: Here  $\text{Spin}^{-7}$  is a non-standard imbedding of  $\text{Spin } 7$  in  $\text{Spin } 9$ , see the discussion in section 3). For the isotropy action of  $G$  on  $T_p M$  the orbits are standard spheres of radius  $r$ ,  $S(r)$ , where  $\exp_p S(r) = S_r$ ,  $r \in (0, \frac{\pi}{2})$ . Let furthermore  $\underline{G} = \underline{H} + \rho'_8$  and  $\underline{H} = \underline{K} + \rho'_7$  be orthogonal decompositions. Since  $\rho'_8|K = \Delta'_7$ , we may now write:

$$(i) \quad \underline{G} = \underline{K} + \rho'_7 + \Delta'_7$$

$$(ii) \quad \underline{F}_4 = \underline{K} + \Delta'_7 + \rho'_7 + \rho_7 + \theta_X + \Delta_7$$

as  $K$ -modules (markings to distinguish different isomorphic submodules). By (i) we identify the tangent space  $T_K(G/K)$  with  $\Delta'_7 + \rho'_7$ ; the resulting isomorphisms (needed later) between the tangent spaces of the  $S_r$ 's, viewed as  $\Delta'_7 + \rho'_7$ , and the orthogonal complement of  $X$  in  $T_p M$  are simultaneously realized in proposition 3.

**Definition:** Define the map  $J(r) : \underline{G} \rightarrow T_{\exp rX} S_r$  by  $J(r)(Z) =$  the Killing field of  $Z$  at the point  $\exp(rX)G$ ,  $r \in (0, \frac{\pi}{2})$ .

**Proposition 3.** Let  $Z \in \underline{G}$  as above. Then we have:

$$(a) \quad J(0)(Z) = 0, \quad \frac{D}{dr} J(0)(Z) = [Z, X], \quad \text{Ker } J(r) = \underline{K}, \quad \text{for } r \in (0, \frac{\pi}{2}).$$

$$(b) \quad \text{For } Z \in \Delta'_7; \quad J(r)(Z) = (\sin r)(\text{Expr } X)_*([Z, X]). \quad \text{For } Z \in \rho'_7 : \\ J(r)(Z) = \frac{1}{2}(\sin 2r)(\text{Expr } X)_*([Z, X])$$

$$(c) \quad adX|_{\Delta'_7} \text{ is an isomorphism onto } \Delta_8^-. \quad adX|_{\rho'_7} \text{ is an isomorphism onto the orthogonal complement of } X \text{ in } \Delta_8^+.$$

(Left translation by  $\exp rX$  has been denoted  $\text{Expr } X$ ).

**Proof:**  $\frac{D}{dr} \frac{\partial}{\partial t} (\exp tZ)(\exp rX)G = \frac{D}{dt} \frac{\partial}{dr} \exp(Ad(\exp tZ)(rX))G = [Z, X]$  at  $r = t = 0$ . The result in (b) is well known from the solution of the Jacobi equation in rank one symmetric spaces. From  $G$ -invariance of the inner product it follows that  $[\underline{G}, X] \subseteq X^\perp$ , from two-point homogeneity it follows that  $[\underline{G}, X]$  has codimension 1, i.e.  $[\underline{G}, X] = X^\perp = \rho_7 + \Delta_7$ . Since  $[\underline{H}, X] \subset \Delta_8^+$  and  $[\underline{K}, X] = 0$ , (c) now follows from (i) and (ii).

q.e.d.

**Corollary.** The Riemannian homogeneous space  $S_r$  is obtained from  $G/K$  by the  $K$ -invariant inner product defined by the two scaling factors  $\frac{1}{2} \sin 2r$  and  $\sin r$  on  $\rho'_7$  and  $\Delta'_7$  respectively, relative to the round unit sphere.

**Remark 1.** For the  $G$ -action on  $T_p M$  we have of course the uniform scaling factor  $r$  on  $\rho'_7 + \Delta'_7$  for  $S(r)$ , and an analogous statement Proposition 3'.

**Remark 2.** Any  $G$ -invariant metric on  $G/K$  is defined by a  $K$ -invariant inner product on  $\rho'_7 + \Delta'_7$ ; by Schur's lemma any such inner product is defined by two scaling factors relative to the unit sphere.

**Proposition 4.**  $G/K$  is a naturally reductive homogenous space ([KN]) if and only if the pair of scaling factor is  $(r, r)$ .

**Proof:** The case  $(r, r)$  admits an extension to an  $SO(T_p M)$ -action, i.e. this is the round sphere. Conversely, let  $r$  be the scaling factor for  $\rho'_7$ , and choose  $X \in \rho'_7$ ,  $Y, V \in \Delta'_7$ . Then  $[V, X] \in \Delta'_7$ ,  $[V, Y] \in \rho'_7 + K$ , hence  $[V, X]_{\underline{M}} \in \Delta_7$ ,  $[V, Y]_{\underline{M}} \in \rho'_7$ , where  $\underline{M} = \rho'_7 + \Delta'_7$ . Hence it follows from the defining equation for natural reductivity:  $\langle [V; X]_{\underline{M}}, Y \rangle + \langle X, [V, Y]_{\underline{M}} \rangle = 0$ , that the other scaling factor must also be  $r$ .

q.e.d.

### 3 The orbital invariants of the cohomogeneity two action.

We now proceed to the computation of the orbital invariants of the cohomogeneity two action of  $H$  on  $M$ . From Proposition 4 it is clear that few general computational techniques are available for the homogeneous spaces  $S_r$ . We again specify for the case  $M = Ca(2)$ , and first recall some of the needed facts about the relevant exceptional groups.

From the triality principle there are three distinct conjugacy classes of Spin 7 in Spin 8; they are conjugate under outer automorphisms of Spin 8 (arising from inner automorphisms of  $F_4$ ). Two of these, denoted by Spin<sup>-7</sup> and Spin<sup>+7</sup> are conjugate in Spin 9, with Spin 9/Spin<sup>±7</sup> =  $S^{15}$ . The last, denoted by Spin<sup>×7</sup>, is in standard position in Spin 9, with Spin 9/Spin<sup>×7</sup> equal to the Stiefel manifold of two-frames in  $\mathbf{R}^9$ . The intersection of two non-conjugate Spin 7 groups in  $H = \text{Spin } 8$  is a  $G_2$ , and Spin 7/ $G_2$  =  $S^7$ . The orbit space  $T_p M/H \times H$  is the first quadrant  $\mathbf{R}_+^2$ , and the unit sphere  $S(1)$  projects to the unit circle in  $\mathbf{R}_+^2$ :  $\{e^{it} | t \in (0; \frac{\pi}{2})\}$ . The orbit types for the  $H \times H$ -action on  $T_p M = \Delta_8^+ + \Delta_8^-$  are:

- (i)  $t = 0 : H \times H/\text{Spin}^{-7} \times H = S^7$

$$(ii) \ t \in (0, \frac{\pi}{2}) : H \times H / \text{Spin}^{-7} \times \text{Spin}^{+7} = S^7 \times S^7$$

$$(iii) \ t = \frac{\pi}{2} : H \times H / H \times \text{Spin}^{+7} = S^7.$$

**Proposition 5.** Let  $\Delta$  be the diagonal imbedding of  $H$  into  $H \times H$ . Then we have: The orbits of the restriction of the  $H \times H$ -action to either: (a) the  $\text{Spin}^{+7} \times \text{Spin}^{-7}$ -action, or (b) the  $\Delta H$ -action, coincide with the full  $H \times H$ -orbits. ( $\Delta H$  is the diagonal imbedding of  $H$  into  $H \times H$ ).

**Proof:** Since  $\text{Spin}^{-7} \cap \text{Spin}^{+7} = G_2$ , we have as principal orbit types: For (a):  $\text{Spin}^{+7} \times \text{Spin}^{-7} / G_2 \times G_2$ ; For (b):  $\text{Spin}8 / G_2$ ; these have full dimension in the  $H \times H$ -orbits. The singular orbits are straightforward.

q.e.d.

**Proposition 6.** Let  $\underline{G} = \underline{K} + \rho'_7 + \Delta'_7$  be the  $K$ -module decomposition of  $\underline{G}$  as before, and let  $Z$  be a unit vector in  $\Delta'_7$  with respect to the standard metric (scaling factors (1,1)). Then: (a) The orbit space  $S_7/H$  is parameterized by  $\exp(\theta Z)K$ ,  $\theta \in [0, \frac{\pi}{2}]$ , with the following orbit types:

$$(i) \ \theta = 0 : H_K = \text{Spin}^{-7}, \ H/H_K = S^7.$$

$$(ii) \ \theta \in (0, \frac{\pi}{2}) : H_{\text{Exp}(tZ)K} = G_2, \ H/G_2 = S^7 \times S^7.$$

$$(iii) \ \theta = \frac{\pi}{2} : H_{\text{Exp}(\frac{\pi}{2}Z)K} = \text{Spin}^{+7}, \ H/\text{Spin}^{+7} = S^7.$$

(b)  $\text{Exp}(\theta Z)K$ ,  $\theta \in \mathbf{R}$ , is a geodesic for any  $G$ -invariant metric on  $G/K$ .

**Proof:** Since (a) does not depend on the choice of metric, it suffices to prove it for the standard metric; i.e. for  $S(1) \subseteq T_p M = \Delta_8^+ + \Delta_8^-$  as an  $H$ -module. This coincides with the  $\Delta H$ -module  $T_p M$  considered in Proposition 5 (where the  $H \times H$ -action is the outer direct sum representation of  $\Delta_8^+$  and  $\Delta_8^-$ ). Since  $S(1)$  is a naturally reductive homogeneous space with respect to the pair  $(K, G)$ , it follows that  $\text{Exp}(tZ)K$  is a geodesic starting at  $X \in \Delta_8^+$  with velocity  $Y = [Z, X]$ . By Proposition 3',  $Y$  is a unit vector in  $\Delta_8^-$ , hence  $\text{Exp}(\frac{\pi}{2}Z)K \in \Delta_8^-$ , and  $\text{Exp}(\theta Z)K$  projects bijectively onto  $S(1)/H$ ,  $\theta \in [0, \frac{\pi}{2}]$ . Moreover,  $\Delta'_7 + \rho'_7|_{G_2} = \mu_7 + \mu_7 + \mathbf{R}Z$ , where  $\mu_7$  is the standard representation of  $G_2$  on the pure Cayley numbers; hence the fixed point set of  $G_2$  is  $\{\text{Exp}(\theta Z)K; \theta \in \mathbf{R}\}$ , which is then totally geodesic in any  $G$ -invariant metric.

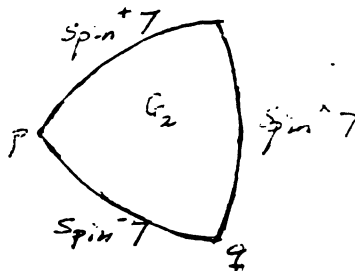
q.e.d.

**Proposition 7.** Let  $\underline{F}_4 = \underline{K} + \Delta'_7 + \rho'_7 + \rho_7 + \theta_X + \Delta_7$  be the  $K$ -module decomposition of  $\underline{F}_4$  as in 2 (ii); and let  $Z$  be a unit vector in  $\Delta'_7$  as above. Then  $\gamma(\theta) = (\text{Exp}\theta Z)(\text{Exp}rX)G$  is a geodesic in  $S_r$ , orthogonal to the  $H$ -orbits.

**Proof:** By the previous proposition  $\gamma(\theta)$  is a geodesic, by the Gauss lemma it is orthogonal to the  $K$ -orbits. Setting  $Y = [Z, X]$  as above, we have from the proof of Proposition 6:  $H_{\text{Exp}(\frac{\pi}{2}Z)(\text{Exp}rX)G} = \text{Spin}^+7$ , hence, by the Gauss lemma applied to the point  $\gamma(\frac{\pi}{2})$ ,  $\gamma(\theta)$  is also orthogonal to the  $\text{Spin}^+7$ -orbits. The tangent spaces of the  $\text{Spin}^-7$  and the  $\text{Spin}^+7$ -orbits together span the tangent spaces of the  $H$ -orbits.

q.e.d

Since  $Z \in \Delta'_7$ , it follows from Proposition 3 that the  $Y$ -direction has scaling factor  $\sin r$ , hence the orbit space  $H \setminus F/G$  is now parameterized by  $(r, \theta) \in (0, \frac{\pi}{2}) \times (0, \frac{\pi}{2})$  with the orbital distance metric  $ds^2 = dr^2 + \sin^2 r d\theta^2$ , i.e. a regular spherical triangle. This can be made more explicit as follows:  $P = \exp_p(\mathbf{R}X + \mathbf{R}Y)$  is a real projective plane; as the fixed point set of  $G_2$  it is totally geodesic in  $M$ ; from the above discussion it must intersect all  $H$ -orbits perpendicularly. It is sufficient to identify  $H$ -orbits in the disk  $D$  of radius  $\frac{\pi}{2}$  in the 2-plane  $\mathbf{R}X + \mathbf{R}Y$  in  $T_pM$ . The centers  $Z(\text{Spin}^-7) = \mathbf{Z}_2$  and  $Z(\text{Spin}^+7) = \mathbf{Z}_2$  act on  $F(G_2)$ ; here  $Z(\text{Spin}^-7)$  acts as  $-Id$  on  $\Delta_8^-$  and  $Z(\text{Spin}^+7)$  acts as  $-Id$  on  $\Delta_8^+$ . Reflections around the  $X$ - and  $Y$ -axes then identify points on the same  $H$ -orbit, and it follows that the intersection of  $D$  with the first quadrant may be used to parameterize the orbit space.



The orbit types are indicated on the above figure. The corners are the fixed points of  $H = \text{Spin}8$ , the edges are the fixed point sets of the three non-conjugate  $\text{Spin} 7$ 's in  $H$ , permuted by the triality automorphisms.

**The volume functional of the cohomogeneity two action.** The scaling factor which determines the relative volumes of the  $G$ -orbits  $S_r$  is given by  $\sin^8 r \sin^7 2r$ . It remains to determine the scaling factors for the  $H$ -orbits  $H \cdot (\text{exp}\theta Z)K$ ,  $\theta \in (0, \frac{\pi}{2})$  in  $S_r$ . The Killing fields defined by  $\underline{H}$  will again span the tangent spaces of the  $H$ -orbits, and hence determine the relative

volumes. Although these Killing fields are well understood as vector fields (those corresponding to  $\text{Spin}^{-7}$  and  $\text{Spin}^{+7}$  are the standard Jacobi-fields on the sphere of  $\sin \theta$ ,  $\cos \theta$ -type, respectively), their metric properties in terms of the  $S_r$ -metric are more complicated. The  $K$ -invariant decomposition  $T_K(G/K) = \rho'_7 \oplus \Delta'_7$  defines a  $G$ -invariant decomposition of the tangent bundle  $T(G/K)$  as a Whitney sum of two subbundles. However, it does not follow that a Killing field defined by  $Z \in \underline{K}$  remains in the  $\Delta'_7$ -summand of scaling factor  $\sin r$  (the Killing field is given by transvections along  $(\exp \theta Z)K$  with respect to the standard metric, i.e. those transvections are in  $SO(T_p M) = SO(16)$ , and do not respect the above splitting of  $T(G/K)$ ). Nevertheless, since we are only interested in the volume element determined by those Killing fields, the result turns out to be easy.

**Proposition 8.** Let  $\exp(\theta Z)K$  be a geodesic in  $S_r$  with  $Z$  a unit vector in  $\Delta'_7$  as before. Then the scaling factor for the relative volumes of the orbits  $H \exp(\theta Z)K$ ,  $\theta \in (0, \frac{\pi}{2})$  is given by  $\sin^7 \theta \cos^7 \theta$ .

**Proof.** Since  $\text{Exp}(-\theta Z)_*$  is an isometry with respect to any  $G$ -invariant metric on  $S_r$ , the Killing fields pull back to vectors in  $T_K(G/K)$  of length  $\sin \theta$  and  $\cos \theta$ , respectively, with respect to the standard metric. Since the orthogonal complements of  $Z$  coincide for all  $K$ -invariant inner products on  $\rho'_7 + \Delta'_7$ , the result follows.

**Corollary:** The volume functional for the  $H$ -action on  $M$  is given by  $V(r, \theta) = \sin^7 \theta \cos^7 \theta \sin^7 r \sin^7 2r$ .

**Proof:** Since  $Z \in \Delta'_7$  corresponds to a factor  $\sin r$ , and the volume of  $S_r$  is  $\sin^8 r \sin^7 2r$ , it follows that the  $r$ -dependance of  $V(r, \theta)$  must be  $\sin^7 r \sin^7 2r$ .

q.e.d.

**Remark:** This is the same result as would be obtained by formal lifting computations in a hypothetical "Hopf fibration"  $S \rightarrow S^4 \rightarrow Ca(2)$ .

We summarize the corresponding results for the remaining cases:

**Complex projective space.**  $CP(n) = U(n+1)/U(n) \times U(1)$ ,  $G = U(n) \times U(1)$ ,  $H = U(n-1) \times U(1) \times U(1)$ . The normal subgroups  $G^1 = U(n)$  and  $H^1 = G^1 \cap H = U(n-1) \times U(1)$  have the same orbits as  $G$  and  $H$  respectively, - hence we study their actions. We have  $K = U(n-1)$  and the  $K$ -module  $\underline{P} = \mu_{n-1} \oplus 2\theta$ , where  $\mu_{n-1}$  is the standard representation of  $U(n-1)$  and  $\theta$  the trivial representation. The scaling factors  $\sin r$  and  $\frac{1}{2} \sin 2r$ , defining the metrics of the Berger-spheres  $S_r^{2n-1}$ , have multiplicities  $2n-2$  and  $1$ , respectively. The principal  $H$ -orbits are of the type  $S^{2n-3} \times S^1$ , and the volume functional is given by  $\sin^{2n-3} r \sin 2r \sin^{2n-3} \theta \cos \theta$ .



**Quaternionic projective space.**  $HP(n) = Sp(n+1)/Sp(n) \times Sp(1)$ ,  $G = Sp(n) \times Sp(1)$ ,  $H = Sp(n-1) \times Sp(1) \times Sp(1)$ . Again we can reduce to the normal subgroups  $Sp(n)$  and  $Sp(n-1) \times Sp(1)$ . We have:  $K = Sp(n-1)$ , the  $K$ -module  $\underline{P} = \nu_{n-1} + 4\theta$ , where  $\nu_{n-1}$  is the standard representation of  $Sp(n-1)$ . The scaling factors  $\sin r$  and  $\frac{1}{2} \sin 2r$  now have multiplicities  $4n-4$  and  $3$ , respectively. The principal  $H$ -orbits are of type  $S^{4n-5} \times S^3$  and the volume functional is given by  $\sin^{4n-5} r \sin^3 2r \sin^{4n-5} \theta \cos^3 \theta$ .

**Change of coordinates:** For convenience we define new spherical coordinates by changing the center to the corner ( $r = 0$ ,  $\theta = \frac{\pi}{2}$ ). (Except for  $n = 2$ , this is not an  $H$ -fixpoint). In these coordinates the volume functionals now become:  $\sin^2 r \cos^{2n-3} r \sin 2\theta$  and  $\sin^6 r \cos^{4n-5} r \sin^3 2\theta$  for  $CP(n)$  and  $HP(n)$ , respectively (for  $Ca(2)$  there is no change).

## 4 The analysis and geometry of the reduced differential equation.

The previous results can be summarized as follows:

Let  $M$  be a projective space of complex, quaternionic, or Cayley-type. Let  $H$  be the isometry group that fixes a pair of conjugate points in  $M$ . Then the following holds:

(1) The orbit space  $M/H$  with the orbital distance metric is a regular spherical triangle parameterized by  $(r, \theta) \in [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$ ;  $ds^2 = dr^2 + \sin^2 r d\theta^2$ .

(2) The volumes of the  $H$ -orbits are given by the functional  $V(r, \theta) = \sin^{2b} r \cos^a r \sin^b 2\theta$ , where:

$$a = 2n - 3, b = 1 \text{ for } CP(n),$$

$$a = 4n - 5, b = 3 \text{ for } HP(n),$$

$$a = b = 7 \text{ for } Ca(2).$$

Let  $\gamma(s) = (r(s), \theta(s))$  be a curve in the interior of  $M/H$ , parameterized by arc length. By Proposition 2 the mean curvature of the corresponding hypersurface of  $M$  is given by  $k(\gamma(s)) - \frac{d}{ds} \ln V$ , where the geodesic curvature  $k = \dot{\alpha} + \dot{\theta} \cos r$ , and  $\bar{n} = -\sin \alpha \frac{\partial}{\partial r} + \cos \alpha \sin^{-1} r \frac{\partial}{\partial \theta}$ ;  $\alpha$  is the angle from  $\frac{\partial}{\partial r}$  to the tangent vector  $\dot{\gamma}(s)$ . Straightforward computation now gives the following result:

**Proposition 9.** The hypersurface corresponding to the curve  $\gamma(s) = (r(s), \theta(s))$  is minimal if and only if the curve  $\Gamma(S) = (r(s), \theta(s), \alpha(s))$  satisfies the following dynamical system:

$$\dot{r} = \cos \alpha$$

$$(\star) \dot{\theta} = \sin \alpha \sin^{-1} r$$

$$\dot{\alpha} = K_r \sin \alpha + 2b \cos \alpha \sin^{-1} r \cot 2\theta,$$

where  $K_r = a \tan r - (2b + 1) \cot r$ .

This is singular along the edges  $\theta = 0, \frac{\pi}{2}$  and  $r = \frac{\pi}{2}$ ; by well known arguments ([T2]) there exists, for each interior point  $P$  of an edge, a unique solution  $\gamma(s)$  with initial value at  $P$ . This solution is analytic and has initial direction orthogonal to the edge; thus it defines a smooth minimal hypersurface in  $M$ . Solution curves with end point at the origin are more complicated, we will need the following observation:

**Proposition 10.** Let  $\Gamma(s) = (r(s), \theta(s), \alpha(s))$  be a solution of  $(\star)$  with  $\lim_{s \rightarrow 0^-} r(s) = 0$ ; assume that the number of  $\theta$ -critical point of  $\Gamma(s)$  is bounded for  $s \in (-\delta, 0)$ . Then  $\Gamma(s)$  is the solution curve  $\theta \equiv \frac{\pi}{4}$ .

**Proof.** a) Since  $\dot{\theta} = \sin \alpha \sin^{-1} r$ , it follows that  $\sin \alpha$  has constant sign in  $(-\epsilon, 0)$  for some positive  $\epsilon$ . Since  $\dot{r}$  at an  $r$ -critical point is easily computed to be  $-\sin^2 \alpha K_r$ , it follows that  $r$ -critical points must be minima for  $r < r_0 = \text{Arctan} \sqrt{(2b + 1)a^{-1}}$  and maxima for  $r > r_0$ . Hence, from the condition  $\lim r(s) = 0$ , we may also assume  $\cos \alpha < 0$  in  $(-\epsilon, 0)$ .

b) Similarly, a computation of  $\ddot{\alpha}$  at a critical point for  $\alpha$  gives:  $\sin \alpha (a \cos^{-2} r \cos \alpha + (2b + 1) \sin^{-2} r \cos \alpha) - 2b \cos \alpha \sin^{-2} r \tan^{-2} 2\theta (\cos r \cos \alpha \tan 2\theta - 2 \sin \alpha \cos^{-2} 2\theta)$ , where both terms have the same constant sign in some  $(-\epsilon, 0)$ . Here the sign of  $\tan 2\theta$  is found by plugging back into the condition  $\dot{\alpha} = 0$ , i.e.  $\tan 2\theta = -2b \cos \alpha \sin^{-1} r K_r^{-1} \sin^{-1} \alpha$ . Hence,  $\dot{\alpha}$  must have constant sign in some interval  $(-\epsilon, 0)$ ; and  $\alpha_1 = \lim_{s \rightarrow 0^-} \alpha(s)$  exists.

c) We prove that  $\alpha_1 = \pi$  and  $\theta_1 = \frac{\pi}{4}$ . We have:  $\frac{d\theta}{dr} = \sin^{-1} r \tan \alpha \approx \sin^{-1} r \tan \alpha_1$  for small  $\epsilon$ , i.e.  $\theta(r) \approx C + \frac{1}{2} (\tan \alpha_1) \ln[(1 - \cos r)(1 + \cos r)^{-1}] \xrightarrow{r \rightarrow 0^+} \infty$  unless  $\tan \alpha_1 = 0$ ; hence  $\alpha_1 = \pi$  follows. Also,  $\frac{d\alpha}{dr} = K_r \tan \alpha + 2b \sin^{-1} r \cot 2\theta \approx 2b \cot 2\theta_1 \sin^{-1} r$  if  $\cot 2\theta_1 \neq 0$ . By a similar integration this would imply  $|\alpha(r)| \xrightarrow{r \rightarrow 0^-} \infty$ , which is a contradiction. Hence  $\theta_1 = \frac{\pi}{4}$ , and the above solution curve must be tangent to  $\theta \equiv \frac{\pi}{4}$  at the origin.

d) To conclude the proof, assume that  $\sin \alpha > 0$  in  $(-\epsilon, 0)$ , hence  $\theta(s) < \frac{\pi}{4}$  in  $(-\epsilon, 0)$ . From  $(\star)$  this implies that  $\dot{\alpha}$  becomes negative as soon as  $(2b + 1) \cot r \sin \alpha$  dominates  $a \tan r \sin \alpha$ ; this contradicts  $\lim_{s \rightarrow 0^-} \alpha(s) = \pi$  unless  $\alpha \equiv \pi$ ,  $\theta \equiv \frac{\pi}{4}$ . The case  $\sin \alpha < 0$  is treated similarly.

q.e.d.

$$\dot{r} = \cos \alpha$$

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$$\dot{\alpha} = K_r \sin \alpha + 2b \cos \alpha \sin^{-1} r \cot 2\theta,$$

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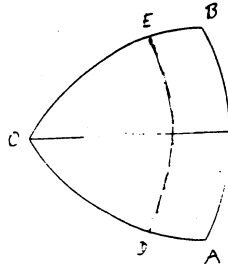
b) Similarly, a computation of  $\ddot{\alpha}$  at a critical point for  $\alpha$  gives:  $\sin \alpha (a \cos^{-2} r \cos \alpha + (2b + 1) \sin^{-2} r \cos \alpha) - 2b \cos \alpha \sin^{-2} r \tan^{-2} 2\theta (\cos r \cos \alpha \tan 2\theta - 2 \sin \alpha \cos^{-2} 2\theta)$ , where both terms have the same constant sign in some  $(-\epsilon, 0)$ . Here the sign of  $\tan 2\theta$  is found by plugging back into the condition  $\dot{\alpha} = 0$ , i.e.  $\tan 2\theta = -2b \cos \alpha \sin^{-1} r K_r^{-1} \sin^{-1} \alpha$ . Hence,  $\dot{\alpha}$  must have constant sign in some interval  $(-\epsilon, 0)$ ; and  $\alpha_1 = \lim_{s \rightarrow 0^-} \alpha(s)$  exists.

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d) To conclude the proof, assume that  $\sin \alpha > 0$  in  $(-\epsilon, 0)$ , hence  $\theta(s) < \frac{\pi}{4}$  in  $(-\epsilon, 0)$ . From  $(\star)$  this implies that  $\dot{\alpha}$  becomes negative as soon as  $(2b + 1) \cot r \sin \alpha$  dominates  $a \tan r \sin \alpha$ ; this contradicts  $\lim_{s \rightarrow 0^-} \alpha(s) = \pi$  unless  $\alpha \equiv \pi$ ,  $\theta \equiv \frac{\pi}{4}$ . The case  $\sin \alpha < 0$  is treated similarly.

q.e.d.

**Definition:** We define the following points in  $M/G$ :  $O = (0, 0)$ ,  $A = (\frac{\pi}{2}, 0)$ ,  $B = (\frac{\pi}{2}, \frac{\pi}{2})$ ,  $C = (\frac{\pi}{2}, \frac{\pi}{4})$ ,  $D = (r_0, 0)$ ,  $E = (r_0, \frac{\pi}{2})$ , with  $r_0 = \text{Arctan}\sqrt{(2b+1)a^{-1}}$ .



**Proposition 11.** The following curves define solutions of  $(\star)$ :

- (i)  $r \equiv r_0 = \text{Arctan}\sqrt{(2b+1)a^{-1}}$ ,
- (ii)  $\theta \equiv \frac{\pi}{4}$ ,
- (iii)  $\sin r \sin \theta \equiv \sqrt{b(a+2b+1)^{-1}}$ .
- (iv)  $\sin r \cos \theta \equiv \sqrt{b(a+2b+1)^{-1}}$ .

**Proof:** We have  $\alpha \equiv \pm\frac{\pi}{2}$  for (i),  $\alpha \equiv 0$  or  $\pi$  for (ii), and the result follows directly by substitution into  $(\star)$ . The solutions (iii) and (iv) correspond to the orbits of maximal volume for the isotropy groups of the points  $B$  and  $A$ , respectively.

q.e.d.

**Remark:** (i), (iii) and (iv) define smooth hypersurfaces of  $M$ , here (iii) and (iv) are "equators", i.e. homogeneous, minimal hyperspheres. Similarly, any solution curve of  $(\star)$  with initial point on the interior of the edge  $OA$  and terminal point on the interior of  $AB$  must define a smooth, minimal hypersphere in  $M$ . The most effective technique so far for proving existence of solution curves which are "closed" in the generalized sense that they have end points on the singular boundary, is to study variations of the pattern of critical points of auxiliary functions along solution curves.

**Definition:** Let  $\Gamma(s) = (r(s), \theta(s), \alpha(s))$  be a solution curve for  $(\star)$ ,  $s \geq 0$ . Let  $r_m^i(\gamma)$  and  $r_M^i(\gamma)$  denote the  $i$ -th local minimum, respectively local maximum, for  $r(s)$ ,  $s > 0$ ; similarly for  $\theta_m^i(\gamma)$  and  $\theta_M^i(\gamma)$ .

**Proposition 12.** Let  $\Gamma(s) = (r(s), \theta(s), \alpha(s))$ ,  $s \geq 0$ , be a solution curve for  $(\star)$  in  $X^* = M^*/H$ , which does not coincide with (i) or (ii). Then the critical points of  $r(s)$  and  $\theta(s)$  are non-degenerate, and no two of them can coincide. Moreover, we have: Any  $r_m^i(\gamma)$  occurs with  $r < r_0$ ,  $r_M^i(\gamma)$  with  $r > r_0$ . Any  $\theta_m^i(\gamma)$  occurs with  $\theta < \frac{\pi}{4}$ ,  $\theta_M^i(\gamma)$  with  $\theta > \frac{\pi}{4}$ .

**Proof:** From  $(\star)$  we have  $\ddot{r} = -(a \tan r - (2b + 1) \cot r)$  at a critical point for  $r$ , hence  $\ddot{r} \neq 0$  unless  $r = r_0$ ; by the uniqueness theorem for differential equations a solution with  $r(s_0) = r_0$ ,  $\cos \alpha(s_0) = 0$  must be (i). Similarly, a critical point for  $\theta$  is non-degenerate unless  $\theta = \frac{\pi}{4}$  and  $\sin \alpha = 0$ , i.e. the solution (ii). At an  $r$ -critical point we have  $\sin \alpha = \pm 1$ , hence  $\dot{\theta} \neq 0$ . Straightforward computation of  $\ddot{r}$  and  $\ddot{\theta}$  concludes the proof.

q.e.d.

**Definition:** The  $(r, \theta)$ -pattern  $P(\Gamma)$  of the above curve  $\Gamma(s)$  is the following sequence: Assign symbols  $r_M^i, r_m^i, \theta_M^i, \theta_m^i$  in the same order as corresponding critical points occur along  $\Gamma(s)$ ;  $s > 0$ . At a boundary point critical points of  $r$  and  $\theta$  coincide, in that case the pair is enclosed in a bracket and considered as one symbol.  $P^N(\Gamma)$  is the segment consisting of the  $N$  first symbols of  $P(\Gamma)$ .

**Definition:** Let  $\Gamma_d(s) = (r_d(s), \theta_d(s), \alpha_d(s))$  denote the solution curve with  $r_d(0) = \frac{\pi}{2}$ ,  $\theta_d(0) = d$ ,  $\alpha_d(0) = -\frac{\pi}{2}$ , and let  $\gamma_d(s) = (r_d(s), \theta_d(s))$ .

We need the following observation:

**Proposition 13.** The curve  $\gamma_d$  depends continuously on  $d \in (0, \frac{\pi}{2})$ .

**Proof:** This is the "continuous dependance of solutions beyond intersections with the singular boundary"; for proof see [HT]. Notice that also  $\Gamma_d$  depends continuously on initial conditions modulo identification of  $\alpha$  with  $\alpha + \pi$  at boundary points.

**Proposition 14.** Let  $I = (b_1 - \epsilon, b_1 + \epsilon) \subseteq (0, \frac{\pi}{2})$ . Assume that  $P^{N-1}(\Gamma_b)$  has no symbol of bracket type for  $b \in I$  and that  $P^N(\Gamma_{b_1})$  has  $N$ -th symbol of bracket type. Then we have:

- (1)  $P^{N-1}(\Gamma_d)$  is constant for  $d \in I$ .
- (2) Let  $p$  be the first intersection point of  $\Gamma_{b_1}(s)$  with the boundary ( $s > 0$ ); the  $N$ -th symbol of  $P^N(\Gamma_{b_1})$  is then as follows:
  - $p \in OD : (r_m^i, \theta_m^j)$
  - $p \in OE : (r_m^i, \theta_M^j)$

$$\begin{aligned} p \in DA \text{ or } AC &: (r_M^i, \theta_n^j) \\ p \in EB \text{ or } BC &: (r_M^i, \theta_M^j). \end{aligned}$$

- (3)  $P^{2N}(\Gamma_{b_1})$  is reflectionally symmetric about its  $N$ -th symbol, and  $P(\Gamma_{b_1})$  is periodic with period  $2N$ .

**Proof:** This follows from Proposition 12, a careful analysis of the sharp turning of solution curves near the singular boundary, estimates on  $\dot{\alpha}$  from ( $\star$ ), and Proposition 13. (For detailed arguments of this type, we refer to [T1]). For (3), just observe that if  $\gamma_{b_1}(s_0)$  is the first boundary intersection point, then the solution curve  $\gamma_{b_1}(s)$ ,  $s \in [0, s_0]$ , can be extended to  $[0, 2s_0]$  by retracing backwards, and then extended to all  $s$  by periodicity.

q.e.d.

**Remark:** In contrast to  $P^N(\Gamma)$ , the whole pattern  $P(\Gamma)$  will not, in general, be constant on any open interval.

We now turn to the most delicate analysis that is needed: the initial part of  $P(\Gamma_d)$  as  $d \rightarrow \frac{\pi}{4}$ ; this is determined by the intersection pattern of  $\gamma_d(s)$  with the solution  $\bar{\theta} \equiv \frac{\pi}{4}$ .

Let  $\bar{\theta} = \theta - \frac{\pi}{4}$ , straightforward computation from ( $\star$ ) gives:

$$(\star\star) \quad \frac{d^2\bar{\theta}}{dr^2} + [\cot r - \cos^{-2} \alpha (a \tan r - (2b+1) \cot r)] \frac{d\bar{\theta}}{dr} - 2b \sin^{-2} r \cos^{-2} \alpha \tan 2\bar{\theta} = 0.$$

By continuous dependance of solution curves on initial conditions it follows that on compact intervals contained in  $[0, \frac{\pi}{2})$ ,  $\alpha_d \rightarrow -\pi$ ,  $\bar{\theta}_d \rightarrow 0$  as  $d \rightarrow \frac{\pi}{4}$ . We can then approximate by the linearization of ( $\star\star$ ) along the solution  $\bar{\theta} \equiv 0$ :

$$(\star\star') \quad \frac{d^2\bar{\theta}}{dr^2} + [2(b+1) \cot r - a \tan r] \frac{d\bar{\theta}}{dr} + 4b \sin^{-2} r = 0.$$

The substitution  $\bar{\theta} = v \cos^{-\frac{a}{2}} r \sin^{-(b+1)} r$  gives:

$$(\star\star\star): \quad \frac{d^2v}{dr^2} + q(r)v = 0, \quad v(\frac{\pi}{2}) = 0.$$

where  $q(r) = \frac{a}{2}(2b+3) + 5b+1 + (3b-b^2) \cot^2 r - \frac{a(a-2)}{4} \tan^2 r$ . As  $r \rightarrow 0+$ , the dominating term is  $b(3-b) \cot^2 r$ , and we have:

For  $CP(n)$ :  $b=1$ ;  $2 \cot^2 r \rightarrow_{r \rightarrow 0+} \infty$

For  $HP(n)$ :  $b=3$ :  $0$

For  $Ca(2)$ :  $b=7$ :  $-28 \cot^2 r \rightarrow_{r \rightarrow 0+} -\infty$ .

Hence the oscillatory behaviour near  $r=0$  of  $\gamma_d$  as  $d \rightarrow \frac{\pi}{4}$  changes drastically with the scalar field: the term forces rapid oscillations in the  $CP(n)$ -case, it counteracts oscillations in the  $Ca(2)$ -case,  $HP(n)$  must be investigated separately.

We estimate the number of zeroes of a non-zero solution of ( $\star\star\star$ ) on the interval  $[\epsilon, T]$  by applying a delicate criterion based on Sturm's comparison method:

**Proposition 15.** Let  $I = [\epsilon, T]$ ,  $\epsilon > 0$ ,  $q(r) \geq 0$  for  $r \in I$ , and let  $N$  be the number of zeroes of a non-zero solution of the equation  $(***)$  on  $I$ . Then we have:  $|\pi N - \int_{\epsilon}^T \sqrt{q(r)} dr| \leq \pi + \frac{1}{4} \int_{\epsilon}^T \frac{|dq(r)|}{q(r)}$ .

**Proof:** This is given in [Ha], Th 5-2, p.347.

**Proposition 16** Let  $M = CP(n)$ ; i.e.  $a = 2n - 3$ ,  $b = 1$ , and  $\delta \in (0, \frac{\pi}{2})$ . Then any solution of  $(***)$  has infinitely many zeroes on  $(0, \delta)$ .

**Proof:** For  $\delta$  sufficiently small we have  $q(r) > p(r) = 1.9 \cot^2 r$  and it suffices to consider  $v'' + p(r)v = 0$ . Here  $\int_{\epsilon}^{\delta} \sqrt{1.9 \cot^2 r} dr = \sqrt{1.9}(\ln(\sin \delta) - \ln(\sin \epsilon))$ , and  $\frac{1}{4} \int_{\epsilon}^{\delta} |d \ln p(r)| = -\frac{1}{2} \ln(\cot \epsilon) + \frac{1}{2} \ln(\cot \delta)$ , hence the difference approaches infinity as  $\epsilon \rightarrow 0+$ , and the result follows from Proposition 15.

q.e.d.

**Theorem 1** There exists infinitely many congruence classes of imbedded, minimal hyperspheres in  $CP(n)$ ,  $n \geq 2$ .

**Proof:** Let  $P_m(\Gamma_d)$  be the segment of the  $(r, \theta)$ -pattern of  $\Gamma_d$  up to and including the first  $r_m$ . From Proposition 16 it follows that the number of intersections of the curve  $\gamma_d(s)$ ,  $s \in [0, \frac{\pi}{2}]$  with  $\theta \equiv \frac{\pi}{4}$  approaches infinity as  $d \rightarrow \frac{\pi}{4}$ . Hence  $P_m(\Gamma_d)$  becomes a  $\theta_m, \theta_M$ -sequence of length  $N$  terminated by an  $r_m$ , where  $N \rightarrow \infty$  as  $d \rightarrow \frac{\pi}{4}$ . By Proposition 14 it follows that as  $d \rightarrow \frac{\pi}{4}$ , it passes through an infinite number of values  $d'$  such that  $\gamma_{d'}$  intersects the singular boundary before the first  $r_m$ . By Proposition 10 this intersection occurs in the interior of one of the edges  $OA$  or  $OB$ ; by Proposition 11  $\gamma_{d'}$  determines a minimal hypersphere in  $M$ . Since  $\gamma_{d'}$  does not have self-intersections before the first  $r_m$  this hypersphere is imbedded.

q.e.d.

For  $HP(n)$  and  $Ca(2)$  it is not possible to obtain the desired solutions from the local oscillation properties around 0. It is needed to obtain long-time control of the solution curves emanating from the boundary  $AB$  to find the intersections with  $OA$  and  $OB$ . The most delicate analytical result required for this is given by the following:

**Proposition 17.** Let  $M = HP(n)$  or  $Ca(2)$ . Then there exists a positive  $\delta$  such that for  $d \in (\frac{\pi}{4}, \frac{\pi}{4} + \delta)$  the solution curve  $\gamma_d(s)$  intersects  $\theta \equiv \frac{\pi}{4}$  twice before  $r_m^1(\gamma_d)$ .

**Proof.** We prove this for the  $HP(n)$  case, and refer the  $Ca(2)$ -case to the Appendix.

From Proposition 15 and standard approximation arguments it follows that it is sufficient to show that:

(A)  $\int_{\epsilon}^T \sqrt{q(r)} dr > 2\pi + \frac{1}{4} \int_{\epsilon}^T \frac{|dq(r)|}{q(r)}$  for some positive  $\epsilon$ . To minimize the computations, we reduce to a previously treated case as follows: The results in [HS], imply that  $Q(r) = \frac{(2d+1)m-4d}{2} - \frac{(m-2)(m-4d)}{4} \tan^2 r$  satisfies (A), (for some  $T \in (0, \frac{\pi}{2})$ ), when  $d = 3$ ,  $m \geq 12$ . It is clear that if (A) holds for some  $Q(r)$ , it also holds for  $\bar{Q}(r) = C_1 Q(r) + C_2$ , where  $C_1 \geq 1$ ,  $C_2 \geq 0$ . Now choose  $C_1 = (4n-5)(4n-7)(m-2)^{-1}(m-12)^{-1}$ ,  $C_2 = 18n - \frac{13}{2} - \frac{1}{2}(7m-12)(4n-5)(4n-7)(m-2)^{-1}(m-12)^{-1}$  and  $d = 3$ . Then  $C_1 Q(r) + C_2 = 18n - \frac{13}{2} - \frac{1}{4}(4n-5)(4n-7) \tan^2 r = q(r)$ . It remains only to observe that choosing  $m = 4n$  makes  $C_1 \geq 1$ ,  $C_2 \geq 0$ .

q.e.d.

**Corollary.** There exists a  $d \in (\frac{\pi}{4}, \text{Arccos}\sqrt{b(a+2b+1)^{-1}})$  such that  $P(\Gamma_d)$  is periodic with period  $(\theta_m, r_m, \theta_M, (\theta_m, r_M), \theta_M, r_m, \theta_m, (\theta_M, r_M))$ .

**Proof:** The solution curve (iv) of Proposition 11 has periodic  $(r, \theta)$ -pattern with period  $((\theta_m, r_m), (\theta_M, r_M))$ . Hence it follows from Propositions 13 that for  $d$  close to  $\sqrt{b(a+2b+1)^{-1}}$  the second intersection point of  $\gamma_d$  with  $\bar{\theta} \equiv \frac{\pi}{4}$  occurs after  $r_m^1(\gamma_d)$ , with  $\alpha \in (0, \frac{\pi}{2})$ . By Proposition 16 and continuous dependence on initial conditions, it follows that for  $d \in (\frac{\pi}{4}, \frac{\pi}{4} + \delta)$  with sufficiently small  $\delta$ , the second intersection point of  $\gamma_d$  with  $\theta \equiv \frac{\pi}{4}$  has  $\alpha \in (\frac{\pi}{2}, \pi)$ . Hence, by continuity,  $\alpha = \frac{\pi}{2}$  at the second intersection point of  $\gamma_d$  with  $\theta \equiv \frac{\pi}{4}$  for some  $d \in (\frac{\pi}{4} + \delta, \text{Arccos}\sqrt{b(a+2b+1)^{-1}})$ . This is then the first  $r_m$ , and the solution is continued through reflection around  $\theta \equiv 0$ , to yield the above periodic  $(r, \theta)$ -pattern.

q.e.d.

**Remark:** Computer pictures of solutions close to this one are shown in the appendix.

**Proposition 18.** There are infinitely many  $d \in (\frac{\pi}{4}, \frac{\pi}{2})$  such that  $\gamma_d$  intersects the boundary arc  $OD$ .

**Proof:** Let  $d_1 = \text{Arccos}\sqrt{b(a+2b+1)^{-1}}$ . Since  $P(\Gamma_{d_1})$  is periodic with period  $((r_m^1, \theta_m^1), (r_M^1, \theta_M^1))$ , it follows from Proposition 13 that for  $d$  sufficiently close to  $d_1$ , the number of  $\theta_m$ 's before  $r_m^{2N}$  in  $P(\Gamma_d)$  is either  $2N-1$  or  $2N$ . Let  $\bar{d}$  be as in the preceding corollary, with the period of  $P(\Gamma_{\bar{d}})$  equal to  $(\theta_m, r_m, \theta_M, (\theta_m, r_M), \theta_M, r_m, \theta_m, (\theta_M, r_M))$ . Again, by Proposition 13, the number of  $\theta_m$ 's before  $r_m^{2N}$  in  $P(\Gamma_d)$  is now  $3N-1$  when  $d$  is sufficiently



close to  $\bar{d}$ . It follows from Proposition 14 that as  $d$  passes through the interval  $(\frac{\pi}{4}, \text{Arccos}\sqrt{b(a+2b+1)^{-1}})$ ,  $\gamma_d$  must have intersected  $OD$  at last  $N-1$  times.

q.e.d.

**Theorem 2.** There exists infinitely many congruence classes of immersed minimal hyperspheres in  $HP(n)$ ,  $n \geq 1$ .

**Theorem 3.** There exists infinitely many congruence classes of immersed, minimal hyperspheres in the Cayley projective plane,  $Ca(2)$ .

**Proof:** These theorems are now corollaries of Proposition 18 and Proposition 11.

## Appendix.

**Estimates for the Cayley projective plane.** In this case we have:

( $\star\star\star$ ):  $\frac{d^2v}{dr^2} + q(r)v = 0$ ,  $v(\frac{\pi}{2}) = 0$ , with  $q(r) = 95.5 - 28 \cot^2 r - 8,75 \tan^2 r$ . This time the now negative term  $b(3-b) \cot^2 r$  limits the region where ( $\star\star\star$ ) is of oscillating type, i.e.  $q(r) > 0$ , to  $[r_1, r_2]$ , where  $r_1 \approx 0,50216$ ,  $r_2 \approx 1,27296$ . Here  $\int_{r_1}^{r_2} \sqrt{q(r)} dr < 2\pi$ , so there is no hope of applying Proposition 15 to prove Proposition 17, as was done for  $HP(n)$ . In that case an argument was needed that applied for all  $n$ . Now, however, we need only the case  $n = 2$ , i.e. the above equation ( $\star\star\star$ ); which opens for brute force estimates. Computer pictures of solution curves close to those of Proposition 17 and its corollary are given at the end. Let  $w = -\frac{1}{v} \frac{dv}{dr}$ , we then have:

( $\star\star\star$ ):  $\frac{dw}{dr} = q + w^2$ .

Recalling that  $v = \bar{\theta} \cos^{\frac{7}{2}} r \sin^8 r$ , it is clear that crossings with  $\bar{\theta} \equiv 0$  corresponds to zeroes of  $v$ , i.e. to  $w$ -crossings with  $\pm\infty$  (notice that we are in the region where  $\alpha \neq \pm\frac{\pi}{2}$ , i.e.  $\frac{dv}{dr} \neq 0$ ). To obtain suitable bounds for  $w$ , we observe the following:

**Lemma.** Let  $w(r)$  and  $\bar{w}(r)$  be solutions of  $\frac{dw}{dr} = q + w^2$  and  $\frac{d\bar{w}}{dr} = C + \bar{w}^2$  respectively, defined on the interval  $[r', r'']$ , and assume that  $q \geq C$  on  $[r', r'']$ ,  $\bar{w}(r'') > w(r'')$ . Then  $\bar{w} > w$  on  $[r', r'']$ .

**Proof:** Notice that  $\frac{d}{dr}(w - \bar{w}) = q - C + w^2 - \bar{w}^2$  is positive whenever  $w = \bar{w}$ ; this contradicts the possibility of the two solutions crossing on  $[r', r'']$ .

q.e.d.

The solution of  $\frac{d\bar{w}}{dr} = -A^2 + \bar{w}^2$  with initial value  $\bar{w}(r') = B$  is given by:

$$(A1): A(e^{2A(r'-r)}(B + A) + B - A)(e^{2A(r'-r)}(B + A) - (B - A))^{-1}.$$

The special case  $\lim_{r \rightarrow r'} \bar{w}(r) = \infty$  by:

$$(A2): A(e^{2A(r'-r)} + 1)(e^{2A(r'-r)} - 1)^{-1}.$$

The solution of  $\frac{d\bar{w}}{dr} = A^2 + \bar{w}^2$  with initial value  $\bar{w}(r') = B$  is given by:

$$(A3): A \tan(A(r - r') + \text{Arctan} \frac{B}{A}).$$

The special case  $\lim_{r \rightarrow r'} \bar{w}(r) = \infty$  by:

$$(A4): A \tan(A(r - r') + \frac{\pi}{2}).$$

We note that  $q(r)$  reaches its maximum at  $r_3 \approx 0,92878$ ; is increasing in  $(0, r_3)$  and decreasing in  $(r_3, \frac{\pi}{2})$ . We will obtain relevant bounds as follows:

- (i) For  $[r', r''] \subset (r_2, \frac{\pi}{2})$ : The comparison equation is  $\frac{d\bar{w}}{dr} = -A^2 + \bar{w}^2$  with  $A^2 \geq -q(r'')$ .
- (ii) For  $[r', r''] \subset (r_3, r_2)$ . The comparison equation is  $\frac{d\bar{w}}{dr} = A^2 + \bar{w}^2$ , with  $A^2 \leq \inf_{r \in [r', r'']} q(r)$ , i.e.  $A^2 \leq q(r'')$ .
- (iii) For  $[r', r''] \subset (r_1, r_3)$  the comparison equation is as in (ii) with  $A^2 \leq q(r')$ .
- (iv) For  $[r', r''] \subset (0, r_1)$ : The comparison equation is as in (i) with  $A^2 \geq -q(r')$ .

We have  $\lim_{r \rightarrow \frac{\pi}{2}-} w(r) = \lim_{r \rightarrow \frac{\pi}{2}-} \frac{d}{dr} \ln v(r) = \infty$ , and we wish to bound  $w(r)$  from above as  $r$  decreases from  $\frac{\pi}{2}$ . We give the steps in detail, the numerical values can now readily be checked on a calculator.

a) We choose to start with the interval  $[1, 55; \frac{\pi}{2} - 0, 01]$ , and apply (i) with  $A^2 = 87398, 7 > q(\frac{\pi}{2} - 0, 01)$ ; since we have no upper estimate for  $w(\frac{\pi}{2} - 0, 01)$ , we choose the case  $\bar{w}(\frac{\pi}{2} - 0, 01) = \infty$ . By (A2) we get:  $\bar{w}(1, 55) = A(e^{2A(\frac{\pi}{2}-1,56)} + 1)(e^{2A(\frac{\pi}{2}-1,56)} - 1)^{-1} < 296, 7$ .

b) We divide  $(1.30, 1.55)$  into intervals  $(r', r'')$  of length at most 0.02, and apply (i), case (A1), where the initial value  $B$  is set equal to the previously obtained upper bound for  $w(r'')$ . At each step, care is taken in rounding off to insure the validity of the upper bound. This gives  $w(1.30) < 10, 77$ ; a final application of this procedure to  $[1, 273, 1.30]$  gives  $w(1, 273) < 8, 78$ .

c) We divide  $(0, 95; 1, 272)$  into intervals of length at most 0,01 and apply (ii), (A3). This gives the estimate  $\bar{w}(0.95) < -28, 5$ . Let  $[r', r''] = [0, 915; 0, 95]$ , then  $\inf_{r \in [r', r'']} q(r) = q(0.95) > 64, 07 = A^2$ . We can then again apply (ii), (A3), and obtain:  $\frac{B}{A} < -\frac{28,5}{8,005}$ ,  $\text{Arctan} \frac{B}{A} < -1, 2969$ ,  $A \cdot (0, 915 - 0, 95) < -0, 28$ ; hence the argument of  $\tan$  in (A3) has already passed  $-\frac{\pi}{2}$ ; hence  $\bar{w}(r)$ , and, by comparison,  $w(r)$ , has passed  $-\infty$  before  $r$  reaches 0.915.

**Corollary.** The first zero of  $v(r)$ , as  $r$  decreases from  $\frac{\pi}{2}$ , occurs for  $r > 0.915$ .

d) On the interval  $[0,91;0,915]$  we apply (iii), (A4), to obtain  $w(0,91) < \bar{w}(0,91) < 200$ .

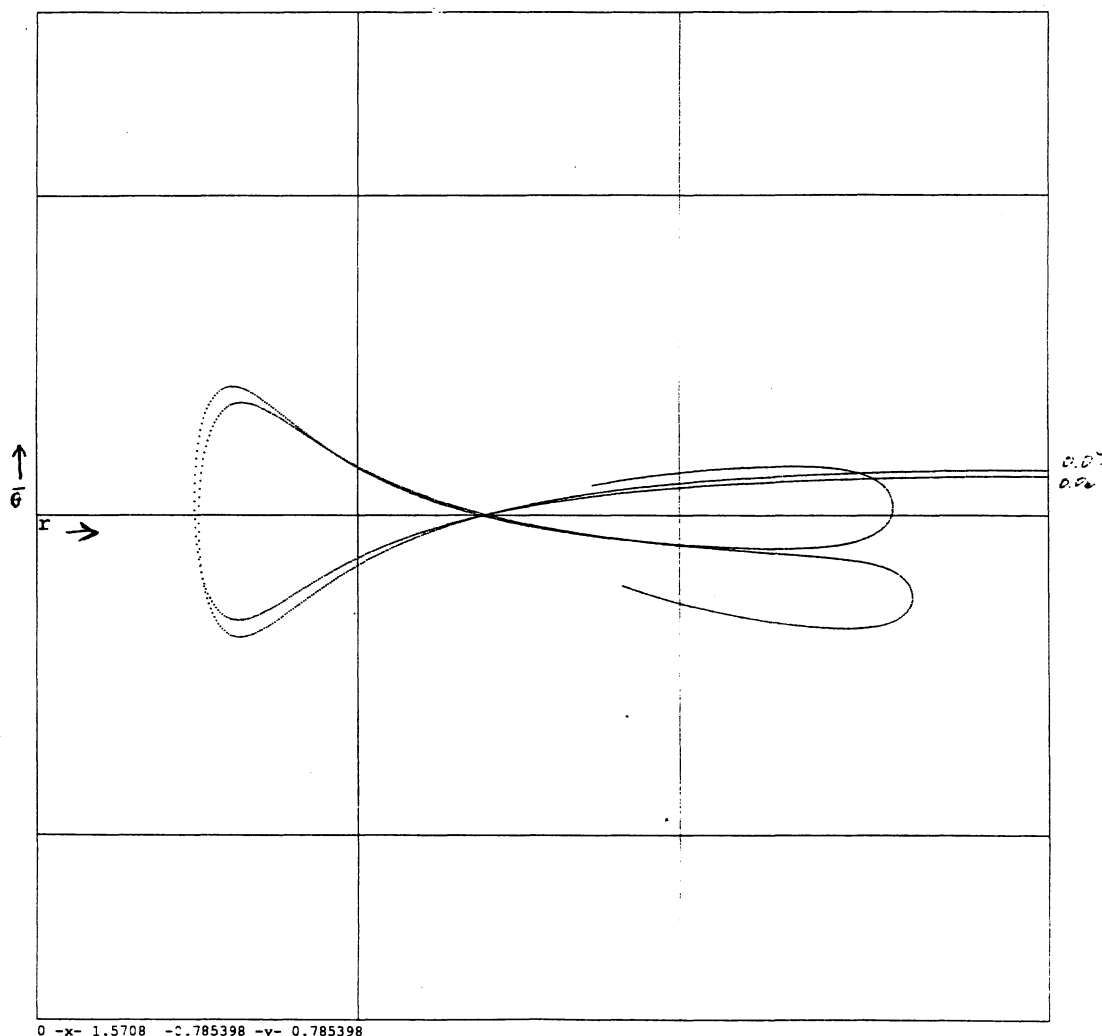
e) Dividing  $(0,52;0,91)$  into intervals of length 0,01, and applying (iii), case (A3), we obtain the bound  $w(0,52) < -10,8$ . Continuing with 3 intervals of length 0,005, we obtain:  $w(0,505) < -12,8$ .

g) It follows that  $w(0,50) < -12,8$ ; we are now in the region  $q < 0$ . Applying (iv), (A1) to the intervals  $[0,48;0,50]$  and  $[0,45;0,48]$ , we obtain  $w(0,45) < -32,4$ . Now consider the denominator  $D(r) = (e^{2A(0,45-r)}(B + A) - (B - A))$  of (A1) on  $[0,40;0,45]$ , with  $A^2 = 62,71 > -q(0,4)$ ,  $B = -32,4$ . Then  $D(0,45) = 2A > 0$ .  $D(0,40) \approx -13,7 < 0$ , hence  $D(r) = 0$  for some  $r \in (0,40;0,45)$ , this implies  $\bar{w}(r)$ , and hence  $w(r)$  reaches  $-\infty$  in this interval.

**Corollary:** The solution  $v(r)$  of (\*\*\*) has at least two zeroes on the interval  $0,40 < r < \frac{\pi}{2}$ . This completes the proof of Proposition 17 for the  $Ca(2)$ -case.

Solutions of(\*) for  $a=7, b=3, : HP(3)$ .

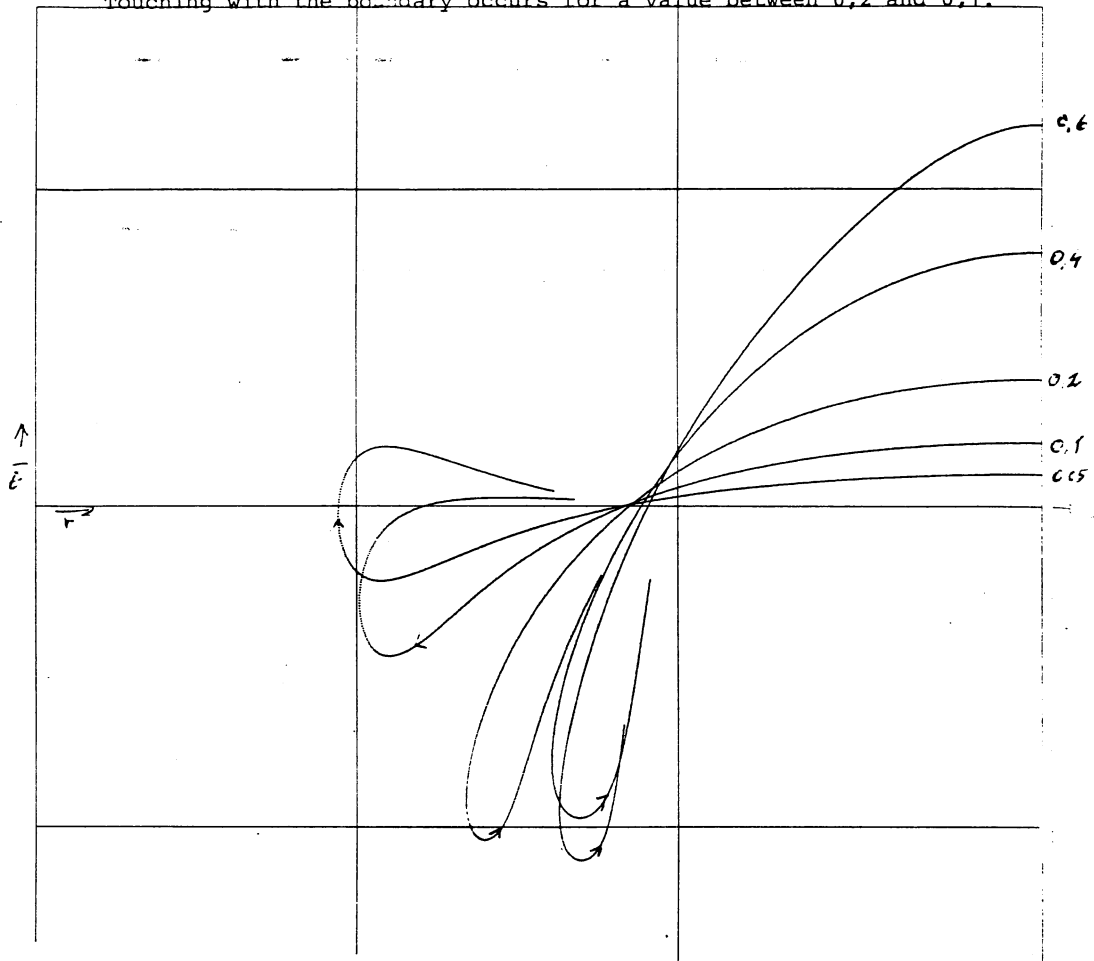
The two solutions shown are for  $d\frac{r}{4} = 0.06$  and  $0.07$ , respectively.



Solutions of (★) for  $a=7, b=7, : Ca(2)$ .

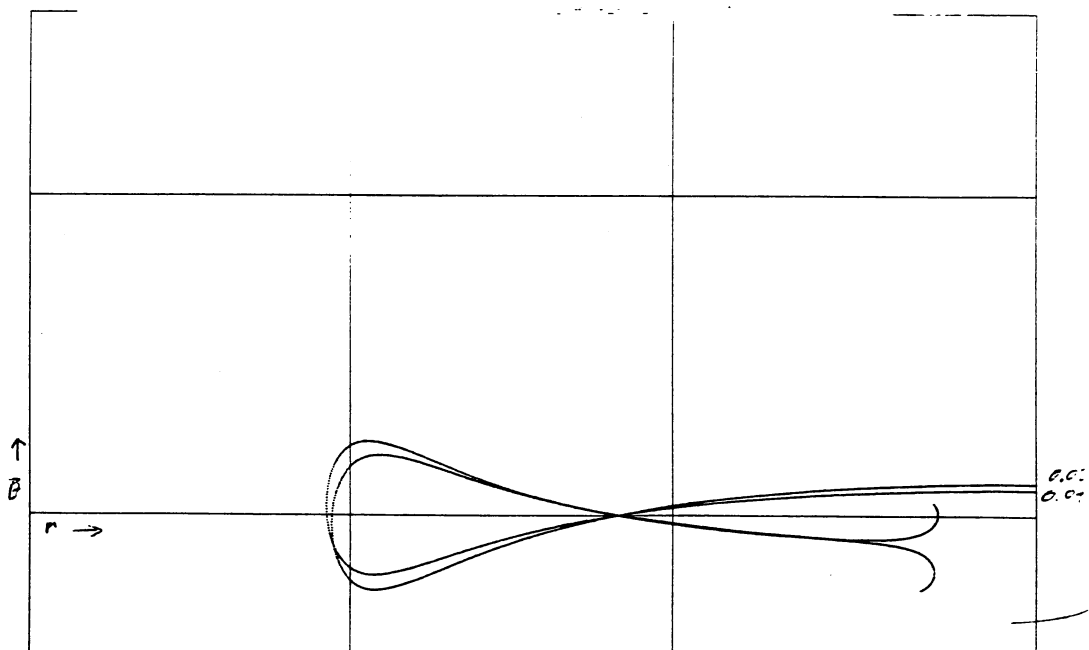
The solutions shown are for  $d-\frac{\pi}{4} = 0.6, 0.4; 0.2 ; 0.1; \text{ and } 0.05$ .

Touching with the boundary occurs for a value between 0.2 and 0.1.



Solutions of (★) for  $a=7, b=7, : Ca(2)$ .

The two solutions shown are for  $d-\frac{\pi}{4} = 0.04$  and  $0.05$ , respectively



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