SINGULAR PERTURBATIONS OF DISCRETE SYSTEMS

TORE M. JONASSEN

ABSTRACT. We introduce a singular perturbation theory for a class of dynamical systems defined on $\mathbb{R}^n \times \mathbb{R}^n$.

1. INTRODUCTION

The Hénon system $(x, y) \mapsto (\alpha - x^2 + \beta y, x)$ has been studied by many authors. It was introduced in [Hen]. There is numerical evidence for a strange attractor at many different parameter values. The most common example is the parameter value $\alpha = 1.4$ and $\beta = 0.3$. A mathematical proof of the existence of strange attractors for very small β is given in [B&C].

The Hénon system $(x, y) \mapsto (\alpha - x^2 + \beta y, x)$ may be viewed as a perturbation of the logistic map to a diffeomorphism in the plane. In this paper we generalize this construction, and study relations between the perturbed and unperturbed system. We will not try to solve the strange attractor problem, but we will show by geometrical methods that there is a very close relationship between the dynamics of smooth maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and a class of diffeomorphisms on $\mathbb{R}^n \times \mathbb{R}^n$ generated by f. We will do this in two steps, first identifying the properties of f with properties of the zero lift, and then use $|\boldsymbol{\beta}| - C^1$-closeness on compact sets of the zero lift and the $\boldsymbol{\beta}$ lift of f.

We have used geometrical arguments, and tried to avoid ad hoc arguments used in [J].

At the end of this paper we have given an example with a map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with a non-wandering set topologically equivalent to a one-sided shift on four symbols. The lifted map $F_{\boldsymbol{\beta}} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ has a non-wandering set topologically equivalent to a full shift on four symbols.

Several computer experiments with $n = 1$ can be found in [J].

2. THE LIFTED DYNAMICAL SYSTEM

We will generalize the construction used to obtain the Hénon family from the logistic family.

Let $\boldsymbol{\beta} \in \mathbb{R}^n$ with $\boldsymbol{\beta} = (\beta_1, \dots, \beta_n)$. We define a n-parameter family of maps

$$\mathcal{L}_{\boldsymbol{\beta}} : C^r(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow C^r(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n)$$

by

$$f \mapsto \mathcal{L}_{\boldsymbol{\beta}}(f) = F_{\boldsymbol{\beta}}$$

where

$$F_{\boldsymbol{\beta}}(x, y) = (f(x) + \boldsymbol{\beta} y, x)$$

Here we think of $\boldsymbol{\beta}$ as a diagonal matrix, and $\boldsymbol{\beta} y$ as the transpose of

$$\begin{bmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

The map $\mathcal{L}_{\boldsymbol{\beta}}(f) = F_{\boldsymbol{\beta}}$ is called the β lift of f. If $\boldsymbol{\beta} = \mathbf{0}$ then $F_{\mathbf{0}}$ is called the zero lift of f. Throughout this paper we will use a capital letter for the lifted map.

We will first state and prove some simple but useful lemmas.

Key words and phrases. bifurcation, horseshoe, hyperbolic structure, shift-map.

Lemma 2.1. *Let $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$ and let $F_{\boldsymbol{\beta}} = \mathcal{L}_{\boldsymbol{\beta}}(f)$. Then $F_{\boldsymbol{\beta}} \in \mathrm{Diff}^r(\mathbb{R}^n)$ if and only if $\beta_i \neq 0$ for $i = 1, 2, \ldots, n$. The inverse is given by the formula $F_{\boldsymbol{\beta}}^{-1}(x, y) = (y, \boldsymbol{\beta}^{-1}(x - f(y)))$. Furthermore, the derivative of $F_{\boldsymbol{\beta}}$ has a constant determinant given by*

$$\det DF_{\boldsymbol{\beta}}(x, y) = (-1)^n \prod_{i=1}^{n} \beta_i.$$

Proof. Consider the equation $(v, w) = (f(x) + \boldsymbol{\beta}y, x)$. We find that $x = v$ and $w = f(x) + \boldsymbol{\beta}y = f(v) + \boldsymbol{\beta}y$. The equation $\boldsymbol{\beta}y = w - f(v)$ has a unique solution $y = \boldsymbol{\beta}^{-1}(w - f(v))$ if and only if the diagonal matrix $\boldsymbol{\beta}$ is invertible, that is $\beta_i \neq 0$ for any i. Hence the inverse map is given by $F_{\boldsymbol{\beta}}^{-1}(x, y) = (y, \boldsymbol{\beta}^{-1}(x - f(y)))$. Furthermore we observe that the smoothness properties of $F_{\boldsymbol{\beta}}$ and $F_{\boldsymbol{\beta}}^{-1}$ depends only on the smoothness properties of f.

The derivative of $DF_{\boldsymbol{\beta}}$ in block matrix form is given by

$$DF_{\boldsymbol{\beta}} = \begin{bmatrix} Df & \boldsymbol{\beta} \\ I & 0 \end{bmatrix}$$

By the Laplace expansion theorem for determinants we find that

$$\det DF_{\boldsymbol{\beta}} = \det \begin{bmatrix} Df & \boldsymbol{\beta} \\ I & 0 \end{bmatrix} = (-1)^n \det \begin{bmatrix} I & 0 \\ Df & \boldsymbol{\beta} \end{bmatrix} = (-1)^n \det I \det \boldsymbol{\beta} = (-1)^n \prod_{i=1}^{n} \beta_i.$$

□

Let π_1 denote the projection $(x, y) \mapsto x$ and π_2 the projection $(x, y) \mapsto y$. Let $f : M \longrightarrow M$, $g : N \longrightarrow N$ and $h : M \longrightarrow N$ be continuous maps. We call f and g semi-conjugate if $h \circ f = g \circ h$.

Lemma 2.2. *f and $F_{\mathbf{0}}$ are semi-conjugate.*

Proof. The diagram

$$\begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^n & \xrightarrow{F_{\mathbf{0}}} & \mathbb{R}^n \times \mathbb{R}^n \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^n \end{array}$$

commutes following arrows since $\pi_1 \circ F_{\mathbf{0}}(x, y) = \pi_1(f(x), x) = f(x)$ and $f \circ \pi_1(x, y) = f(x)$. □

Lemma 2.3. *Let $K \subset \mathbb{R}^n \times \mathbb{R}^n$ be a compact set contained in the ball $\{z \in \mathbb{R}^n \times \mathbb{R}^n : \|z\| \leq k\}$ for some $k \geq 1$. Then $F_{\mathbf{0}}$ and $F_{\boldsymbol{\beta}}$ are $k|\boldsymbol{\beta}| - C^1$-close on K.*

Proof. We will first find the C^0-size of $F_{\boldsymbol{\beta}} - F_{\mathbf{0}}$ on K. Let $z = (x, y) \in K$. Then

$$\|F_{\boldsymbol{\beta}}(z) - F_{\mathbf{0}}(z)\| = \|(f(x) + \boldsymbol{\beta}y - f(x), x - x)\| = \|\boldsymbol{\beta}y\| \leq \|\boldsymbol{\beta}\|k.$$

Let v denote a vector of norm 1 in the tangent space of $\mathbb{R}^n \times \mathbb{R}^n$ over some point $z = (x, y)$ in K. Then

$$\|(DF_{\boldsymbol{\beta}}(z) - DF_{\mathbf{0}}(z))v\| = \left\| \begin{bmatrix} 0 & \boldsymbol{\beta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{2n} \end{bmatrix} \right\| \leq \|\boldsymbol{\beta}\|.$$

Hence the C^1-size of $F_{\boldsymbol{\beta}} - F_{\mathbf{0}}$ on K is bounded by $\max\{k\|\boldsymbol{\beta}\|, \|\boldsymbol{\beta}\|\} = k\|\boldsymbol{\beta}\|$ since $k \geq 1$. □

Lemma 2.4. *Let $\eta_m(\lambda)$ denote the characteristic polynomial of $Df^m(x)$. Then the characteristic polynomial of $DF_{\mathbf{0}}^m(x,y)$ is given by $\xi_m(\lambda) = \lambda^n \eta_m(\lambda)$.*

Proof. A direct calculation together with the Laplace expansion theorem for determinants shows that

$$\xi_m(\lambda) = \det(\lambda I - DF_{\mathbf{0}}^m(x,y)) = \det \begin{bmatrix} \lambda I - Df^m(x) & 0 \\ -I & \lambda I \end{bmatrix} = \det(\lambda I - Df^m(x)) \det(\lambda I) = \lambda^n \eta_m(\lambda).$$

□

Let $O^+(f, x_0)$ denote the forward orbit of x_0 under iterations by f. A sequence $\{y_i\}_{i=0}^{\infty}$ is called a α-pseudo-orbit for f if $\|y_{i+1} - f(y_i)\| < \alpha$ for all $i \geq 0$. An orbit $O^+(f, x_0)$ γ-shadows the sequence $\{y_i\}_{i=0}^{\infty}$ if $\|f^i(x_0) - y_i\| < \gamma$ for all $i \geq 0$.

Lemma 2.5. *If $O^+(F_{\boldsymbol{\beta}}, p_0) \subset K$ where K is a compact set of size less than k, then $O^+(F_{\boldsymbol{\beta}}, p_0)$ is $k|\boldsymbol{\beta}|$-shadowed by a pseudo-orbit from the system generated by $F_{\mathbf{0}}$.*

Proof. This is an immediate consequence of lemma 2.3 since $F_{\boldsymbol{\beta}}$ and $F_{\mathbf{0}}$ are $k\|\boldsymbol{\beta}\|$-C^1-close on K. □

3. Fixed points and periodic orbits

Suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ has a fixed point x_0. It is then easily seen that $F_{\mathbf{0}}$ has a fixed point in (x_0, x_0). If f has a n-periodic orbit $\{x_0, x_1, \ldots, x_{n-2}, n_{n-1}\}$, where the points on the orbit are indexed such that $f(x_i) = x_{i+1}$ modulo n, we see that the corresponding periodic orbit for $F_{\mathbf{0}}$ is given by $\{(x_0, x_{n-1}), (x_1, x_0), \ldots, (x_{n-2}, x_{n-3}), (x_{n-1}, x_{n-2})\}$. We get the following lemma by the implicit function theorem:

Lemma 3.1. *Suppose $\{x_0, \ldots, x_{n-1}\}$ is a periodic orbit of f. If $1 \notin \operatorname{spec}(Df^n(x_0))$, then there exists a neighborhood B of $\boldsymbol{\beta} = \mathbf{0}$ such that $F_{\boldsymbol{\beta}}$ has at least one n-periodic orbit near the n-periodic orbit of the zero lift. The stability properties of the periodic orbit may not be preserved.*

If $\operatorname{spec}(Df^n(x_0)) \cap S^1 = \varnothing$ then there exists a neighborhood B of $\boldsymbol{\beta} = \mathbf{0}$ such that $F_{\boldsymbol{\beta}}$ has a unique n-periodic orbit near the n-periodic orbit of the zero lift. Furthermore, if the periodic orbit of f is stable then the periodic orbit of $F_{\boldsymbol{\beta}}$ is stable. If the periodic orbit of f is unstable or of saddle type then the periodic orbit of $F_{\boldsymbol{\beta}}$ is of saddle type.

Proof. Let $F_{\boldsymbol{\beta}}^n(x,y) = F^n(x,y,\boldsymbol{\beta})$ and define

$$H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{N} \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

by

$$H(x,y,\boldsymbol{\beta},n) = F^n(x,y,\boldsymbol{\beta}) - Id(x,y).$$

Then $H(x,y,\mathbf{0},n) = (f^n(x) - x, f^{n-1}(x) - y)$ and if $\{x_0, \ldots, x_{n_0-1}\}$ is a n_0-periodic orbit of f we have $H(x_0, x_{n_0-1}, \mathbf{0}, n_0) = (\mathbf{0}, \mathbf{0})$. We find that the derivative of $H(x,y,\mathbf{0},n_0)$ with respect to (x,y) is given by

$$DH_{(x,y)}(x_0, x_{n_0-1}, \mathbf{0}, n_0) = \begin{bmatrix} Df^{n_0}(x_0) - I & 0 \\ Df^{n_0-1}(x_0) & -I \end{bmatrix}$$

We find that

$$\det DH_{(x,y)}(x_0, x_{n_0-1}, \mathbf{0}, n_0) = \det(Df^{n_0}(x_0) - I)\det(-I) = (-1)^n \det(Df^{n_0}(x_0) - I).$$

Hence $\det DH_{(x,y)}(x_0, x_{n_0-1}, \mathbf{0}, n_0) = 0$ if and only if $1 \in \operatorname{spec}(Df^{n_0}(x_0))$. Now the implicit function theorem gives us the first part of the lemma. The second part of the lemma follows from the implicit function theorem together with lemma 2.4 and lemma 2.5 noting that $F_{\boldsymbol{\beta}}$ has at least n eigenvalues close to zero for small $\|\boldsymbol{\beta}\|$. □

4. The relation between some invariant sets for the map and zero lift

We call a set Λ weak f-invariant if $f(\Lambda) \subset \Lambda$. A set Λ is called f-invariant if $f(\Lambda) = \Lambda$. Let

$$\mathcal{G}(f, K) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = f(y), y \in K\}.$$

We have $F_0(\mathbb{R}^n \times \mathbb{R}^n) = \mathcal{G}(f, \mathbb{R}^n)$ of course.

Lemma 4.1. *Suppose $\Gamma \subset \mathbb{R}^n$. Then Γ is weak f-invariant if and only if $\pi_1^{-1}(\Gamma)$ is weak F_0-invariant.*

Proof. Suppose Γ is weak f-invariant. Let $(x, y) \in \pi_1^{-1}(\Gamma)$. Then $F_0(x, y) = (f(x), x) \in \pi_1^{-1}(\Gamma)$ since $f(x) \in \Gamma$.

Suppose $\pi_1^{-1}(\Gamma)$ is weak F_0-invariant. Let $x \in \Gamma$. Now $(x, y) \in \pi_1^{-1}(\Gamma)$ implies that $(f(x), xy) \in \pi_1^{-1}(\Gamma)$ so $f(x) \in \Gamma$. □

Corollary 4.1. *If Λ is weak f-invariant then $\pi^{-1}(\Lambda)$ is weak F_0-invariant. In particular, $\Omega(F_0) \subset \Omega(f) \times \mathbb{R}^n$. Moreover, we have a one-to-one correspondence between weak f-invariant sets $\Lambda \subset \mathcal{G}(f, \mathbb{R}^n)$ and weak F_0-invariant sets Γ given by $\Gamma(\Lambda) = \mathcal{G}(f, \Lambda)$.*

If $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear isomorphism such that $\operatorname{spec}(L) \cap S^1 = \varnothing$ we call L a hyperbolic isomorphism. If L is a hyperbolic isomorphism then there exists a splitting $\mathbb{R}^n = E_L^s \oplus E_L^u$ in L-invariant subspaces such that $L|_{E_L^s}$ is a contraction and $L|_{E_L^u}$ is an expansion. The subspace E_L^s is called the stable subspace of L and E_L^u is called the unstable subspace of L. The subspace E_L^s (E_L^u) is the generalized eigenspace corresponding to the (possible complex) eigenvalues of norm less than 1 (greater than 1).

Suppose $x_0 \in \mathbb{R}^n$ is a fixed point of f and $Df(x_0)$ is a hyperbolic isomorphism. Then by the inverse function theorem f is a local diffeomorphism in some neighborhood V_{x_0} of x_0. By the local invariant manifold theorem [P&M] there exist C^r-discs, $W_{loc}^s(f, x_0)$ and $W_{loc}^u(f, x_0) \subset V_{x_0}$ such that

$$W_{loc}^s(f, x_0) = \{x \in V_{x_0} : f^n(x) \to x_0 \text{ and } f^n(x) \in V_{x_0} \text{ for all } n \geq 0\}$$
$$W_{loc}^u(f, x_0) = \{x \in V_{x_0} : f^{-n}(x) \to x_0 \text{ and } f^{-n}(x) \in V_{x_0} \text{ for all } n \geq 0\}$$

for some neighborhood V_{x_0} of x_0. We have $\dim W_{loc}^s(f, x_0) = \dim E_{Df(x_0)}^s = s$ and $\dim W_{loc}^u(f, x_0) = \dim E_{Df(x_0)}^u = u$. Furthermore the tangent spaces at x_0 are given by $T_{x_0} W_{loc}^s(f, x_0) = E_{Df(x_0)}^s$ and $T_{x_0} W_{loc}^u(f, x_0) = E_{Df(x_0)}^u$.

Unfortunately F_0 has a singularity in every point of its domain of definition, so the considerations above do not carry over directly. We see by lemma 2.4 that $\operatorname{rank}(DF_0(z)) \leq n$ for all $z \in \mathbb{R}^n \times \mathbb{R}^n$. However, the considerations above are valid if F_0 is restricted to $\mathcal{G}(f, \mathbb{R}^n)$. We give $\mathbb{R}^n \times \mathbb{R}^n$ a trivial foliation with sets of form $\{x\} \times \mathbb{R}^n$ as the leaves. We note that the F_0-image of each leaf consists of a single point $(f(x), x) \in \mathcal{G}(f, \mathbb{R}^n)$. Hence we define local stable and unstable sets at the fixed point (x_0, x_0) of F_0 as

$$W_{loc}^s(F_0, (x_0, x_0)) = W_{loc}^s(f, x_0) \times \mathbb{R}^n$$
$$W_{loc}^u(F_0, (x_0, x_0)) = W_{loc}^u(f, x_0) \times \mathbb{R}^n$$

These local stable and unstable sets are not well-behaved due to the singularity of F_0 in (x_0, x_0) since

$$\dim W_{loc}^s(F_0, (x_0, x_0)) + \dim W_{loc}^u(F_0, (x_0, x_0)) = (s + n) + (u + n) = (s + u) + 2n = 3n.$$

The unstable set is "too big" as seen later. Trivially we have

$$W_{loc}^s(F_0|_{\mathcal{G}(f,\mathbb{R}^n)}, (x_0, x_0)) = \mathcal{G}(f, W_{loc}^s(f, x_0))$$
$$W_{loc}^u(F_0|_{\mathcal{G}(f,\mathbb{R}^n)}, (x_0, x_0)) = \mathcal{G}(f, W_{loc}^u(f, x_0))$$

We find the the tangent space of these sets at (x_0, x_0) by mapping vectors in $E_{Df(x_0)}^s$ and $E_{Df(x_0)}^u$ with the linear map

$$v \mapsto \begin{bmatrix} Df(x_0) \\ I \end{bmatrix} v.$$

The above remarks are also true for periodic orbits, replacing f by a power of f.

5. Some remarks on stable and unstable sets for the map versus stable and unstable manifolds for the diffeomorphism

We will now discuss the relationship between the stable and unstable sets $W^s_{loc}(F_{\mathbf{0}},(x_0,x_0))$ and $W^u_{loc}(F_{\mathbf{0}},(x_0,x_0))$ and the local invariant manifolds $W^s_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta})))$ and $W^u_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta})))$ when $\beta_i \neq 0$, $i=1,2,\dots,n$, and $|\boldsymbol{\beta}|$ is near zero. Before giving all technical details we will give some heuristic arguments for the relationship.

Suppose $x_0 \in \mathbb{R}^n$ is a hyperbolic non-degenerate fixed point of f, that is $\operatorname{spec}(Df(x_0)) \cap (S^1 \cup \{0\}) = \varnothing$. Then by lemma 3.1 $F_{\boldsymbol{\beta}}$ has a hyperbolic fixed point $(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))$ for all $\boldsymbol{\beta} \in B_0$, where B_0 is some open neighborhood of $\mathbf{0} \in \mathbb{R}^n$. If $\beta_i \neq 0$ for $i=1,\dots,n$, then $F_{\boldsymbol{\beta}}$ is a diffeomorphism of $\mathbb{R}^n \times \mathbb{R}^n$, and by the local invariant manifold theorem there exist C^r-discs $W^s_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta})))$ and $W^u_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta})))$ with properties as described in section 4. The dimension of these sets are given by the dimension of the stable subspace $E^s_{DF_{\boldsymbol{\beta}}(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))}$ and the unstable subspace $E^u_{DF_{\boldsymbol{\beta}}(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))}$ with

$$\dim W^s_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))) = \dim E^s_{DF_{\boldsymbol{\beta}}(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))}$$
$$\dim W^u_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))) = \dim E^u_{DF_{\boldsymbol{\beta}}(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))}.$$

By lemma 2.4 the characteristic polynomial of $DF_{\mathbf{0}}(x_0,x_0)$ is given by $\xi_1(\lambda) = \lambda^n \eta_1(\lambda)$ where $\eta_1(\lambda)$ is the characteristic polynomial of $Df(x_0)$. Since the eigenvalues vary continuously with $\boldsymbol{\beta}$ it follows that none of the zeroes in ξ_1 cross S^1 for $|\boldsymbol{\beta}|$ near zero, and we see that ξ_1 has n zeros close to zero (in $\mathbb{C}$). Hence we conclude that $\dim E^s_{DF_{\boldsymbol{\beta}}(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))} = s+n$ and $\dim E^u_{DF_{\boldsymbol{\beta}}(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))} = u$ where s is the dimension of the stable subspace of $Df(x_0)$ and u is the dimension of the unstable subspace of $Df(x_0)$.

In the case of $\boldsymbol{\beta} = 0$ the vectors $w^{(1)},\dots,w^{(n)} \in \mathbb{R}^{2n}$ where $w^{(j)}_i = 0$ if $i \neq n+j$ and $w^{(j)}_i = 1$ if $i = n+j$ are eigenvectors corresponding to the zero eigenvalue of multiplicity n. We should expect that there are n eigenvectors (possible complex) $w^{(1)}(\boldsymbol{\beta}),\dots,w^{(n)}(\boldsymbol{\beta}) \in \mathbb{R}^{2n}$ such that $\|w^{(j)}(\boldsymbol{\beta}) - w^{(j)}\|$ is small.

Let $D^n_\epsilon(x_0)$ denote the open n-disc of radius ϵ with center at x_0, From the above remarks together with the location of the stable and unstable sets of $F_{\mathbf{0}}$ at (x_0,x_0) we should expect the local stable and unstable manifolds at $(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))$ to be $|\boldsymbol{\beta}| - C^1$-close to the sets $W^s_{loc}(f,x_0) \times D^n_\epsilon(x_0)$ and $\mathcal{G}(f, W^u_{loc}(f,x_0))$, that is

$$W^s_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))) \approx W^s_{loc}(f,x_0) \times D^n_\epsilon(x_0)$$
$$W^u_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))) \approx \mathcal{G}(f, W^u_{loc}(f,x_0))$$

The terms $|\boldsymbol{\beta}| - C^1$-close and approximately equal will be given presice meaning below.

We will use the following definition for C^r-closeness of submanifolds. It is taken from [P&M].

Definition. Let S and S' be C^r-submanifolds of a manifold M, and let $\epsilon > 0$. We say that S and S' are $\epsilon - C^r$-close if there exists a C^r-diffeomorphism $h : S \longrightarrow S' \subset M$ such that $i' \circ h$ is ϵ-close to i in the C^r-topology. The maps $i : s \longrightarrow M$ and $i' : S' \longrightarrow M$ denote the inclusion maps.

Theorem 5.1. *Suppose $x_0 \in \mathbb{R}^n$ is a non-degenerate hyperbolic fixed point of $f \in C^r(\mathbb{R}^n,\mathbb{R}^n)$ with $r \geq 1$. Suppose $\beta_i \neq 0$ for $i=1,\dots,n$, and let $F_{\boldsymbol{\beta}}$ denote the $\boldsymbol{\beta}$ lift of f. Let*

$$i : W^s_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))) \subset \mathbb{R}^n \times \mathbb{R}^n$$
$$i' : W^s_{loc}(f,x_0) \times D^n_\epsilon(x_0) \subset \mathbb{R}^n \times \mathbb{R}^n$$
$$j : W^u_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))) \subset \mathbb{R}^n \times \mathbb{R}^n$$
$$j' : \mathcal{G}(f, W^u_{loc}(f,x_0)) \subset \mathbb{R}^n \times \mathbb{R}^n$$

denote the inclusion maps. Then there exist C^r-diffeomorphisms, $h : W^s_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))) \longrightarrow W^s_{loc}(f,x_0) \times D^n_\epsilon(x_0) \subset \mathbb{R}^n \times \mathbb{R}^n$ and $g : W^u_{loc}(F_{\boldsymbol{\beta}},(x_0(\boldsymbol{\beta}),y_0(\boldsymbol{\beta}))) \longrightarrow \mathcal{G}(f, W^u_{loc}(f,x_0))$ such that i and $i' \circ h$ are $|\boldsymbol{\beta}| - C^1$-close, and j and $j' \circ g$ are $|\boldsymbol{\beta}| - C^r$-close.

Proof. Since $F_{\mathbf{0}}$ is singular we can not apply the local invariant manifold theorem directly. To show closeness of the local stable manifolds we use Irwins proof [P&M] of the local stable manifold theorem, this proof is valid also for maps. To show closeness of the local unstable manifolds we use the local non-linear graph transform [S], which is also valid for maps. The reason for this is that both technics involve only forward iterates of the map, which are well-defined.

By assumption $\mathrm{spec}(Df(x_0)) \cap S^1 = \varnothing$. Then by lemma 2.4 $\mathrm{spec}(DF_{\mathbf{0}}(x_0, x_0)) \cap S^1 = \varnothing$. By lemma 3.1 there exists a neighborhood B of $\boldsymbol{\beta} = \mathbf{0}$ such that $F_{\boldsymbol{\beta}}$ has a hyperbolic fixed point $z_f(\boldsymbol{\beta})$. Hence there is a direct sum splitting $\mathbb{R}^n \times \mathbb{R}^n = E^s_{\boldsymbol{\beta}} \oplus E^u_{\boldsymbol{\beta}}$, associated with the derivative at the fixed point, depending smoothly on $\boldsymbol{\beta}$, such that $DF_{\boldsymbol{\beta}}(z_f)|_{E^s_{\boldsymbol{\beta}}}$ is a contraction and $DF_{\boldsymbol{\beta}}(z_f)|_{E^u_{\boldsymbol{\beta}}}$ is an expansion for all $\boldsymbol{\beta} \in B$. Associated with this splitting there are numbers $\lambda_s(\boldsymbol{\beta}) < 1$ and $\lambda_u(\boldsymbol{\beta}) > 1$ such that $\|DF_{\boldsymbol{\beta}}(z_f)w\| < \lambda_s(\boldsymbol{\beta})\|w\|$ if $w \in E^s_{\boldsymbol{\beta}}$ and $\|DF_{\boldsymbol{\beta}}(z_f)w\| > \lambda_u(\boldsymbol{\beta})\|w\|$ if $w \in E^u_{\boldsymbol{\beta}}$.

Now Irwins proof of the local stable manifold theorem applies where the local stable manifold is obtained as a graph of a function obtained by the implicit function theorem for functions on Banach spaces observing that the construction of the suitable functions depends only on forward iterates of $F_{\mathbf{0}}$. Moreover, the function we obtain varies smoothly with perturbations of $F_{\mathbf{0}}$.

We obtain the unstable manifold as a fixed point from the local non-linear graph transform observing again that we use only forward iterates of $F_{\mathbf{0}}$. Also in this case the fixed point varies smoothly with perturbations of $F_{\mathbf{0}}$. □

6. Simple bifurcations

We will discuss the relationship between bifurcations in the map f and the $\boldsymbol{\beta}$ lift. We will restrict this discussion to three types, the saddle-node, the period-doubling, and the Hopf bifurcation.

The relation between bifurcations for f and the $\boldsymbol{\beta}$ lift will be discussed in terms of transversality theory in a suitable jet space. At the end of this paper we give an example with maps $f : \mathbb{R} \longrightarrow \mathbb{R}$ lifted to plane diffeomorphisms, using the implicit function theorem in a constructive proof for the saddle-node and the period doubling bifurcation. Example 2 below provides an alternative proof. The Hopf bifurcation does not occur in dissipative plane diffeomorphisms.

Suppose $x_0 \in \mathbf{Per}(f)$ with period n_0. We will assume that the derivative $Df^{n_0}(x_0)$ has a single real eigenvalue on S^1 or a single pair of eigenvalues on $S^1 \setminus \{-1, 1\}$. We also assume that f^{n_0} is non-singular at x_0.

Since f^{n_0} is non-singular at x_0, with $f^{n_0}(x_0) = x_0$, f^{n_0} is a diffeomorphism in some neighborhood of x_0. If $Df^{n_0}(x_0)$ has a single eigenvalue $\lambda_1 = -1$ or $\lambda_1 = 1$ and all other eigenvalues off S^1, then there is a one dimensional center manifold tangent to the eigenspace $E^c_{x_0}$ associated with λ_1 at x_0. If $Df^{n_0}(x_0)$ has a single pair of eigenvalues $\lambda_1 = \overline{\lambda_2}$ on $S^1 \setminus \{-1, 1\}$ and all other eigenvalues off S^1, then there is a two dimensional center manifold tangent to the eigenspace $E^c_{x_0}$ associated with λ_1, λ_2 at x_0.

The following two examples show the idea. We then prove the general case.

Example 1 (Saddle-node and period-doubling for one-dimensional maps). Consider C^3-maps $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$. We view the first coordinate as the state variable, and the second coordinate as a parameter. Assume $f(x_0, \alpha_0) = x_0$ and $f_x(x_0, \alpha_0) = 1$. Let $p = (x_0, \alpha_0)$. Consider the 2-jet extension

$$j^2 f : \mathbb{R} \times \mathbb{R} \longrightarrow J^2(\mathbb{R} \times \mathbb{R}, \mathbb{R}).$$

We equip $J^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with coordinates $(x, \alpha, f, f_x, f_\alpha, f_{xx}, f_{x\alpha}, f_{\alpha\alpha})$. Let

$$q = (x_0, \alpha_0, f(p), f_x(p), f_\alpha(p), f_{xx}(p), f_{x\alpha}(p), f_{\alpha\alpha}(p))$$

In this coordinate system we have

$$D(j^2 f) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_\alpha \\ f_{xx} & f_{x\alpha} \\ f_{x\alpha} & f_{\alpha\alpha} \\ f_{xxx} & f_{xx\alpha} \\ f_{xx\alpha} & f_{x\alpha\alpha} \\ f_{x\alpha\alpha} & f_{\alpha\alpha\alpha} \end{bmatrix}$$

Hence the space $D(j^2 f)_p(\mathbb{R} \times \mathbb{R})$ is spanned by

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ f_{xx}(p) \\ f_{x\alpha}(p) \\ f_{xxx}(p) \\ f_{xx\alpha}(p) \\ f_{x\alpha\alpha}(p) \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} 0 \\ 1 \\ f_\alpha(p) \\ f_{x\alpha}(p) \\ f_{\alpha\alpha}(p) \\ f_{xx\alpha}(p) \\ f_{x\alpha\alpha}(p) \\ f_{\alpha\alpha\alpha}(p) \end{bmatrix}$$

We define the surface $B_{SN}^{(2,1)}$ in $J^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ as the set $x = f$ and $f_x = 1$. This set has codimension 2 in $J^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, and a basis for $T_q B_{SN}^{(2,1)}$ is given by

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and } v_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let w_1 and w_2 be as above. In order to have a stable intersection between $j^2 f$ and $B_{SN}^{(2,1)}$ at $f(x_0, \alpha_0) = x_0$ and $f_x(x_0, \alpha_0) = 1$ we must have $(j^2 f) \pitchfork_p B_{SN}^{(2,1)}$. As this intersection is non-empty we must have

$$D(j^2 f)_p(\mathbb{R} \times \mathbb{R}) + T_q B_{SN}^{(2,1)} = T_q(J^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})) \simeq \mathbb{R}^8.$$

Let $M_{B_{SN}^{(2,1)}}$ be the matrix defined by $M_{B_{SN}^{(2,1)}} = [w_1, w_2, v_1, v_2, \dots, v_6]$. Hence the transversality condition is $\operatorname{rank}(M_{B_{SN}^{(2,1)}}) = 8$, which is equivalent to $\det(M_{B_{SN}^{(2,1)}}) \neq 0$. The matrix $M_{B_{SN}^{(2,1)}}$ is given by

$$M_{B_{SN}^{(2,1)}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & f_\alpha(p) & 1 & 0 & 0 & 0 & 0 & 0 \\ f_{xx}(p) & f_{x\alpha}(p) & 0 & 0 & 0 & 0 & 0 & 0 \\ f_{x\alpha}(p) & f_{\alpha\alpha}(p) & 0 & 0 & 1 & 0 & 0 & 0 \\ f_{xxx}(p) & f_{xx\alpha}(p) & 0 & 0 & 0 & 1 & 0 & 0 \\ f_{xx\alpha}(p) & f_{x\alpha\alpha}(p) & 0 & 0 & 0 & 0 & 1 & 0 \\ f_{x\alpha\alpha}(p) & f_{\alpha\alpha\alpha}(p) & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We find that $\det(M_{B_{SN}^{(2,1)}}) = -f_\alpha(p) f_{xx}(p)$, and the transversality condition in terms of conditions on derivatives of f is $f_\alpha(p) f_{xx}(p) \neq 0$.

The same calculation may be done in the case when $f(x_0, \alpha_0) = x_0$ and $f_x(x_0, \alpha_0) = -1$. Here we define a surface $B_{PD}^{(2,1)}$ by $x = f$ and $f_x = -1$ in $J^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. It is easily seen that $B_{PD}^{(2,1)}$ has codimension 2 in $J^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and a basis for $T_p B_{PD}^{(2,1)}$ is given by $\{v_1, \dots, v_6\}$, where v_i is as above. Let w_2 be as above and define w_1 with $f_x(p) = -1$. The transversality condition

$$D(j^2 f)_p(\mathbb{R} \times \mathbb{R}) + T_q B_{PD}^{(2,1)} = T_q(J^2(\mathbb{R} \times \mathbb{R}, \mathbb{R}))$$

becomes that the determinant of the matrix

$$M_{B_{PD}} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & f_\alpha(p) & 1 & 0 & 0 & 0 & 0 & 0 \\ f_{xx}(p) & f_{x\alpha}(p) & 0 & 0 & 0 & 0 & 0 & 0 \\ f_{x\alpha}(p) & f_{\alpha\alpha}(p) & 0 & 0 & 1 & 0 & 0 & 0 \\ f_{xxx}(p) & f_{xx\alpha}(p) & 0 & 0 & 0 & 1 & 0 & 0 \\ f_{xx\alpha}(p) & f_{x\alpha\alpha}(p) & 0 & 0 & 0 & 0 & 1 & 0 \\ f_{x\alpha\alpha}(p) & f_{\alpha\alpha\alpha}(p) & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is non-zero. This determinant is given by $\det(M_{B_{PD}}) = -(2f_{x\alpha}(p) + f_\alpha(p)f_{xx}(p))$ so the transversality condition in terms of f is $2f_{x\alpha}(p) + f_\alpha(p)f_{xx}(p) \neq 0$.

Example 2. Consider C^3-maps $h : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. We view the two first coordinates as the state variables, and the third and fourth coordinates as parameters. Assume for simplicity that $h(x, y, \alpha, \beta) = (f(x, \alpha), g(x, \alpha))$. Let $p = (x_0, y_0, \alpha_0, 0)$. Again we consider the 2-jet extension

$$j^2h : C^3(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2) \longrightarrow J^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2).$$

We equip $J^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ with coordinates

$$(x, y, \alpha, \beta, f, g, f_x, f_y, f_\alpha, f_\beta, g_x, g_y, g_\alpha, g_\beta, f_{xx}, \dots, f_{\beta\beta}, g_{xx}, \dots, g_{\beta\beta})$$

Note that $T_q(J^2(\mathbb{R}^4, \mathbb{R}^2)) \simeq \mathbb{R}^{34}$. Let $q = (j^2h)(p)$. The tangent map of the 2-jet extension is given by

$$D(j^2h) = \begin{bmatrix} I_{4\times 4} \\ Dh \\ D^2h \\ D^3h \end{bmatrix}$$

The space $D(j^2h)_p(\mathbb{R}^4)$ is spanned by the column space of $D(j^2h)_p$. As above we define a set $B_{SN}^{(4,2)}$ by the equations $x = f$, $y = g$, and $1 - (f_x + g_y) - f_y g_x = 0$. The codimension of $B_{SN}^{(4,2)}$ in $J^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ is 3. Suppose we have $f(x_0, \alpha_0) = x_0$ and $f_x(x_0, \alpha_0) = 1$, and that the transversality condition for f in $J^2(\mathbb{R}^2, \mathbb{R})$ above is satisfied, $f_\alpha(x_0, \alpha_0) f_{xx}(x_0, \alpha_0) \neq 0$. Furthermore $f_y(p) = g_y(p) = 0$. Let

$$\begin{aligned} G_1(x, y, \dots, g_{\beta\beta}) &= x - f \\ G_2(x, y, \dots, g_{\beta\beta}) &= y - g \\ G_3(x, y, \dots, g_{\beta\beta}) &= f_x + g_y + f_y g_x - 1 \end{aligned}$$

The tangent space $T_q B_{SN}^{(4,2)}$ is given by

$$T_q B_{SN}^{(4,2)} = \{v :< \nabla G_i(q), v >= 0\}$$

Here

$$\begin{aligned} \nabla G_1(x, y, \dots, g_{\beta\beta}) &= (1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, \dots, 0) \\ \nabla G_2(x, y, \dots, g_{\beta\beta}) &= (0, 1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, \dots, 0) \\ \nabla G_3(x, y, \dots, g_{\beta\beta}) &= (0, 0, 0, 0, 0, 0, 1, g_x, 0, 0, f_y, 1, 0, \dots, 0) \end{aligned}$$

Evaluated in q we have

$$\nabla G_1(q) = (1,0,0,0,-1,0,0,0,0,0,0,0,\dots,0)$$
$$\nabla G_2(q) = (0,1,0,0,0,-1,0,0,0,0,0,0,\dots,0)$$
$$\nabla G_3(q) = (0,0,0,0,0,0,1,g_x(p),0,0,0,1,0,\dots,0)$$

Let e_i denote the standard unit basis vectors in $\mathbb{R}^m$. From the above we see that

$$< \nabla G_j(q), e_i >= 0 \text{ for } j = 1,2,3 \text{ and } i = 3,4,9,10,11,13,14,15,\dots 33,34.$$

From $< \nabla G_1(q), v_1 >= 0$ we find $v_1 = (1,0,0,0,1,0,\dots,0)$, and from $< \nabla G_2(q), v_2 >= 0$ we find $v_2 = (0,1,0,0,0,1,0,\dots,0)$. In addition from $< \nabla G_3(q), v_i >= 0$ we get

$$v_3 = (0,0,0,0,0,0,-1,0,0,0,0,1,0,\dots,0) \text{ and } v_4 = (0,0,0,0,0,0,-g_x(p),1,0,0,\dots,0).$$

This is totality a set of 31 linearly independent vectors, and the set

$$\{v_1, v_2, v_3, v_4, e_3, e_4, e_9, e_{10}, e_{11}, e_{13}, e_{14}, \dots, e_{33}, e_{34}\}$$

is a basis for $T_q B_{SN}^{(4,2)} \simeq \mathbb{R}^{31}$.

The transversality condition

$$D(j^2 h)_p(\mathbb{R}^2 \times \mathbb{R}^2) + T_q B_{SN}^{(4,2)} = T_q(J^2(\mathbb{R}^4, \mathbb{R}^2))$$

is equivalent to $\det(M_{B_{SN}^{(4,2)}}) \neq 0$, where $M_{B_{SN}^{(4,2)}}$ is given by

$$M_{B_{SN}^{(4,2)}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & f_\alpha(p) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_x(p) & 0 & g_\alpha(p) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_{xx}(p) & 0 & f_{x\alpha}(p) & 0 & 0 & 0 & -1 & -g_x(p) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ f_{x\alpha}(p) & 0 & f_{\alpha\alpha}(p) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ g_{xx}(p) & 0 & g_{x\alpha}(p) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The determinant of this matrix is $\det(M_{B_{SN}^{(4,2)}}) = -f_\alpha(p) f_{xx}(p)$. We observe that this is the same transversality condition on f we had for the corresponding problem in $J^2(\mathbb{R}^2, \mathbb{R})$.

We define $B_{PD}^{(4,2)}$ by the equations $x = f$, $y = g$, and $-1 + (f_x + g_y) - f_y g_x = 0$. The codimension of $B_{PD}^{(4,2)}$ in $J^2(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ is 3. Suppose we have $f(x_0, \alpha_0) = x_0$ and $f_x(x_0, \alpha_0) = -1$. The same calculations as above can be done here and we obtain that the transversality condition

$$D(j^2 h)_p(\mathbb{R}^2 \times \mathbb{R}^2) + T_q B_{PD}^{(4,2)} = T_q(J^2(\mathbb{R}^4, \mathbb{R}^2))$$

is equivalent to $\det(M_{B^{(4,2)}_{PD}}) \neq 0$, where $M_{B^{(4,2)}_{PD}}$ is given by

$$M_{B^{(4,2)}_{PD}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & f_\alpha(p) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g_x(p) & 0 & g_\alpha(p) & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ f_{xx}(p) & 0 & f_{x\alpha}(p) & 0 & 0 & 0 & -1 & g_x(p) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ f_{x\alpha}(p) & 0 & f_{\alpha\alpha}(p) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ g_{xx}(p) & 0 & g_{x\alpha}(p) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The determinant of this matrix is $\det(M_{B^{(4,2)}_{PD}}) = -(2f_{x\alpha}(p) + f_\alpha(p))f_{xx}(p)$. We observe again that this is the same transversality condition on f we had for the corresponding problem in $J^2(\mathbb{R}^2, \mathbb{R})$.

The computations above can be pictured in the following diagram

$$\begin{array}{ccccc} \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} & \xrightarrow{j^2h} & J^2(\mathbb{R}^4,\mathbb{R}^2) & & \supset B^{(4,2)} \\ \downarrow{\scriptstyle \pi_{13}} & & \downarrow{\scriptstyle \pi} & & \\ \mathbb{R}\times\mathbb{R} & \xrightarrow[j^2f]{} & J^2(\mathbb{R}^2,\mathbb{R}) & & \supset B^{(2,1)} \end{array}$$

Here $B^{(4,2)} = B^{(4,2)}_{SN}$ or $B^{(4,2)}_{PD}$, and $B^{(2,1)} = B^{(2,1)}_{SN}$ or $B^{(2,1)}_{PD}$, $\pi_{1,3}$ denotes the projection from first and third component and π denotes the natural projection. It is easily seen that $\pi(B^{(4,2)}) \supset B^{(2,1)}$.

We summarize the preceding computations in the following lemma:

Lemma 6.1. *Let $f, g : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, and let $h : \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by the formula $h(x, y, \alpha, \beta) = (f(x, \alpha), g(x, \alpha))$. Let $B^{(4,2)}$, $B^{(2,1)}$, p_i and q_i be as above. If $j^2f(p_1) \pitchfork_{q_1} B^{(2,1)}$ then $j^2h(p_2) \pitchfork_{q_2} B^{(4,2)}$.*

We want to find the bifurcation set in the parameter space of h. Consider the following diagram of inclusions and maps:

$$\begin{array}{ccccc} & & & & C(h) = (j^2h(\mathbb{R}^4) \cap B^{(4,2)} \\ & & & & \parallel \\ D(h) & & (j^2h)^{-1}(C(h)) & & C(h) \\ \cap & & \cap & & \cap \\ \mathbb{R}^2 & \xleftarrow{\pi_{34}} & \mathbb{R}^4 & \xrightarrow{j^2h} & J^2(\mathbb{R}^4,\mathbb{R}^2) \end{array}$$

The bifurcation surface is given in the jet-space by $C(h) = j^2h(\mathbb{R}^4) \cap B^{(4,2)}$. The relevant bifurcation set in the parameter space is found by taking the inverse image of $C(h)$ by j^2h, and then projecting this set to the parameter space:

$$D(h) = \pi_{34}((j^2h)^{-1}(C(h)))$$

Here we see that

$$\dim(C(h)) = \dim((j^2h)(\mathbb{R}^4)) - \operatorname{codim}(B^{(4,2)}) = 4 - 3 = 1$$

since $j^2h(p_2) \pitchfork_{q_2} B^{(2,1)}$. The map j^2h is injective so $\dim((j^2h)^{-1}(C(h))) = 1$, and hence $\dim(D(h)) = 1$.

Now since $j^2h(p_2) \pitchfork_{q_2} B^{(2,1)}$ we have by Thoms transversality theorem that $j^2h_\epsilon(p) \pitchfork_q B^{(2,1)}$ for all small perturbations h_ϵ of h. Hence, by the remarks above we have proved that the β lift of f, has a non-degenerate saddle-node or period doubling bifurcation for small $|\beta|$ if f has one. The dimension considerations above is still valid, so the bifurcation set in the parameter space is a curve through the point $(\alpha_0, 0)$.

We will now apply the construction above to bifurcations of the $\boldsymbol{\beta}$-lift on $\mathbb{R}^n \times \mathbb{R}^n$. Consider the diagram

$$\begin{array}{ccc}
\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n & \xrightarrow{\ j^2F\ } & J^2(\mathbb{R}^{3n+1}, \mathbb{R}^{2n}) \supset B^{(3n+1,2n)} \\
\Big\downarrow \pi_{sp} & & \Big\downarrow \pi \\
\mathbb{R}^n \times \mathbb{R} & \xrightarrow[\ j^2f\ ]{} & J^2(\mathbb{R}^{n+1}, \mathbb{R}^n) \ \supset B^{(n+1,n)}
\end{array}$$

Here $B^{(3n+1,2n)} = B_{SN}^{(3n+1,2n)}$, $B_{PD}^{(3n+1,2n)}$ or $B_H^{(3n+1,2n)}$, and $B^{(n+1,n)} = B_{SN}^{(n+1,n)}$, $B_{PD}^{(n+1,1)}$ or $B_H^{(n+1,1)}$. π_{sp} denotes the projection on the first n state variables and the parameter space and π denotes the natural projection. The sets B_H is defined below.

Let $B_H^{(n+1,n)}$ be the set in $J^2(\mathbb{R}^{n+1}, \mathbb{R}^n)$ such that $x_i = f^i$ and $\det(\exp(i\theta)I - DM) = 0$, $\theta \in (0, \pi)$, where DM is the matrix

$$DM = \begin{bmatrix} f^1_{x_1} & \cdots & f^1_{x_n} \\ \vdots & & \vdots \\ f^1_{x_1} & \cdots & f^n_{x_n} \end{bmatrix}.$$

Let $B_H^{(3n+1,2n)}$ be the set in $J^2(\mathbb{R}^{3n+1}, \mathbb{R}^{2n})$ such that $x_i = f^i$, $y_i = g_i$ and $\det(\exp(i\theta)I - DN) = 0$, $\theta \in (0, \pi)$, where DN is the matrix

$$DM = \begin{bmatrix} f^1_{x_1} & \cdots & f^1_{x_n} & f^1_{y_1} & \cdots & f^1_{y_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ f^n_{x_1} & \cdots & f^n_{x_n} & f^n_{y_1} & \cdots & f^n_{y_n} \\ g^1_{x_1} & \cdots & g^1_{x_n} & g^1_{y_1} & \cdots & g^1_{y_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ g^n_{x_1} & \cdots & g^n_{x_n} & g^n_{y_1} & \cdots & g^n_{y_n} \end{bmatrix}.$$

It is easily seen that $\operatorname{codim}(B_H^{(n+1,n)}) = n + 1$ and $\operatorname{codim}(B_H^{(3n+1,2n)}) = 2n + 1$ since the determinant involves a one-parameter family of a pair of complex conjugate eigenvalues.

The dimension of the space $J^2(\mathbb{R}^n, \mathbb{R}^m)$ is given by $n + m + nm + nm(n + 1)/2$. Hence the dimension of $J^2(\mathbb{R}^{n+1}, \mathbb{R}^n)$ and $J^2(\mathbb{R}^{3n+1}, \mathbb{R}^{2n})$ is given by

$$\dim(J^2(\mathbb{R}^{n+1}, \mathbb{R}^n)) = 1 + 4n + \frac{5}{2}n^2 + \frac{n^3}{2}$$
$$\dim(J^2(\mathbb{R}^{3n+1}, \mathbb{R}^{2n})) = 1 + 9n + 15n^2 + 9n^3$$

We will first compute a basis for the tangent space $T_qB^{(3n+1,2n)}$. Since $\operatorname{codim}(B^{(3n+1,2n)}) = 2n + 1$ we see that $T_qB^{(3n+1,2n)} \simeq \mathbb{R}^{7n+15n^2+9n^3}$. We will choose the basis such that as many basis vectors as possible are equal to standard unit vectors in $\mathbb{R}^m$. In this construction it turns out that we can choose $1 + 5n + 11n^2 + 9n^3$ vectors of this form. Hence there are $4n^2 + 2n - 1$ in a non-standard form.

From the fixed point equation we obtain $2n$ basis vectors written in a $2n \times (1+9n+15n^2+9n^3)$-matrix as column vectors:

$$\begin{bmatrix} I_n & 0_n \\ 0_n & I_n \\ 0 & 0 \\ 0_n & 0_n \\ I_n & 0_n \\ 0_n & I_n \\ 0_n & 0_n \\ \vdots & \vdots \\ 0_n & 0_n \end{bmatrix}$$

From the eigenvalue equation involving the determinant of the Jacobian of F with respect to the state variables we obtain $4n^2 - 1$ vectors of the form

$$(O_n, 0_n, 0, 0_n, 0_n, 0_n, A_{f,1,x}, A_{f,1,y}, 0, 0_n, \dots, A_{f,n,x}, A_{f,n,y}, 0, 0_n,$$
$$A_{g,1,x}, A_{g,1,y}, 0, 0_n, \dots, A_{g,n,x}, A_{g,n,y}, 0, 0_n, 0_n, 0_n, \dots, 0_n)$$

Here the symbols $A_{f,n,x}$ means a block of size n. Furthermore there are $n+1$ standard unit vectors with 1 on the parameter coordinates, and $2n^2+2n$ standard unit vectors with 1 on the coordinates for derivatives with respect to the parameters. Finally there are $n(3n+1)(3n+2)$ standard unit vectors with 1 on all coordinates representing derivatives of order two. Clearly the set of vectors above is linear independent, and contained in the tangent space. Since $2n + (4n^2 - 1) + (n+1) + (2n^2 + 2n) + n(3n+1)(3n+2) = 7n + 15n^2 + 9n^3$ the set is a basis for $T_q B^{(3n+1,2n)}$.

We will also need a basis for $T_q B^{(n+1,n)}$. From the fixed point equations we get n basis vectors written in a $n \times (1 + 4n + \frac{5}{2}n^2 + \frac{n^3}{2})$-matrix as column vectors:

$$\begin{bmatrix} I_n \\ 0 \\ I_n \\ 0_n \\ 0 \\ \vdots \\ 0_n \\ 0 \\ \vdots \\ 0_n \end{bmatrix}$$

From the eigenvalue equation involving the determinant of the Jacobian of f with respect to the state variables we obtain $n^2 - 1$ vectors of the form

$$(0_n, 0, 0_n, A^1, 0, A^2, 0, \dots, A^n, 0, 0_n, \dots, 0_n)$$

Furthermore there is one standard unit vector with 1 on the parameter coordinate, and n standard unit vectors with 1 on the coordinates for derivatives with respect to the parameter. Finally there are $n(n+1)(n+2)/2$ standard unit vectors with 1 on all coordinates representing derivatives of order two. Clearly the set of vectors above is linear independent, and contained in the tangent space. By counting the number of vectors we see that the set is a basis for $T_q B^{(n+1,n)}$.

We also need a basis for the range of the tangent maps $D(j^2F)$ and $D(j^2f)$. We find the range from the Jacobians, and since j^2 is injective, the set of column vectors is a basis.

We write down the basis vectors from $T_q B^{(n+1,n)}$ and $D(j^2f)$ in a matrix written in block form. After deleting equal columns we obtain the following $(1 + 4n + \frac{5}{2}n^2 + \frac{n^3}{2}) \times (1 + 4n + \frac{5}{2}n^2 + \frac{n^3}{2})$-matrix:

$$
\begin{bmatrix}
I & 0 & 0 & 0 & \cdots & 0 & I & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\
I & 0 & 0 & 0 & \cdots & 0 & D_x f & D_\alpha f \\
0 & 0 & K^1 & 0 & \cdots & 0 & D^2 f^1_{xx} & D^2 f^1_{x\alpha} \\
0 & 0 & 0 & 1 & \cdots & 0 & D^2 f^1_{x\alpha} & D^2 f^1_{\alpha\alpha} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & K^n & 0 & \cdots & 0 & D^2 f^n_{xx} & D^2 f^1_{x\alpha} \\
0 & 0 & 0 & 0 & \cdots & 1 & D^2 f^n_{x\alpha} & D^2 f^1_{\alpha\alpha}
\end{bmatrix}
$$

We write down the basis vectors from $T_q B^{(3n+1,2n)}$ and $D(j^2F)$ in a matrix written in block form. After deleting equal columns we obtain the following $(1+9n+15n^2+9n^3)\times(1+9n+15n^2+9n^3)$-matrix:

$$
\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & I & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & I & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & I & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D_x f & 0 & D_\alpha f \\
0 & I & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D_x g & 0 & D_\alpha g \\
0 & 0 & 0 & 0 & A^1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^2 f^1_{xx} & 0 & D^2 f^1_{x\alpha} \\
0 & 0 & 0 & 0 & B^1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^2 f^1_{x\alpha} & 0 & D^2 f^1_{\alpha\alpha} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & A^n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^2 f^n_{xx} & 0 & D^2 f^n_{x\alpha} \\
0 & 0 & 0 & 0 & B^n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^2 f^n_{x\alpha} & 0 & D^2 f^n_{\alpha\alpha} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C^1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^2 g^1_{xx} & 0 & D^2 g^1_{x\alpha} \\
0 & 0 & 0 & 0 & D^1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^2 g^1_{x\alpha} & 0 & D^2 g^1_{\alpha\alpha} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & C^n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & D^2 g^n_{xx} & 0 & D^2 g^n_{x\alpha} \\
0 & 0 & 0 & 0 & D^n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & D^2 g^n_{x\alpha} & 0 & D^2 g^n_{\alpha\alpha} \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I & \multicolumn{3}{c}{\text{Third order block}}
\end{bmatrix}
$$

There is a hidden identity block in the dots in the "zero-row" in row number seven from the bottom. We are interested only in the determinants of these matrices. Hence we can delete columns consisting of a single I-block, and the corresponding rows, and vise versa. The reduced matrices take the form:

$$M_1 = \begin{bmatrix} I & 0 & I & 0 \\ I & 0 & D_x f & D_\alpha f \\ 0 & A^1 & D^2 f^1_{xx} & D^2 f^1_{x\alpha} \\ 0 & B^1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & A^n & D^2 f^n_{xx} & D^2 f^n_{x\alpha} \\ 0 & B^n & 0 & 0 \\ 0 & C^1 & D^2 g^1_{xx} & D^2 g^1_{x\alpha} \\ 0 & D^1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & C^n & D^2 g^n_{xx} & D^2 g^n_{x\alpha} \\ 0 & D^n & 0 & 0 \end{bmatrix} \qquad M_2 = \begin{bmatrix} I & 0 & I & 0 \\ I & 0 & D_x f & D_\alpha f \\ 0 & K^1 & D^2 f^n_{xx} & D^2 f^1_{x\alpha} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & K^n & D^2 f^n_{xx} & D^2 f^n_{x\alpha} \end{bmatrix}$$

We now need the structure of the second block column in the first matrix above. This structure is found from the fact that the tangent space of $B^{(3n+1,2n)}$ is determined from a gradient of a determinant. Hence we must look at the cofactor matrix of

$$\begin{bmatrix} Df - \lambda I & 0 \\ Dg & \lambda I \end{bmatrix}$$

A small calculation using the fact that $\det(Df(p) - \lambda I) = 0$ shows that the cofactor matrix is of the form

$$\begin{bmatrix} \lambda^n \mathrm{cof}(Df) & X \\ 0 & 0 \end{bmatrix}$$

where X is some "ugly" $n \times n$-matrix. Hence the matrix M_1 reduces to

$$\begin{bmatrix} I & 0 & I & 0 & 0 & 0 \\ I & 0 & D_x f & D_\alpha f & 0 & 0 \\ 0 & K_1 & D^2 f^1_{xx} & D^2 f^1_{x\alpha} & \tilde{K}_1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & K_n & D^2 f^n_{xx} & D^2 f^n_{x\alpha} & \tilde{K}_n & 0 \\ 0 & 0 & 0 & 0 & \tilde{B}_1 & B_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \tilde{B}_n & B_n \end{bmatrix}$$

where the first column is of size n, the second of size $n^2 - 1$, the third of size n, the fourth and fifth of size 1 and the last column is of size $n^2 - 1$. All rows are of size n. There is a block of zeroes in the lower left corner, and hence the determinant is given by the product of the determinant of M_2, in the upper left corner, and the determinant of the $n^2 \times n^2$-matrix

$$\begin{bmatrix} \tilde{B}_1 & B_1 \\ \vdots & \vdots \\ \tilde{B}_n & B_n \end{bmatrix}$$

in the lower right corner. This last matrix is easily seen to be non-singular, so we see that M_1 is singular if and only if M_2 is singular.

By the above we have the following lemma:

Lemma 6.2. *Let $f, g : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be at least C^3, and let $h : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ be defined by the formula $h(x, y, \alpha, \beta) = (f(x, \alpha), g(x, \alpha))$. Let $B^{(3n+1,2n)}$, $B^{(n+1,n)}$, p_i and q_i be as above. Then $j^2 f(p_1) \pitchfork_{q_1} B^{(n+1,n)}$ if and only if $j^2 h(p_2) \pitchfork_{q_2} B^{(3n+1,2n)}$.*

The same considerations as above can be done about the bifurcation set in the parameter space of h: Consider the following diagram of inclusions and maps:

$$\begin{array}{ccccc}
 & & & & C(h) = (j^2h)(\mathbb{R}^{3n+1}) \cap B^{(3n+1,2n)} \\
 & & & & \parallel \\
D(h) & & (j^2h)^{-1}(C(h)) & & C(h) \\
\cap & & \cap & & \cap \\
\mathbb{R}^{n+1} & \xleftarrow{\ \pi\ } & \mathbb{R}^{3n+1} & \xrightarrow{\ j^2h\ } & J^2(\mathbb{R}^{3n+1}, \mathbb{R}^{2n})
\end{array}$$

The bifurcation surface is given in the jet-space by $C(h) = j^2h(\mathbb{R}^{3n+1}) \cap B^{(3n+1,2n)}$. The relevant bifurcation set in the parameter space is found by taking the inverse image of $C(h)$ by j^2h, and then projecting this set to the parameter space:

$$D(h) = \pi((j^2h)^{-1}(C(h)))$$

Here we see that

$$\dim(C(h)) = \dim((j^2h)(\mathbb{R}^{3n+1})) - \operatorname{codim}(B^{(3n+1,2n)}) = 3n + 1 - (2n + 1) = n$$

since $j^2h(p_2) \pitchfork_{q_2} B^{(n+1,n)}$. The map j^2h is injective so $\dim((j^2h)^{-1}(C(h))) = n$, and hence $\dim(D(h)) = n$.

By Thom's transversality theorem we have the following theorem.

Theorem 6.1. *If f undergoes a saddle-node, period-doubling or Hopf bifurcation, then the β-lift of f undergoes a saddle-node, period-doubling or Hopf bifurcation if $\|\beta\|$ small.*

Proof. The saddle-node, period-doubling and Hopf bifurcation conditions on fixed points of f are given in terms of conditions on the first order derivatives of f together with transversality conditions which appear as conditions on the second order derivatives. For the Hopf bifurcation there are some additional resonance conditions, but these are closed subsets of the surface $B_H^{(n+1,n)}$. Hence the theorem follows by lemma 6.2 and Thom's transversality theorem. □

7. Homoclinic and heteroclinic orbits

Let $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, and let $x_0 \in \mathbf{Fix}(f)$. Let $\omega(f, y)$ denote the ω-limit set of the orbit through y. Assume that $\operatorname{rank}(Df(x_0)) = n$, then by the previous sections there exist local stable and unstable manifolds, $W^s_{loc}(f, x_0)$ and $W^u_{loc}(f, x_0)$, associated with x_0. Assume that $\dim(W^u_{loc}(f, x_0)) \geq 1$, and that there is a point $x_h \in W^u_{loc}(f, x_0)$ such that $\omega(f, x_h) = x_0$. We say that f has a homoclinic orbit associated with x_0.

There are three cases to consider:

(1) $\dim(W^u_{loc}(f, x_0)) = n$ with $x_h \in W^u_{loc}(f, x_0)$ and $f^{n_0}(x_h) = x_0$ for some $n_0 \in \mathbb{N}$.
(2) $\dim(W^u_{loc}(f, x_0)) < n$ with

$$\bigcup_{n=0}^{n_0} f^n(W^u_{loc}(f, x_0)) \cap W^s_{loc}(f, x_0) \neq \varnothing$$

for some $n_0 \in \mathbb{N}$.

(3) $\dim(W^u_{loc}(f, x_0)) < n$ with $x_h \in W^u_{loc}(f, x_0)$ and $f^{n_0}(x_h) = x_0$ for some $n_0 \in \mathbb{N}$.

In case (1) we call the homoclinic orbit non-degenerate if $\text{rank}(Df^{n_0}(x_h)) = n$. It is easily seen that a degenerate homoclinic orbit of this type may disappear under arbitrarily small perturbations of f.

In case (2) we call the homoclinic orbit non-degenerate if the intersection at some

$$x \in \bigcup_{n=0}^{n_0} f^n(W^u_{loc}(f, x_0)) \cap W^s_{loc}(f, x_0)$$

is transversal. It is easily seen that non-transversal intersections may disappear under arbitrarily small perturbations of f.

In case (3) we call the homoclinic orbit non-degenerate if the intersection of $W^s_{loc}(f, x_0)$ and $f^{n_0}(W_{x_h})$, where $W_{x_h} \subset W^u_{loc}(f, x_0)$ is a neighborhood of x_h in $W^u_{loc}(f, x_0)$, is transversal.

Lemma 7.1. *If f has a non-degenerate homoclinic orbit associated with $x_0 \in \mathbf{Fix}(f)$ then the corresponding image of unstable sets and the local stable manifold of $(x_0, x_0) \in \mathbf{Fix}(F_0)$ have a non-empty transversal intersections.*

Proof. By assumption f is a local diffeomorphism at x_0. Hence the leaf $x_0 \times \mathbb{R}^n$ intersect the graph of f transversally in $\mathbb{R}^n \times \mathbb{R}^n$. □

Theorem 7.1. *If f has a non-degenerate homoclinic orbit associated with $x_0 \in \mathbf{Fix}(f)$, then there exists a $\epsilon > 0$ such that the stable and unstable manifold of $(x(\boldsymbol{\beta}), y(\boldsymbol{\beta}))$ has a non-empty transversal intersection for all $F_{\boldsymbol{\beta}}$ with $0 < |\beta_i| < \epsilon$, $i = 1, \ldots, n$.*

Proof. All transversality conditions above depend only on a finite number of f-iterates. Hence by $|\beta| - C^1$-closeness of $F_{\mathbf{0}}$ and $F_{\boldsymbol{\beta}}$ on bounded sets, theorem 5.1 and lemma 7.1 the result follows by the weak transversality theorem [Ar] applied to the inclusion maps. □

Similar results hold for heteroclinc orbits as well.

8. One-sided k-shifts in the n-dimensional map

In this section we discuss some sufficient conditions on a map $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \leq 1$, such that f has a non-wandering set topologically equivalent to a one-sided shift on k symbols.

In the following let $\|\cdot\|$ denote the max-norm on $\mathbb{R}^n$, and let $B(\Delta) = \{x \in \mathbb{R}^n : \|x\| \leq \Delta, \text{ where } \Delta > 0\}$ denote the cube of size Δ with center in $\mathbf{0}$. We will omit the explicit reference to Δ, and simply write B.

Let $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, with the following properties:

(1) f is norm-expanding outside B, that is, $\|f(x)\| > \|x\|$ for all $x \in \mathbb{R}^n \setminus B$.

(2) f overflows B, that is, int $(f(B)) \supset B$. The set $B \cap f^{-1}(B)$ consists of k disjoint connected components, $K_1, \ldots, K_k$, such that $f(K_j) = B$, and such that the restriction

$$f : K_j \subset W_{K_j} \longrightarrow V_{K_j} \supset B$$

is a diffeomorphism for some neighborhoods W_{K_j} of K_j and V_{K_j} of B.

(3) For each K_j there is a number n_j such that

$$\min_{x \in K_j} \{|\lambda| : \lambda \in \text{spec}(Df^{n_j}(x))\} > 1.$$

We will now look at some consequences of the properties above, starting with property (1).

Lemma 8.1. *f increases the norm along any forward orbit outside B, that is, if $x_0 \in \mathbb{R}^n \setminus B$ then*

$$\|f^k(x_0)\| > \|f^{k-1}(x_0)\| > \cdots > \|f(x_0)\| > \|x_0\|.$$

Proof. Let $y_0 = f(x_0)$ with $x_0 \in \mathbb{R}^n \setminus B$. Clearly $y_0 \in \mathbb{R}^n \setminus B$ so

$$\|f^2(x_0)\| = \|f(y_0)\| > \|y_0\| = \|f(x_0)\| > \|x_0\|,$$

and the result follows by induction. □

Lemma 8.2. $\Omega(f) \subset B$.

Proof. Let $x_0 \in \mathbb{R}^n \setminus B$, and let $V_{x_0} \subset \mathbb{R}^n \setminus B$ be a neighborhood of x_0. Define a continuous map $k : V_{x_0} \times V_{x_0} \longrightarrow \mathbb{R}$ by $k(x, y) = \|f(x)\| - \|y\|$. We see that $k(x_0, x_0) = \|f(x_0)\| - \|x_0\| > 0$. Hence there is a neighborhood $W_{(x_0,x_0)} \subset V_{x_0} \times V_{x_0}$ such that $k(x, y) > 0$ for all $(x, y) \in W_{(x_0,x_0)}$. The neighborhood $W_{(x_0,x_0)}$ contains neighborhoods of the form $U_{x_0} \times U_{x_0}$, where U_{x_0} is a neighborhood of x_0. Hence

$$\inf_{y\in f(U_{x_0})} \{\|y\|\} > \sup_{x\in U_{x_0}} \{\|x\|\}$$

so $f(U_{x_0}) \cap U_{x_0} = \varnothing$. By lemma 8.1 $\|f^k(x)\| > \|x\|$ for all $k \geq 1$ with $x \in \mathbb{R}^n \setminus B$ so

$$\inf_{y\in f^k(U_{x_0})} \{\|y\|\} > \sup_{x\in U_{x_0}} \{\|x\|\}$$

proving that $f^k(U_{x_0}) \cap U_{x_0} = \varnothing$ for all $k \geq 1$. □

From property (2) we get the following lemma:

Lemma 8.3. *The set*

$$f^{-m}(B) \cap f^{-(m-1)}(B) \cap \cdots \cap f^{-1}(B) \cap B$$

is a disjoint union of k^m closed connected sets

$$\bigcap_{i=0}^{m} f^{-i}(B) = \bigcup_{\substack{1\leq i_j\leq k \\ 1\leq j\leq m}} K_{i_1 i_2\cdots i_m}$$

with the property that $K_{i_1i_2\cdots i_m} \subset K_{i_1i_2\cdots i_{m-1}}$, $f(K_{i_1i_2\cdots i_m}) = K_{i_1i_2\cdots i_{m-1}}$ and $f^m(K_{i_1i_2\cdots i_m}) = B$. The restriction

$$f^m : K_{i_1i_2\cdots i_m} \subset W_{K_{i_1i_2\cdots i_m}} \longrightarrow V_{K_{i_1i_2\cdots i_m}} \supset B$$

is a diffeomorphism for some neighborhoods $W_{K_{i_1i_2\cdots i_m}}$ of $K_{i_1i_2\cdots i_m}$ and $V_{K_{i_1i_2\cdots i_m}}$ of B.

Proof. We will prove this lemma by induction. By property (2) the lemma is true for $m = 1$, with the obvious modifications in notation.

Assume the lemma is true for $m = l - 1$. Then

$$f^{l-1}(K_{i_1i_2\cdots i_{l-1}}) = B \supset K_j \text{ for } j = 1, \ldots, k$$

homeomorphically. Hence there exist k closed connected and disjoint sets

$$K_{i_1i_2\cdots i_l} \subset K_{i_1i_2\cdots i_{l-1}} \text{ where } i_l = 1, \ldots, k$$

such that

$$f^{l-1}(K_{i_1i_2\cdots i_l}) = K_{i_l}.$$

Hence $f^l(K_{i_1i_2\cdots i_l}) = f(f^{l-1}(K_{i_1i_2\cdots i_l})) = f(K_{i_l}) = B$. Furthermore we have by construction that $f(K_{i_1i_2\cdots i_l}) = K_{i_1i_2\cdots i_{l-1}}$. This map extends by the inclusion above to a diffeomorphism of some neighborhoods of $K_{i_1i_2\cdots i_l}$ and $K_{i_1i_2\cdots i_{l-1}}$, and we obtain the diffeomorphism in the lemma by composition.

Hence the result follows by induction on m. □

We will prove a simple lemma needed to obtain a Cantor set when intersecting some suitable preimages of B.

Lemma 8.4. *Suppose $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, and $\{A_i\}_{i=0}^{\infty}$ is a nested sequence of non-empty closed sets such that $f(A_n) = A_{n-1}$ and $\lambda = \min_{x \in A_0}\{|\text{spec}Df(x)|\} > 1$. Then there exists a unique point $x_f \in A_0$ such that*

$$\bigcap_{n \geq 0} A_n = \{x_f\}.$$

Proof. Nested intersections of non-empty closed sets are non-empty. Let d be the usual metric on $\mathbb{R}^n$, and let $\delta_n = \text{diam}A_n = \sup_{x,y \in A_n} d(x,y)$. Since the sequence $\{A_i\}_{i=1}^{\infty}$ is nested it is clear that $0 \leq \delta_n \leq \delta_{n-1} \leq \delta_0$. Hence $\{\delta_n\}$ has a limit in $[0, \delta_0]$. By the intermediate value theorem in $\mathbb{R}^n$ we see that $\delta_{n+1} \leq \lambda^{-1}\delta_n$ so by induction $d_n \leq \lambda^{-n}\delta_0$. Now $\lambda > 1$ so the sequence converges to 0. □

Corollary 8.1. *Suppose $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$ and $\{A_i\}_{i=1}^{\infty}$ is a nested sequence of non-empty closed sets such that $f(A_n) = A_{n-1}$ and $\lambda = \min_{x \in A_0}\{|\text{spec}Df^{k_0}(x)|\} > 1$ for some $k_0 \in \mathbb{N}$. Then there exists a unique point $x_f \in A_0$ such that*

$$\bigcap_{n \geq 0} A_n = \{x_f\}.$$

Proof. We apply lemma 8.4 to the sequence $A_0 \supset A_{k_0} \supset A_{2k_0} \supset \cdots$ and f^{k_0}. □

We will now use lemma 8.3 together with property (3) and corollary 8.1 to obtain an f-invariant Cantor set.

We observe from lemma 8.3 that

$$K_{i_1} \supset K_{i_1 i_2} \supset \cdots \supset K_{i_1 i_2 \cdots i_m} \supset \cdots$$

such that $f(K_{i_1 i_2 \cdots i_m}) = K_{i_1 i_2 \cdots i_{m-1}}$. By property (3) and corollary 8.1 the intersection of this nested sequence of inclusions is a unique point. Let $\Lambda(f, B)$ be the union of all such intersections:

$$\Lambda(f,B) = \bigcup_{\substack{\text{All possible} \\ \text{combinations of} \\ i_1 i_2 \cdots i_m \\ i_j \in \{1,\ldots,k\}}} \left(\bigcap_{m \geq 1} K_{i_1 i_2 \cdots i_m} \right)$$

By construction $\Lambda(f, B)$ is weak f-invariant. Let Σ_k^+ denote the one sided shift space of k symbols, and σ the left shift operator on Σ_k^+.

Theorem 8.1. *If $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, satisfy property (1),(2) and (3) above then there exists a f-invariant set $\Lambda(f, B) \subset B$ and a homeomorphism $h : \Lambda(f, B) \longrightarrow \Sigma_k^+$ such that the diagram*

$$\begin{array}{ccc} \Lambda(f,B) & \xrightarrow{f} & \Lambda(f,B) \\ \Big\downarrow h & & \Big\downarrow h \\ \Sigma_k^+ & \xrightarrow{\sigma} & \Sigma_k^+ \end{array}$$

commutes. The set $\Lambda(f, B)$ is the largest f-invariant set contained in B.

Proof. By the standard construction where we for $p \in \Lambda(f, B)$ define the itinerary of p as the sequence $h(p) = i_1 k_2 k_3 \ldots$ where $i_n = j$ if $f(p) \in K_j$. □

Combining theorem 8.1 and lemma 8.2 we have the following theorem:

Theorem 8.2. *If $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, satisfy property (1),(2) and (3) above then the non-wandering set of f, $\mathbf{\Omega}(f)$, is contained in B, and the restriction of f to $\mathbf{\Omega}(f)$ is topologically equivalent to a one-sided shift on k symbols.*

Proof. By lemma 8.2 $\mathbf{\Omega}(f) \subset B$, and from theorem above we have $\mathbf{\Omega}(f) = \Lambda(f, B)$. □

9. Full k-shifts in the diffeomorphism

We will now study what happens to the non-wandering set of f described in the preceding section when f is lifted to $F_{\boldsymbol{\beta}}$. We use the same notation as in section 8. We will replace property (1) by a stronger condition to gain control of the iterates of $F_{\boldsymbol{\beta}}$ and $F_{\boldsymbol{\beta}}^{-1}$ outside some compact set. This condition is only used to show that the non-wandering set is contained in a set S defined by $S = B \times B$, and can be replaced by some other conditions. The new condition is

(1')

$$\|f(x) + \boldsymbol{\beta} y\| > \|x\| \text{ if } \|x\| \geq \|y\|$$
$$\|\boldsymbol{\beta}^{-1}(x - f(y))\| > \|y\| \text{ if } \|y\| \geq \|x\|,\ \boldsymbol{\beta} \neq \mathbf{0}$$

for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus S$.

We note that the first part of property (1') implies property (1) in section 8.

In the following we assume that property (1'), (2) and (3) hold for $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$ with $r \geq 1$. Then

$$f^{-1}(B) \cap B = \bigcup_{i=1}^{k} K_i, \text{ where } K_i \cap K_j = \varnothing \text{ if } i \neq j.$$

Furthermore $f(K_i) = B$ for $i = 1, \ldots, k$. If $x \in B$ then $f(x) \in B$ if and only if $x \in K_1 \cup \cdots \cup K_k$. We define a set $B_{\boldsymbol{\beta}}$ by

$$B_{\boldsymbol{\beta}} = \{x \in \mathbb{R}^n : x = v + \boldsymbol{\beta} w \text{ where } v, w \in B\}.$$

$B_{\boldsymbol{\beta}}$ is a closed set and $B \subset B_{\boldsymbol{\beta}}$.

Lemma 9.1. *There exists an $\epsilon > 0$ and k disjoint connected sets $\hat{K}_i(\boldsymbol{\beta}_0) \supset K_i$, $i = 1, \ldots, k$, such that $f(\hat{K}_i(\boldsymbol{\beta}_0)) = B_{\boldsymbol{\beta}_0}$ if $\|\boldsymbol{\beta}_0\| < \epsilon$.*

Proof. We have $B_{\boldsymbol{\beta}_0} \supset B$ with $B_0 = B$. Then by property (2) f overflows B and the restriction is a diffeomorphism in some neighborhoods of K_i and B. □

By the map $x \mapsto (f(x), x)$ we see that there exist k disjoint connected sets $\hat{H}_i \subset S$, $i = 1, \ldots, k$ such that

$$F_{\mathbf{0}}(S) \cap S = \bigcup_{i=1}^{k} \hat{H}_i.$$

The topological dimension of the sets $\hat{H}_i$ is n, and $\hat{H}_i$ is homeomorphic to B. If $(x, y) \in S$ then $F_{\mathbf{0}}(x, y) \in S$ if and only if $x \in K_1 \cup \cdots \cup K_k$. Hence $F_{\mathbf{0}}(x, y) \notin S$ if $x \in \partial \hat{K}_i(\boldsymbol{\beta}_0)$, $i = 1, \ldots, k$ if $|\beta_j| > 0$.

Lemma 9.2. *Suppose $|\beta_j| < |(\boldsymbol{\beta}_0)_j|$ for $j = 1, \ldots, n$, and $C \subset B$. Then $\pi_1 \circ F_{\boldsymbol{\beta}}(x, y) \notin B$ if $x \in \bigcup_{i=1}^{k}(\text{int } (\hat{K}_i(\boldsymbol{\beta}_0)))$ and $y \in C$. In particular, for fixed $y_0 \in C$ there exist k disjoint closed sets $\tilde{K}_i(y_0)$ such that $\pi_1 \circ F_{\boldsymbol{\beta}}(\tilde{K}_i(y_0), y_0) = B$.*

Proof. Let $d(x, B)$ denote the distance between x and the set B, b the radius of B and β_{max} the absolute value of the largest component in $\boldsymbol{\beta}_0$. We note that $f(\partial \hat{K}_i(\boldsymbol{\beta}_0)) = \partial B(\boldsymbol{\beta}_0)$. Let $x_0 \in \bigcup_{i=1}^{k}(\text{int } (\hat{K}_i(\boldsymbol{\beta}_0)))$. Then $d(f(x_0), B) \geq b\beta_{max}$. Since $C \subset B$ and $|\beta_j| < |(\boldsymbol{\beta}_0)_j|$ the set

$$C_{x_0} = \{x \in \mathbb{R}^n : x = f(x_0) + \boldsymbol{\beta} y \text{ where } y \in C\}$$

is contained in a ball of radius less than $b\beta_{max}$. Hence $C_{x_0} \cap B = \varnothing$. The last statement is easily seen from lemma 9.1. □

Let $T \subset S$. We define the projections π_H an π_V by

$$\pi_H : T \longrightarrow B \text{ by } (x, y) \mapsto (x, 0)$$
$$\pi_V : T \longrightarrow B \text{ by } (x, y) \mapsto (0, y)$$

Definition. A closed connected set $T \subset S = B \times B$ is called a horizontal set if the projection $\pi_H : T \longrightarrow B$ is surjective and the fiber $\pi_H^{-1}(x) \subset T$ is connected for each $x \in B$. A closed connected set $T \subset S = B \times B$ is called a vertical set if the projection $\pi_V : T \longrightarrow B$ is surjective and the fiber $\pi_V^{-1}(y) \subset T$ is connected for each $y \in B$.

Definition. A closed connected set $T \subset S = B \times B$ with a piecewise smooth boundary is called a horizontal slice if T is a horizontal set and the fiber $\pi_H^{-1}(x) \subset T$ is homeomorphic to B for each $x \in B$. A closed connected set $T \subset S = B \times B$ with a piecewise smooth boundary is called a vertical slice if T is a vertical set and the fiber $\pi_V^{-1}(y) \subset T$ is homeomorphic to B for each $y \in B$.

Lemma 9.3. *Each connected component of $F_{\mathbf{0}}(S) \cap S$ is a horizontal set.*

Proof. The connected components have form $H_i = \{(f(x), x) : x \in K_i\}$. We have $\pi_H H_i = B$ since $f(K_i) = B$, and $\pi_H^{-1}(x) = \{(f(z), z)\}$ for some unique $z \in B$. □

Lemma 9.4. *Suppose $0 < |\beta_j| < |(\beta_0)_j|$ and T is a horizontal slice. Then $F_{\boldsymbol{\beta}}(T) \cap S$ is a disjoint union of k horizontal slices.*

Proof. Since T is a horizontal slice we have that $\pi_H(T) = B$ and $\pi_H^{-1}(x_0) = T_{x_0} \simeq B$. The set T_{x_0} is a closed set of dimension n. Since $\pi_H(T) = B$ we see from lemma 9.2 that there are points $(x, y) \in T$ such that $f(x) + \boldsymbol{\beta} y = x_0$ if $x_0 \in B$. Hence $\pi_H(F_{\boldsymbol{\beta}}(T) \cap S) = B$. Since $F_{\boldsymbol{\beta}}$ is a diffeomorphism we see that $F_{\boldsymbol{\beta}}(T) \cap S$ is a closed set with a piecewise smooth boundary. By lemma 9.2 we see that for each fixed $y_0 \in B$ there exist k disjoint sets $\tilde{K}_i(y_0)$ such that $\pi_H \circ F_{\boldsymbol{\beta}}(\tilde{K}_i(y_0)) = B$. We note that $\pi_H \circ F_{\mathbf{0}}(K_i \times B) = B$, and by the above remark there are k disjoint sets M_i close to $K_i \times B$ such that $\pi_H \circ F_{\boldsymbol{\beta}}(M_i) = B$, and the sets M_i are vertical slices. Now, since T is a horizontal slice we obtain k disjoint sets, $M_i \cap T$, such that $\pi_H \circ F_{\boldsymbol{\beta}}(M_i \cap T) = B$. Hence $F_{\boldsymbol{\beta}}(M_i \cap T)$, $i = 1, \ldots, k$, are k disjoint horizontal slices. □

Lemma 9.5. *Suppose $0 < |\beta_j| < |(\beta_0)_j|$. Then the set*

$$\bigcap_{j=0}^{m} F_{\boldsymbol{\beta}}^{j}(S)$$

consists of k^m disjoint horizontal slices.

Proof. We observe that S is a horizontal slice, and the lemma follows by induction using lemma 9.4. □

Lemma 9.6. *Suppose $0 < |\beta_j| < |(\beta_0)_j|$. Then the set $F_{\boldsymbol{\beta}}^{-1}(S) \cap S$ consists of k disjoint vertical slices.*

Proof. From lemma 9.5 we see that $F_{\boldsymbol{\beta}}(S) \cap S$ is a disjoint union of k horizontal slices. We find

$$S \cap F_{\boldsymbol{\beta}}^{-1}(S) = F_{\boldsymbol{\beta}}^{-1}(F_{\boldsymbol{\beta}}(S)) \cap F_{\boldsymbol{\beta}}^{-1}(S) = F_{\boldsymbol{\beta}}^{-1}(F_{\boldsymbol{\beta}}(S) \cap S) = F_{\boldsymbol{\beta}}^{-1}(\bigcup_{i=1}^{k} H_i) = \bigcup_{i=1}^{k} F_{\boldsymbol{\beta}}^{-1}(H_i) = \bigcup_{i=1}^{k} V_i$$

Hence $S \cap F_{\boldsymbol{\beta}}^{-1}(S)$ consists of k disjoint connected components. Consider the image of the set $L_{y_0} = \{(x, y) \in S : y = y_0\}$ given by $F_{\boldsymbol{\beta}}(L_{y_0}) = \{(x, y) \in S : x = f(x) + \boldsymbol{\beta} y_0,\ y = x\}$. It is clear that $F_{\boldsymbol{\beta}}(L_{y_0}) \cap H_i \neq \varnothing$, $i = 1, \ldots, k$ since there are k disjoint sets in B such that $f(x) + \boldsymbol{\beta} y_0$ overflows B on each set. Hence the inverse image of H_i intersect every set of the form L_{y_0} with $y_0 \in B$, and therefore $\pi_V V_i = B$. From $F_{\boldsymbol{\beta}}^{-1}(x, y) = (y, \boldsymbol{\beta}^{-1}(x - f(y)))$ we see that the fiber $\pi_V^{-1}(y_0) \subset V_i$ is homeomorphic to B. □

We denote the horizontal slices from lemma 9.5 by

$$\bigcap_{j=0}^{m} F_{\boldsymbol{\beta}}^{j}(S) = \bigcup_{\substack{i_j=1 \\ j=1,\ldots,m}}^{k} H_{i_1 i_1 \cdots i_m}$$

Lemma 9.7. *Suppose* $0 < |\beta_j| < |(\beta_0)_j|$. *Then the set*

$$\bigcap_{j=0}^{m} F_{\beta}^{-j}(S)$$

consists of k^m *disjoint vertical slices.*

Proof. We find that

$$F_{\beta}^{-m}(\bigcap_{j=0}^{m} F_{\beta}^{j}(S)) = \bigcap_{j=0}^{m} F_{\beta}^{j-m}(S) = \bigcap_{j=0}^{m} F_{\beta}^{-j}(S).$$

On the other hand, we find as in lemma 9.6 that

$$F_{\beta}^{-m}(\bigcap_{j=0}^{m} F_{\beta}^{j}(S)) = F_{\beta}^{-m}(\bigcup_{\substack{i_j=1\\ j=1,\dots,m}}^{k} H_{i_1 i_1 \cdots i_m}) = \bigcup_{\substack{i_j=1\\ j=1,\dots,m}}^{k} V_{i_1 i_1 \cdots i_m}$$

where each $V_{i_1 i_1 \cdots i_m}$ is a vertical slice. Hence

$$\bigcap_{j=0}^{m} F_{\beta}^{-j}(S) = \bigcup_{\substack{i_j=1\\ j=1,\dots,m}}^{k} V_{i_1 i_1 \cdots i_m}$$

□

Definition. The vertical size $d_V(H)$ of a horizontal slice H is the supremum over the diameter of the fibers $\pi_H^{-1}(x_0) \subset H$ taken over $x_0 \in B$. The horizontal size $d_H(V)$ of a horizontal slice V is the supremum over the diameter of the fibers $\pi_V^{-1}(y_0) \subset V$ taken over $y_0 \in B$.

A horizontal slice H and a vertical slice V intersects in a set $H \cap V$ of topological dimension $2n$. It is clear that the diameter of the set $H \cap V$ is less or equal to $\max(d_v(H), d_H(V))$.

Lemma 9.8. *The vertical size of the horizontal slices in*

$$\bigcap_{j=0}^{m} F_{\beta}^{j}(S) = \bigcup_{\substack{i_j=1\\ j=1,\dots,m}}^{k} H_{i_1 i_1 \cdots i_m}$$

and the horizontal size of vertical slices in

$$\bigcap_{j=0}^{m} F_{\beta}^{-j}(S) = \bigcup_{\substack{i_j=1\\ j=1,\dots,m}}^{k} V_{i_1 i_1 \cdots i_m}$$

tends to zero as $m \to \infty$.

Proof. The horizontal slices are nested so the vertical diameter of $H_{i_1 i_1 \cdots i_m}$ is less than b/k^m. Hence it tends to zero as $m \to \infty$. The same is true for the vertical slices. □

Theorem 9.1. *Suppose $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, satisfy (1'), (2) and (3) such that the restriction of f to the non-wandering set $\boldsymbol{\Omega}(f)$ is topologically conjugate to a one-sided shift on k symbols. Then there exist $\epsilon > 0$ such that the non-wandering set of $F_{\boldsymbol{\beta}}$ is contained in $B \times B$ and the restriction of the lift $F_{\boldsymbol{\beta}}$ to $\boldsymbol{\Omega}(F_{\boldsymbol{\beta}})$ is topologically conjugate to a full shift on k symbols for all $\boldsymbol{\beta}$ with $|\beta_i| \neq 0$, $i = 1, \dots, n$, and $\|\boldsymbol{\beta}\| < \epsilon$.*

Proof. By property (1') we obtain as in lemma 8.2 that $\boldsymbol{\Omega}(F_{\boldsymbol{\beta}}) \cap U_1 = \varnothing$ where $U_1 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\| \geq \|y\|, (x, y) \notin S\}$, and $\boldsymbol{\Omega}(F_{\boldsymbol{\beta}}^{-1}) \cap U_2 = \varnothing$ where $U_2 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\| \leq \|y\|, (x, y) \notin S\}$. Hence $\boldsymbol{\Omega}(F_{\boldsymbol{\beta}}) \subset S$.

The maximal invariant set in S is given by

$$\bigcap_{j=-\infty}^{\infty} F_{\boldsymbol{\beta}}(S)$$

This set is obtained as a nested intersection of boxes each being an intersection of a horizontal and a vertical slice. The diameter of these boxes tends to zero, so in the limit we obtain a unique point. Each point is uniquely coded by a bi-infinite sequence on k symbols, and we obtain a symbolic dynamics in the usual manner.

Remark. Property (2) of f is only necessary to obtain a nice invariant set for the dynamical system generated by f on $\mathbb{R}^n$. It is easy to construct an example on the real line with an interval of fixed points such that all except one is destroyed in the lift.

10. Hyperbolic structures

In section 3 we proved that hyperbolic periodic orbits for f had hyperbolic counterparts in the $\boldsymbol{\beta}$ lift. We will in this section discuss hyperbolic structures for non-finite f-invariant sets.

Definition. Let $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, and let Λ be a compact f-invariant set. We call Λ expanding hyperbolic if

$$\max_{p \in \Lambda}\{|\lambda| : \lambda \in \operatorname{spec}(Df(p))\} > 1.$$

Our first result is that expanding hyperbolic invariant sets give a hyperbolic structure on the corresponding invariant set in the $\boldsymbol{\beta}$-lift.

Theorem 10.1. *Suppose $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, satisfy (1'), (2) and (3), and that f is expanding hyperbolic on the non-wandering set. Then there exists $\epsilon > 0$ such that $\boldsymbol{\Omega}(F_{\boldsymbol{\beta}})$ has a hyperbolic structure for all $\boldsymbol{\beta}$ with $\|\boldsymbol{\beta}\| < \epsilon$ and $\beta_i \neq 0$ for $i = 1, \dots, n$*

Proof. Theorem 9.1 gives the existence of a non-wandering set Λ such that the restriction of $F_{\boldsymbol{\beta}}$ to this set is topologically conjugate to a full shift on k symbols.

The assumption that f is expanding hyperbolic implies that there exists a constant $k' > 1$ such that $\|Df_p v\| \geq k'\|v\|$ for all $v \in \mathbb{R}^n$ and all $p \in \boldsymbol{\Omega}(f)$. The set $\boldsymbol{\Omega}(f)$ is compact so the inequality

$$\|Df_p v\| \geq k\|v\| \text{ where } k > 1$$

holds on a neighborhood of $\boldsymbol{\Omega}(f)$.

In the following let $\|\cdot\|$ denote the Euclidean norm, and $< \cdot, \cdot >$ the Euclidean inner product on $\mathbb{R}^m$. The tangent space $T_q(\mathbb{R}^n \times \mathbb{R}^n)$ is given by $\mathbb{R}^n \times \mathbb{R}^n$. For $w \in T_q(\mathbb{R}^n \times \mathbb{R}^n)$ we write $w = (u, v)$. We define cones $C_1(q)$ and $C_2(q)$ by $C_1(q) = \{w \in T_q(\mathbb{R}^n \times \mathbb{R}^n) : \|u\| \geq \|v\|\}$ and $C_2(q) = \{w \in T_q(\mathbb{R}^n \times \mathbb{R}^n) : \|u\| \leq \|v\|\}$. In order to establish a hyperbolic structure on Λ we must show that $DF_{\boldsymbol{\beta}}(q)$ maps $C_1(q)$ to $C_1(F_{\boldsymbol{\beta}}(q))$, $DF_{\boldsymbol{\beta}}^{-1}(q)$ maps $C_2(q)$ to $C_2(F_{\boldsymbol{\beta}}^{-1}(q))$ and that they expand the cones. See [New].

If $q \in \mathbb{R}^n \times \mathbb{R}^n$ we write $q = (p, r)$.

Suppose $w \in C_1(q)$. We note that

$$DF_{\beta}(q)w = \begin{bmatrix} Df_p & \beta \\ I & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Df_p u + \beta v \\ u \end{bmatrix}.$$

Cauchy-Schwarz inequality and the fact that f is expanding hyperbolic implies that

$$| < Df_p u, \beta v > | \leq \|Df_p u\| \|\beta v\| \leq k\|\beta\| \|u\| \|v\| \leq k\|\beta\| \|u\|^2$$

We find

$$\begin{aligned} \|Df_p u + \beta v\|^2 &= < Df_p u + \beta v, Df_p u + \beta v > = \|Df_p u\|^2 + 2 < Df_p u, \beta v > + \|\beta v\|^2 \\ &\geq \|Df_p u\|^2 + 2 < Df_p u, \beta v > \geq \|Df_p u\|^2 - 2k\|\beta\| \|u\|^2 \\ &\geq k^2\|u\|^2 - k\|\beta\| \|u\|^2 = (k^2 - 2k\|\beta\|)\|u\|^2 > \|u\|^2 \end{aligned}$$

if $(k^2 - 2k\|\beta\|) > 1$. Hence DF_{β} maps the cone $C_1(q)$ to the cone $C_1(F_{\beta}(q))$. To see that the restriction of DF_{β} to $C_1(q)$ is an expansion we simply note from the above that

$$\|DF_{\beta}(q)w\|^2 = \|Df_p u + \beta v\|^2 + \|u\|^2 > 2\|u\|^2 \geq \|u\|^2 + \|v\|^2 = \|w\|^2.$$

Suppose $w \in C_2(q)$. Consider

$$DF_{\beta}^{-1}(q)w = \begin{bmatrix} 0 & I \\ \beta^{-1} & -\beta^{-1}Df_p \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v \\ \beta^{-1}(u - Df_r v) \end{bmatrix} = \begin{bmatrix} v \\ z \end{bmatrix}$$

Then

$$\begin{aligned} \|\beta z\|^2 = \|u - Df_r v\|^2 &= < u - Df_r v, u - Df_r v > \\ &= \|u\|^2 - 2 < u, Df_r v > + \|Df_r v\|^2 \geq \|Df_r v\|^2 - 2\|Df_r v\| \|u\| + \|u\|^2 \\ &= (\|Df_r v\| - \|u\|)^2 \geq (\|Df_r v\| - \|v\|)^2 \\ &\geq (k\|v\| - \|v\|)^2 = (k-1)^2\|v\|^2 \end{aligned}$$

Now

$$\|\beta\|^2\|z\|^2 \geq \|\beta z\|^2 \geq (k-1)^2\|v\|^2$$

so

$$\|z\|^2 \geq \left(\frac{k-1}{\|\beta\|}\right)^2 \|v\|^2 > \|v\|^2$$

if

$$\frac{k-1}{\|\beta\|} > 1$$

Hence DF_{β}^{-1} maps the cone $C_2(q)$ to the cone $C_2(F_{\beta}^{-1}(q))$. To see that the restriction of DF_{β}^{-1} to $C_2(q)$ is an expansion we simply note from the above that

$$\|DF_{\beta}^{-1}(q)w\|^2 = \|v\|^2 + \|\beta^{-1}(u - Df_r v)\|^2 > 2\|v\|^2 \geq \|u\|^2 + \|v\|^2 = \|w\|^2.$$

□

From the results on homoclinic orbits together with the Smale-Birkhoff homoclinic theorem we get the following theorem:

Theorem 10.2. *Suppose $f \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq 1$, has a non-degenerate homoclinic orbit associated with a fixed-point (or a periodic orbit). Then there exists $\epsilon > 0$ such that F_{β} has a hyperbolic invariant set for all β with $\|\beta\| < \epsilon$ and $\beta_i \neq 0$ for $i = 1, \ldots, n$, on which f is topologically conjugate to a subshift of finite type.*

Proof. We simply note that if f has a non-degenerate homoclinic orbit then F_{β} has a transversal homoclinic point for $\|\beta\|$ small. Hence the Smale-Birkhoff homoclinic theorem [G&H] applies. □

11. An example

We will give an example of a map $f_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where the non-wandering set is a Cantor set $\Lambda(f_\alpha)$ such that f_α restricted to $\Lambda(f_\alpha)$ is topologically conjugate to a one-sided shift on four symbols for $\alpha > 2$. The lift of f_α, $F_{(\alpha,\beta)} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ has a non-wandering set $\Lambda(F_{(\alpha,\beta)})$ such that the restriction of $F_{(\alpha,\beta)}$ to $\Lambda(F_{(\alpha,\beta)})$ is topologically conjugate to a full shift on four symbols.

In the following let $\|\cdot\|$ denote the max-norm on $\mathbb{R}^n$, let Σ_4^+ denote the space of all infinite sequences of four symbols equipped with its usual metric, let Σ_4 denote the space of all bi-infinite sequences of four symbols equipped with its usual metric and let σ denote the shift map on Σ_4^+ and Σ_4.

Let $f_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $(x,y) \mapsto (\alpha - y^2, \alpha - x^2)$. Let $p \in \mathbb{R}^2$. We see that the fiber $f_\alpha^{-1}(p)$ generically is empty or consists of four points. Let $\alpha > 2$, $R_\alpha \in ((1+\sqrt{1+4\alpha})/2, \alpha)$ and $S(R_\alpha) = [-R_\alpha, R_\alpha]^2$. We find that if $p \in \mathbb{R}^2 \setminus S(R_\alpha)$ then $\|f_\alpha^n(p)\| \to \infty$ as $n \to \infty$, and $f_\alpha(S(R_\alpha)) \supset S(R_\alpha)$. Hence the non-wandering set of f_α is contained in the square $S(R_\alpha)$.

We see that $f_\alpha^{-1}(S(R_\alpha)) \cap S(R_\alpha)$ consists of four disjoint rectangles L_i, $i = 1,2,3,4$. Let $E_n = \{p \in S(R_\alpha) : f_\alpha^k(p) \in S(R_\alpha) \text{ for } 0 \le k \le n \text{ but } f_\alpha^k(p) \notin S(R_\alpha)\}$. The non-wandering set of f_α is given by

$$\Lambda(f_\alpha) = S(R_\alpha) \setminus \bigcup_{n \ge 1} E_n.$$

$\Lambda(f_\alpha)$ is non-empty and is a Cantor set. For $p \in \Lambda(f_\alpha)$ we define the itinerary of p as the sequence $h(p) = k_0 k_1 k_2 \ldots$ where $k_n = j$ if $f_\alpha(p) \in L_j$. By the standard method we find that h is a map $h : \Lambda(f_\alpha) \longrightarrow \Sigma_4^+$, and it is not hard to establish that h is in fact a homeomorphism such that $h \circ f_\alpha = \sigma \circ h$.

Consider the lift $F_{\alpha,\beta} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2$ defined by $(x,y) \mapsto (f_\alpha(x) + \beta y, x)$. It can be shown that there is an $\alpha_0(\beta)$ such that for $\alpha > \alpha_0(\beta)$ there is a $R_{\alpha,\beta} > 0$ such that $\|f_\alpha(x) + \beta y\| > \|x\|$ and $\|y - f_\alpha(x)\| > |\beta|\|x\|$ if $\|x\| > R_{\alpha,\beta}$.

Let

$$\begin{aligned} S(R_{\alpha,\beta}) &= \{p \in \mathbb{R}^4 : \|p\| < R_{\alpha,\beta}\} \\ K_1 &= \{p = (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|x\| \ge \|y\|\} \\ K_2 &= \{p = (x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|x\| \le \|y\|\} \\ M_i &= (\mathbb{R}^4 \setminus S(R_{\alpha,\beta})) \cap K_i \text{ where } i = 1,2. \end{aligned}$$

We find that $F_{\alpha,\beta}(M_1) \subset M_1$ and $F_{\alpha,\beta}^{-1}(M_2) \subset M_2$. Furthermore we find that if $p \in M_1$ then $\|F_{\alpha,\beta}^n(p)\| \to \infty$ when $n \to \infty$ and if $p \in M_2$ then $\|F_{\alpha,\beta}^{-n}(p)\| \to \infty$ when $n \to \infty$. We conclude that the non-wandering set of $F_{\alpha,\beta}$ is contained in the cube $S(R_{\alpha,\beta})$.

Consider $S(R_{\alpha,\beta}) \cap F_{\alpha,\beta}(S(R_{\alpha,\beta}))$. We claim that this set consists of four topological cubes cutting completely through $S(R_{\alpha,\beta})$. To see this let

$$T(\xi,\eta) = \{\xi\} \times \{\eta\} \times [-R_{\alpha,\beta}, R_{\alpha,\beta}]^2.$$

Then

$$F_{\alpha,\beta}(T(\xi,\eta)) = [\alpha - \eta^2 - |\beta|R_{\alpha,\beta}, \alpha - \eta^2 + |\beta|R_{\alpha,\beta}] \times [\alpha - \xi^2 - |\beta|R_{\alpha,\beta}, \alpha - \xi^2 + |\beta|R_{\alpha,\beta}] \times \{\xi\} \times \{\eta\}$$

Let ξ_i be the four solutions of the two equations $\alpha - \xi^2 - |\beta|R_{\alpha,\beta} = R_{\alpha,\beta}$ and $\alpha - \xi^2 + |\beta|R_{\alpha,\beta} = -R_{\alpha,\beta}$, and η_i be the four solutions of the two equations $\alpha - \eta^2 - |\beta|R_{\alpha,\beta} = R_{\alpha,\beta}$ and $\alpha - \eta^2 + |\beta|R_{\alpha,\beta} = -R_{\alpha,\beta}$. These points define four disjoint rectangles in the $\xi\eta$-plane such that if (ξ_0, η_0) is not in any of these rectangles then there is topological cube defined by

$$C_1 = \bigcup_{(\xi,\eta) \in J_1} F_{\alpha,\beta}(T(\xi,\eta)) \text{ or } C_2 = \bigcup_{(\xi,\eta) \in J_2} F_{\alpha,\beta}(T(\xi,\eta))$$

where $J_1 = (\xi_0 - \epsilon, \xi_0 + \epsilon) \times [-R_{\alpha,\beta}, R_{\alpha,\beta}]$ and $J_2 = [-R_{\alpha,\beta}, R_{\alpha,\beta}] \times (\eta_0 - \epsilon, \eta_0 + \epsilon)$, such that $S(R_{\alpha,\beta}) \cap C_1 = \varnothing$ or $S(R_{\alpha,\beta}) \cap C_2 = \varnothing$ for some $\epsilon > 0$.

Similarly we find that $S(R_{\alpha,\beta}) \cap F_{\alpha,\beta}^{-1}(S(R_{\alpha,\beta}))$ consists of four topological cubes cutting completely through $S(R_{\alpha,\beta})$.

We see that

$$P_n = \bigcap_{i=-n}^{n} F_{\alpha,\beta}^{i}(S(R_{\alpha,\beta}))$$

is 4^n disjoint topological cubes whose diameter tends to zero for increasing n. The non-wandering set of $F_{\alpha,\beta}$ is given by

$$\Lambda(F_{\alpha,\beta}) = \bigcap_{i=-\infty}^{\infty} F_{\alpha,\beta}^{i}(S(R_{\alpha,\beta})),$$

and the construction of a homeomorphism $h : \Lambda(F_{\alpha,\beta}) \longrightarrow \Sigma_4$ such that $h \circ F_{\alpha,\beta} = \sigma \circ h$ is standard like for Smale's horseshoe as in [Dev].

12. Bifurcations in maps of the line lifted to the plane

This section contains a constructive proof for the existence of saddle-node- and periode doubling bifurcations in the lifted system in the case $n = 1$. This is an alternative method of those applied in section 6.

We first state the period doubling bifurcation- and the saddle-node bifurcation theorem for maps on the real line. For a proof see [G&H].

Period doubling bifurcations for one dimensional maps. *Let $f_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ be a one-parameter family of mappings such that f_{μ_0} has a fixed point x_0 with eigenvalue -1. Assume*

$$\left(\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial \mu}\right) = \frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x} - 1\right)\frac{\partial^2 f}{\partial x \partial \mu} \neq 0 \text{ at } (x_0, \mu_0) \tag{A1}$$

and let

$$s = \left(\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 f}{\partial x^3}\right)\right) \text{ at } (x_0, \mu_0). \tag{A2}$$

Then there is a smooth curve of fixed points of f_μ passing through (x_0, μ_0), the stability of which changes at (x_0, μ_0). There is also a smooth curve γ passing through (x_0, μ_0) so that $\gamma - \{(x_0, \mu_0)\}$ is a union of hyperbolic period 2 orbits. The curve γ has quadratic tangency with the line $\mathbb{R} \times \{\mu_0\}$ at (x_0, μ_0). The sign of s determines the the stability and direction of the bifurcation of the orbit of period two. If $s > 0$ the orbits are stable, and if $s < 0$ the orbits are unstable.

Saddle-node bifurcations for one dimensional maps. *Let $f_\mu : \mathbb{R} \longrightarrow \mathbb{R}$ be a one-parameter family of mappings such that f_{μ_0} has a fixed point x_0 with eigenvalue 1. Assume*

$$\frac{\partial^2 f}{\partial x^2} \neq 0 \text{ at } (x_0, \mu_0) \tag{A3}$$

$$\frac{\partial f}{\partial \mu} \neq 0 \text{ at } (x_0, \mu_0). \tag{A4}$$

Let

$$s = \frac{\partial^2 f}{\partial x^2}(x_0, \mu_0)\frac{\partial f}{\partial \mu}(x_0, \mu_0).$$

Then there is a smooth curve γ of fixed points of f_μ passing through (x_0, μ_0), the stability of which changes at (x_0, μ_0). The curve γ has quadratic tangency with the line $\mathbb{R} \times \{\mu_0\}$ at (x_0, μ_0). If $s < 0$ then there exist an $\epsilon > 0$ such that f has no fixed point near (x_0, μ) for $\mu \in (\mu_0 - \epsilon, \mu_0)$ and two hyperbolic *fixed points near (x_0, μ) for $\mu \in (\mu_0, \mu_0 + \epsilon)$. If $s > 0$ then there exist an $\epsilon > 0$ such that f has two hyperbolic fixed points near (x_0, μ) for $\mu \in (\mu_0 - \epsilon, \mu_0)$ and no fixed point near (x_0, μ) for $\mu \in (\mu_0, \mu_0 + \epsilon)$.*

Let

$$F_{a,b}(x, y) = F(x, y, a, b) = (F_1(x, y, a, b), F_2(x, y, a, b)).$$

Suppose F is independent of y at $b = 0$, that is $F(x, y, a, 0) = G(x, a)$. Clearly all iterates of F at $b = 0$ is independent of y, and all fixed-points and periodic points of F are determined by $G_1(x, a)$. Taylor expansion of of each component in F with respect to b at $b = 0$ gives

$$F_i(x, y, a, b) = G_i(x, a) + \frac{\partial F_i}{\partial b}(x, y, a, 0) \cdot b + \frac{1}{2}\frac{\partial^2 F_i}{\partial b^2}(x, y, a, \xi_i(b)) \cdot b^2$$

where $i = 1, 2$ and $0 < \xi_i(b) < b$ for $b > 0$ and $b < \xi_i(b) < 0$ for $b < 0$. We write

$$\frac{\partial F_i}{\partial b}(x, y, a, 0) = H_i(x, y, a) \quad \text{and} \quad \frac{1}{2}\frac{\partial^2 F_i}{\partial b^2}(x, y, a, \xi_i(b)) = K_i(x, y, a, b)$$

and therefore

$$F(x, y, a, b) = G(x, a) + H(x, y, a) \cdot b + K(x, y, a, b) \cdot b^2. \tag{1}$$

Let

$$M(x, y, a, b) = F(x, y, a, b) - \mathrm{Id}_{\mathbb{R}^2}(x, y). \tag{2}$$

Then

$$\frac{\partial M}{\partial(x,y)} = \begin{bmatrix} \frac{\partial M_1}{\partial x} & \frac{\partial M_1}{\partial y} \\ \frac{\partial M_2}{\partial x} & \frac{\partial M_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x} - 1 & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} - 1 \end{bmatrix}.$$

We will use M and its derivative throughout this section.

The period doubling bifurcations.

Suppose $F(x_0, y_0, a_0, 0) = (x_0, y_0)$ with $\frac{\partial F_1}{\partial x}(x_0, y_0, a_0, 0) = \frac{\partial G_1}{\partial x}(x_0, a_0) = -1$, and that G_1 has a period doubling bifurcation at (x_0, a_0) viewed as a one-dimensional system. Then $M(x_0, y_0, a_0, 0) = (0, 0)$ and

$$\det \frac{\partial M}{\partial(x,y)}(x_0, y_0, a_0, 0) = \begin{bmatrix} -2 & 0 \\ \frac{\partial F_2}{\partial x} & -1 \end{bmatrix} = 2 \neq 0.$$

By the implicit function theorem there exist neighborhoods $U_{(a_0,0)}$ of $(a_0, 0)$ and $V_{(x_0,y_0)}$ of (x_0, y_0) and a map $\Psi : U_{(a_0,0)} \longrightarrow V_{(x_0,y_0)}$ with $\Psi(a_0, 0) = (x_0, y_0)$ such that $M(\Psi(a, b), a, b) = M(x(a, b), y(a, b), a, b) = (0, 0)$. We will return to the problem of estimating the size of the neighborhood $U_{(a_0,0)}$.

We define p by the equation

$$p(x, y, a, b, \lambda) = \det \begin{bmatrix} \lambda - \frac{\partial F_1}{\partial x} & -\frac{\partial F_1}{\partial y} \\ -\frac{\partial F_2}{\partial x} & \lambda - \frac{\partial F_2}{\partial y} \end{bmatrix} = \left(\lambda - \frac{\partial F_1}{\partial x}\right)\left(\lambda - \frac{\partial F_2}{\partial y}\right) - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x}.$$

By using (1) we find the following formula for p:

$$\begin{aligned} p(x,y,a,b,\lambda) &= \lambda^2 - \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) + \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial x} \\ &= \lambda^2 - \left(\frac{\partial G_1}{\partial x} + \left(\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y}\right)\cdot b + \left(\frac{\partial K_1}{\partial x} + \frac{\partial K_2}{\partial y}\right)\cdot b^2\right)\lambda \\ &+ \left(\frac{\partial G_1}{\partial x}\frac{\partial H_2}{\partial y} - \frac{\partial G_2}{\partial x}\frac{\partial H_1}{\partial y}\right)\cdot b \\ &+ \left(\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y} - \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x} + \frac{\partial G_1}{\partial x}\frac{\partial K_2}{\partial y} - \frac{\partial G_2}{\partial x}\frac{\partial K_1}{\partial y}\right)\cdot b^2 \\ &+ \left(\frac{\partial H_1}{\partial x}\frac{\partial K_2}{\partial y} - \frac{\partial H_2}{\partial x}\frac{\partial K_1}{\partial y} + \frac{\partial H_2}{\partial y}\frac{\partial K_1}{\partial x} - \frac{\partial H_1}{\partial y}\frac{\partial K_2}{\partial x}\right)\cdot b^3 \\ &+ \left(\frac{\partial K_1}{\partial x}\frac{\partial K_2}{\partial y} - \frac{\partial K_1}{\partial y}\frac{\partial K_2}{\partial x}\right)\cdot b^4 \end{aligned}$$

Therefore p has the form

$$p(x,y,a,b,\lambda) = \lambda^2 - \left(\frac{\partial G_1}{\partial x}(x,a) + \xi(x,y,a,b)\cdot b\right)\lambda + \eta(x,y,a,b)\cdot b. \tag{3}$$

We now define a map $q : U_{(a_0,0)} \longrightarrow \mathbb{R}$ by the formula

$$(a,b) \mapsto q(a,b) = p(\Psi(a,b),a,b,-1).$$

We note that a point (a,b) is in the zero-set of q if and only if $\Psi(a,b)$ is a fixed-point of F and -1 is an eigenvalue of DF at the fixed-point. In particular we have that

$$q(a_0,0) = p(\Psi(a_0,0),a_0,0,-1) = p(x_0,y_0,a_0,0,-1) = 1 + (-1) = 0.$$

From the expression for p we see that

$$q(a,b) = 1 + \frac{\partial G_1}{\partial x}(x(a,b),a) + (\eta(x(a,b),y(a,b),a,b) - \xi(x(a,b),y(a,b),a,b))\cdot b.$$

We find that

$$\frac{\partial q}{\partial a}(a_0,0) = \frac{\partial}{\partial a}\left(\frac{\partial G_1}{\partial x}(x(a,b),a)\bigg|_{a=a_0}\right) = \frac{\partial^2 G_1}{\partial x^2}(x_0,a_0)\frac{\partial x}{\partial a}(a_0,0) + \frac{\partial^2 G_1}{\partial a \partial x}(x_0,a_0).$$

We want to apply the implicit function theorem to the equation $q(a,b) = 0$ to obtain a function $a = \psi(b)$ with $\psi(0) = a_0$. In order to apply the theorem we must show that

$$\frac{\partial^2 G_1}{\partial x^2}(x_0,a_0)\frac{\partial x}{\partial a}(a_0,0) + \frac{\partial^2 G_1}{\partial a \partial x}(x_0,a_0) \neq 0.$$

At a fixed point on the line $b = 0$ we have $x(a) = G_1(x(a),a)$. We find

$$\frac{\partial x}{\partial a} = \frac{\partial G_1}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial G_1}{\partial a}.$$

At (x_0,a_0) we find

$$2\frac{\partial x}{\partial a} = \frac{\partial G_1}{\partial a}$$

so

$$\frac{\partial^2 G_1}{\partial x^2}(x_0,a_0)\frac{\partial x}{\partial a}(a_0,0) + \frac{\partial^2 G_1}{\partial a \partial x}(x_0,a_0) = \frac{1}{2}\frac{\partial^2 G_1}{\partial x^2}(x_0,a_0)\frac{\partial G_1}{\partial a}(x_0,a_0) + \frac{\partial^2 G_1}{\partial a \partial x}(x_0,a_0) \neq 0$$

from (A1) in the period doubling theorem. Now by the implicit function theorem there exists neighborhoods W_0 and W_{a_0} with $W_{a_0} \times W_0 \subset U_{(a_0,0)}$ and a function $a = \psi(b)$ with $\psi(0) = a_0$ such that $q(\psi(b),b) = 0$.

The saddle-node bifurcations.

Suppose $F(x_0, y_0, a_0, 0) = (x_0, y_0)$ with $\frac{\partial F_1}{\partial x}(x_0, y_0, a_0, 0) = \frac{\partial G_1}{\partial x}(x_0, a_0) = 1$ and that G_1 has a saddle-node bifurcation at (x_0, a_0) viewed as a one-dimensional system.

Let $p = p(x, y, a, b, \lambda)$ be as in (3). We define a map $r : \mathbb{R}^4 \longrightarrow \mathbb{R}$ by the formula

$$(x, y, a, b) \mapsto r(x, y, a, b) = p(x, y, a, b, 1).$$

We note that r has a zero at $(x_0, y_0, a_0, 0)$, and that $\frac{\partial F}{\partial(x, y)}(x, y, a, b)$ has an eigenvalue 1 if and only if (x, y, a, b) is in the zero-set of r, but $r(x_1, y_1, a_1, b_1) = 0$ **does not** imply $M(x_1, y_1, a_1, b_1) = 0$ in (2). (3) implies that r has the form

$$r(x, y, a, b) = 1 - \frac{\partial G_1}{\partial x}(x, a) - \xi(x, y, a, b) \cdot b + \eta(x, y, a, b) \cdot b.$$

The partial derivative of r with respect to x at $(x, y, a, b) = (x_0, y_0, a_0, 0)$ is given by

$$\left.\frac{\partial r}{\partial x}\right|_{(x_0, y_0, a_0, 0)} = -\frac{\partial^2 G_1}{\partial x^2}(x_0, a_0).$$

By (A3) we have

$$\frac{\partial^2 G_1}{\partial x^2}(x_0, a_0) \neq 0 \text{ so } \frac{\partial r}{\partial x}(x_0, y_0, a_0, 0) \neq 0,$$

so the implicit function theorem implies that there exist neighborhoods $U_{(y_O, a_0, 0)}$ of $(y_0, a_0, 0)$ and V_{x_0}, and a function $\Gamma : U_{(y_O, a_0, 0)} \longrightarrow V_{x_0}$ with $\Gamma(y_0, a_0, 0) = x_0$ and $r(\Gamma(y, a, b), y, a, b) = 0$.

Consider the map M in (2). We define a map $N : U_{(y_0, a_0, 0)} \longrightarrow \mathbb{R}^2$ by the formula

$$(y, a, b) \mapsto N(y, a, b) = M(\Gamma(y, a, b), y, a, b).$$

We note that $N(y_0, a_0, 0) = M(\Gamma(y_0, a_0, 0), y_0, a_0, 0) = M(x_0, y_0, a_0, 0) = 0$. The Jacobi matrix of N with respect to (y, a) is given by

$$\frac{\partial N}{\partial(y, a)} = \begin{bmatrix} \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial a} \\ \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial a} \end{bmatrix} = \begin{bmatrix} \frac{\partial M_1}{\partial x}\frac{\partial \Gamma}{\partial y} + \frac{\partial M_1}{\partial y} & \frac{\partial M_1}{\partial x}\frac{\partial \Gamma}{\partial a} + \frac{\partial M_1}{\partial a} \\ \frac{\partial M_2}{\partial x}\frac{\partial \Gamma}{\partial y} + \frac{\partial M_2}{\partial y} & \frac{\partial M_2}{\partial x}\frac{\partial \Gamma}{\partial a} + \frac{\partial M_2}{\partial a} \end{bmatrix}.$$

We want to show that this matrix is non-singular at $(y_0, a_0, 0)$. Using the definition of M we find

$$\begin{aligned} \frac{\partial M_1}{\partial x}(x_0, y_0, a_0, 0) &= \frac{\partial F_1}{\partial x}(x_0, y_0, a_0, 0) - 1 = \frac{\partial G_1}{\partial x}(x_0, a_0) - 1 = 0 \\ \frac{\partial M_1}{\partial y}(x_0, y_0, a_0, 0) &= \frac{\partial F_1}{\partial y}(x_0, y_0, a_0, 0) = 0 \\ \frac{\partial M_1}{\partial a}(x_0, y_0, a_0, 0) &= \frac{\partial F_1}{\partial a}(x_0, y_0, a_0, 0) = \frac{\partial G_1}{\partial a}(x_0, a_0) \neq 0 \\ \frac{\partial M_2}{\partial y}(x_0, y_0, a_0, 0) &= \frac{\partial F_2}{\partial y}(x_0, y_0, a_0, 0) - 1 = -1. \end{aligned}$$

Furthermore we have $r(\Gamma(y, a, b), y, a, b) = 0$ so

$$\frac{\partial r}{\partial x}\frac{\partial \Gamma}{\partial y} + \frac{\partial r}{\partial y} = 0.$$

Evaluating at $(y_0, a_0, 0)$ we see that $\frac{\partial \Gamma}{\partial y}(y_0, a_0, 0) = 0$. We find by the above that

$$\frac{\partial N}{\partial(y,a)}(y_0, a_0, 0) = \begin{bmatrix} 0 & \frac{\partial G_1}{\partial a}(x_0, a_0) \\ -1 & \frac{\partial N_2}{\partial a}(y_0, a_0, 0) \end{bmatrix},$$

so $\frac{\partial N}{\partial(y,a)}(y_0, a_0, 0)$ is non-singular since

$$\det \frac{\partial N}{\partial(y,a)}(y_0, a_0, 0) = \frac{\partial G_1}{\partial a}(x_0, a_0) \neq 0.$$

Now, by the implicit function theorem there exist neighborhoods Z_0 of 0 and $Z_{(y_0,a_0)}$ and a map $\gamma : Z_0 \longrightarrow Z_{(y_0,a_0)}$ with $\gamma(0) = (y_0, a_0)$ and $N(\gamma(b), b) = 0$. Define

$$\phi = \pi_2 \circ \gamma : Z_0 \longrightarrow Z_{a_0},$$

where π_2 is the projection on the second component. Then for $(b, a) \in \mathrm{Graph}(\phi)$ there exists a non-hyperbolic fixed point for F near (x_0, y_0) with an eigenvalue 1.

The above may be formulated in the following theorem:

Theorem 12.1. *Suppose $F_{a,b}(x,y) = F(x,y,a,b)$ is in $C^r(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ such that*

$$F_{a,b_0}(x,y) = F(x,y,a,b_0) = G(x,a),$$

and $F_{a,b} \in \mathrm{Diff}^2(\mathbb{R}^2)$ for all $b \neq b_0$. If $x \mapsto G_1(x,a)$ has a saddle-node- or period doubling bifurcation at (x_p, a_p), (then G has a saddle-node- or period doubling bifurcation at (x_p, y_p, a_p) where $y_p = G_2(x_p, a_p)$), then there exist an $\epsilon_p > 0$ and a C^r function $\phi = \phi(b)$ defined in $(b_0 - \epsilon_p, b_0 + \epsilon_p)$ with $a_p = \phi(b_0)$ such that $F_{\phi(b),b}$ has a saddle-node- or period doubling bifurcation.

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Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo
E-mail address: tojonass@math.uio.no