ENUMERATIVE GEOMETRY FOR PLANE CUBIC CURVES WITH j-INVARIANT 0 IN CHARACTERISTIC 2

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ABSTRACT. Consider plane cubic curves with j-invariant zero over an algebraically closed field of characteristic 2. By blowing up the parameter space along the locus of the nonreduced curves we obtain a variety B of complete cubics. We then calculate the characteristic numbers for the family by intersecting divisors on B. We also obtain characteristic numbers for more specialized families, in particular the cuspidal cubics.

1. Introduction. The problem of finding the characteristic numbers for plane cubic curves was originally solved by Maillard and Zeuthen in the early 1870s. It took more than a century until these numbers were rigorously confirmed. The articles by Kleiman and Speiser [**7, 8, 9**] and Aluffi [**1, 2**] give two different ways of attacking the problem. Kleiman and Speiser use the classical degeneration method. They specialize to degenerate families and use previously obtained results. Aluffis method is more direct. By a sequence of five blow-ups of $\mathbf{P}^9$ he constructs a variety of complete cubics, and the characteristic numbers are obtained by intersecting certain divisors on this variety. These papers assume that the characteristic is different from 2 and 3. The only published work dealing with enumerative geometry over characteristic p is Vainsenchers "Conics in characteristic 2" [**11**].

In this paper we hope to fill in some of the gap. We consider plane cubic curves with j-invariant zero over an algebraically closed field of characteristic 2. These curves are parametrized by $\mathbf{P}^8$, and the nonreduced curves are parametrized by the image L of the Segre embedding $r\colon \mathbf{P}^2 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^8$. To obtain the characteristic numbers $N_{\alpha,\beta}$ for our family (the number of curves passing through α given points and tangent to β given lines when $\alpha + \beta = 8$) we need to intersect α point conditions and β line conditions in $\mathbf{P}^8$ and count the intersection points outside L. By blowing up $\mathbf{P}^8$ along L we get a variety of complete cubics, and the characteristic numbers are now equal to the intersection numbers of the strict transforms of the point and line conditions. These numbers can be found by computing the total Segre class of the normal bundle of the inclusion $L \to \mathbf{P}^8$. The cuspidal cubics are parametrized by a hypersurface of degree 3 in $\mathbf{P}^8$, so without too much more work we also find the characteristic numbers for the cuspidal cubics in characteristic 2.

Most of the material in this paper is part of the authors cand.scient. thesis written under the guidance of R.Piene. It is a pleasure to thank R.Piene for proposing the problem and for many helpful suggestions.

2. Plane cubics in characteristic 2. In this and the next two sections we look at some elementary properties of plane cubic curves in characteristic 2.

Proposition 2.1. *Let $C \subset \mathbf{P}^2$ be a nonsingular cubic given by* $F(x,y,z) = ax^3 + by^3 + cz^3 + dx^2y + ex^2z + fxy^2 + gy^2z + hxz^2 + iyz^2 + jxyz$. *Then the following are equivalent:*
(1) C is projectively equivalent to the curve with equation $x^3 + y^3 + z^3 = 0$
(2) $j = 0$ in the equation for C
(3) C has Hasse-invariant 0
(4) C has j-invariant 0

Proof. If $C \sim D$ (projective equivalence) and $j_D = 0$ then it is easy to verify that $j_C = 0$, so we have (1) $\Rightarrow$ (2). (By j_D we mean the coefficient of xyz in the equation for D)
To prove (2) $\Rightarrow$ (1) we need to know that $C \sim \{x^3 + y^3 + z^3 + txyz = 0\}$ for some t. This is well known when the characteristic is 0. The proof in [3] also works in characteristic 2 provided we have at least two points of inflection, but this follows from [10], Theorem 9. By the argument used in the proof of (1) $\Rightarrow$ (2) we see that t must be zero, so we have (2) $\Rightarrow$ (1)
(2) $\Leftrightarrow$ (3) is a special case of [6], IV Prop.4.21.
(3) $\Leftrightarrow$ (4) follows from [6], IV.4.23 (note to corollary). □

The cubics described in Proposition 2.1 we call j-curves. We next show that the cuspidal cubics are degenerate j-curves. First we need some lemmas.

Lemma 2.2. *Let C be a cubic with $j_C = 0$. Let H be the matrix* $\begin{pmatrix} a & f & h \\ d & b & i \\ e & g & c \end{pmatrix}$. *Then:*
C is nonsingular $\Leftrightarrow$ $rk(H) = 3$
C is singular and reduced $\Leftrightarrow$ $rk(H) = 2$
C is nonreduced $\Leftrightarrow$ $rk(H) = 1$

Proof. Note that $F = xF_x + yF_y + zF_z$, so the singular locus is precisely the set of points (x,y,z) such that all the partial derivatives are zero, or equivalently: (x^2, y^2, z^2) belongs to the nullspace of H. The lemma now follows by elementary linear algebra. □

Lemma 2.3. *Let C be a singular cubic with equation $ax^3 + \ldots + jxyz = 0$. Then:*
C is cuspidal (possibly degenerate) $\Longleftrightarrow$ $j = 0$ in the equation for C

Proof. Choose a $B \in PGL(2)$ which sends a singularity of C to $(0,0,1)$. Let $D = B(C)$ and introduce affine coordinates $x' = \frac{x}{z}$, $y' = \frac{y}{z}$. The affine equation of D can be written as $f(x', y') = 0$. Let $f = f_3 + f_2 + f_1 + f_0$ where f_i is homogeneous of degree i. Since D is singular at $(0,0)$ we have $f_1 = f_0 = 0$ and $f_2 = e_D x'^2 + g_D y'^2 + j_D x'y'$ is the equation of the tangent cone. D is cuspidal exactly when the tangent cone is a double line and that happens exactly when $j_D = 0$. Since $j_C = 0 \Leftrightarrow j_D = 0$ the lemma follows. □

Proposition 2.4. *Let $C \subset \mathbf{P}^2$ be a cubic with equation $ax^3 + by^3 + \ldots + jxyz = 0$. Then: C is cuspidal (possibly degenerate) $\iff$ $\det H = j = 0$*

Proof. If C is cuspidal, then $j = 0$ by (2.3) and $\det H = 0$ by (2.2).
$\det H = j = 0$ implies that C is singular by (2.2) and cuspidal by (2.3). □

Let C be a nondegenerate cuspidal cubic given by $F(x,y,z) = ax^3 + by^3 + \ldots + jxyz$. By (2.2) we have that $\mathrm{rk}H = 2$. Then $\mathrm{rk}(\mathrm{cof}(H)) = 1$ so that nonzero rows (resp. columns) of $\mathrm{cof}(H)$ define the same point in $\mathbf{P}^2$.

$$\mathrm{cof}(H) = \begin{pmatrix} bc + gi & cd + ei & dg + be \\ cf + gh & ac + eh & ag + ef \\ fi + bh & ai + dh & ab + df \end{pmatrix}$$

Let P be the point defined by the columns, and let Q be the point defined by the square root of the rows: If $(\alpha, \beta, \gamma) \neq (0,0,0)$ is a row, then $Q = (\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma})$. This is well defined since there is only one square root in characteristic 2.

Before we proceed we need to make precise the notions of tangent and flex. A tangent is a line intersecting the curve with multiplicity at least two at a point (not necessarily nonsingular). If the touching point is nonsingular we call the tangent proper. A flex is a nonsingular point where the tangent intersects with at least multiplicity three.

Proposition 2.5. *Let C, P and Q be as above. Then Q is the cusp of C, and P is the only flex of C. Also P is a strange point, that is: every proper tangent of C contains P.*

Proof. Suppose $Q = (\sqrt{bc + gi}, \sqrt{cd + ei}, \sqrt{dg + be})$ is given by the first row of $\mathrm{cof}H$. Remembering that $\det H = 0$ we easily see that $F_x(Q) = F_y(Q) = F_z(Q) = 0$ so Q is the cusp of C. Now the last part: The tangent at $(u_0, u_1, u_2) \in C$ is given by

$$(au_0^2 + fu_1^2 + hu_2^2)x + (du_0^2 + bu_1^2 + iu_2^2)y + (eu_0^2 + gu_1^2 + cu_2^2)z = 0$$

or equivalently $(x\ y\ z)H(u_0^2\ u_1^2\ u_2^2)^t = 0$. We must show that this holds for $(x,y,z) = P$ a column in $\mathrm{cof}H$ or a row in $\mathrm{adj}H$. But this follows immediately from the identity $(\mathrm{adj}H)H = I\det H = 0$.

To prove that $P \in C$ just note that $F = (x\ y\ z)H(x^2\ y^2\ z^2)^t$ and use the same argument. If the tangent at P meets the curve at another point S then (since P is a strange point) this tangent would be a bitangent which is impossible for cubics. This proves that P is a flex. The tangent at other nonsingular points all contain P so there cannot be more flexes. □

3. The parameter space. The set of j-curves (including degenerate curves) are parametrized by $\mathbf{P}^8$ when we to a point $(a,b,c,..,i)$ in $\mathbf{P}^8$ associate the curve with equation $ax^3 + by^3 + \ldots + iyz^2$. We have seen that the cuspidal cubics are parametrized by a hypersurface of degree 3 in $\mathbf{P}^8$.

Proposition 3.1. *The cubics with cusp (resp. flex) on a given line are parametrized by a 6-fold of degree 3 in* $\mathbf{P}^8$.

Proof. Assume the line is given by $x = 0$. By the first part of (2.5) we find the desired locus to be given by $bc + gi = cf + gh = fi + bh = 0$ which the computer program Macaulay tells us has degree 3 and codimension 2 in $\mathbf{P}^8$. The case with the flex is similar. □

Proposition 3.2. *Let* $L \subset \mathbf{P}^8$ *parametrize the nonreduced cubics. Then* L *is given by* $\mathrm{rk}H = 1$ *, and equals the image of the Segre embedding* $r \colon \mathbf{P}^2 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^8$.

Proof. That L is given by $\mathrm{rk}H = 1$ is immediate from (2.2). Hence L is a determinantal variety, and it is well known that such a variety is the image of a Segre embedding, in this case $r \colon \mathbf{P}^2 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^8$. □

4. The dual map. Let C be a nonreduced cubic given by $F(x,y,z) = ax^3 + by^3 + \ldots + jxyz$. We now want to find an equation for the dual curve $\check{C}$. Introduce affine coordinates $x' = \frac{x}{z}, y' = \frac{y}{z}$, and consider the affine curve $C \subset \mathbf{A}^2$ given by $f(x',y') = F(\frac{x}{z}, \frac{y}{z}, 1) = 0$. For each line $\{y' = mx' + t\} \subset \mathbf{A}^2$ tangent to C we associate the point $(m,t) \in \check{\mathbf{A}}^2$. We need to find all such points.

Substitute $mx' + t$ for y' in the affine equation for C and obtain a polynomial $g(x') = f(x', mx' + t)$. Now the line $y' = mx' + t$ is a tangent exactly when $g(x')$ has multiple roots, and that happens when the discriminant $\Delta g(x')$ is zero.

So the affine equation of $\check{C}$ in $\check{\mathbf{A}}^2$ is $\Delta g(x') = 0$. In characteristic 2 the discriminant of $ax^3 + bx^2 + cx + d$ is $ad + bc$. $(m,t) \in \check{\mathbf{A}}^2$ corresponds to $(m,1,t) \in \check{\mathbf{P}}^2$ so if we use (x,y,z) as homogeneous coordinates for $\check{\mathbf{P}}^2$ we obtain the following homogeneous equation for $\check{C}$:

$$\begin{aligned}
&(bc + gi)x^3 + (ac + eh)y^3 + (ab + df)z^3 + (cf + gh + ij)x^2y \\
&+(fi + bh + gj)x^2z + (cd + ei + hj)xy^2 + (ai + dh + ej)y^2z \\
&+(dg + be + fj)xz^2 + (ag + ef + dj)yz^2 + j^2xyz = 0
\end{aligned}$$

Restricting to the $\mathbf{P}^8$ of j-curves we obtain the dual map $d : \mathbf{P}^8 \to \check{\mathbf{P}}^8$ associating to each j-curve (in fact each reduced curve) its dual. This map is given by

$$(a,b,c,\ldots,i) \mapsto (bc+gi, ac+eh, ab+df, cf+gh, fi+bh, cd+ei, ai+dh, dg+be, ag+ef)$$

5. Characteristic numbers. In this section, which is independent of the characteristic of the base field k, we define the characteristic numbers and give their basic properties. Let $\mathbf{P}^{\frac{1}{2}d(d+3)}$ parametrize all curves of degree d, and let $\mathbf{P}^N \subseteq \mathbf{P}^{\frac{1}{2}d(d+3)}$ be a linear subspace. We will later study the $\mathbf{P}^8$ of j-curves inside the $\mathbf{P}^9$ of all cubics.

Definition. *A point condition on* $\mathbf{P}^N$ *is a hyperplane* H *parametrizing the curves containing a given point. A line condition is a hypersurface* M *parametrizing the curves tangent to a given line.*

Let $R \subset \mathbf{P}^N$ be a r-dimensional subvariety parametrizing a family of generically reduced curves. Suppose we have α points and β lines in general position, with $\alpha+\beta=r$. Let H_i and M_j be the corresponding point and line conditions on $\mathbf{P}^N$ and let $L \subset \mathbf{P}^N$ parametrize the nonreduced curves.

Definition. *We define the total characteristic numbers for R to be:*

$$\Gamma_{\alpha,\beta} = \sum_{P \in \mathbf{P}^N \setminus L} i(P, R \cdot H_1 \cdots H_\alpha \cdot M_1 \cdots M_\beta)$$

that is the weightet number of reduced curves passing through the α points and tangent to the β lines.

The characteristic numbers for R are defined to be:

$$N_{\alpha,\beta} = \sum_{P \in Q} i(P, R \cdot H_1 \cdots H_\alpha \cdot M_1 \cdots M_\beta)$$

where $Q = \{x \in \mathbf{P}^N : (singC_x) \cap l_j = \varnothing \;\; \forall j\}$. ($C_x$ is the curve parametrized by x, and l_j is one of the β lines.) $N_{\alpha,\beta}$ counts the curves passing through the α points and properly tangent to the β lines.

It is shown in [4, section 2] that these numbers are well defined and that the curves counted by a given characteristic number $N_{\alpha,\beta}$ all appear with the same multiplicity $m = p^e$, a power of the characteristic ($m = 1$ in characteristic zero).

Lemma 5.1. *Suppose $S \subset \mathbf{P}^N$ is an irreducible curve parametrizing generically irreducible curves, and let $x \in S$ be a general point. Then there exist at most finitely many point conditions H_p tangent to S at x.*

Proof. Let T be the tangent line to S at x. Then: H_p is tangent to S at $x \Rightarrow T \subset H_p \Rightarrow p \in C_y \;\; \forall y \in T$. Clearly only a finite number of such p can exist. □

Proposition 5.2. *Suppose $R \subset \mathbf{P}^N$ is a subvariety parametrizing a family of genrerically irreducible curves. Then a general point condition will intersect R transversally (by transversal we always mean that the scheme theoretical intersection has no nonreduced components).*

Proof. Since the set of points p such that H_p does not have the required property is closed it is enough to show the existence of one H_p that has. Suppose that all point conditions intersect R in a nonreduced component. Then the union of these components will cover R. Let $S \subset R$ be a general curve, and let $x \in S$ be a general point. Since the set of point conditions is 2-dimensional there will be infinitely many point conditions tangent to R at x. These will also be tangent to S, contradicting the lemma. □

Proposition 5.3. *Suppose $R \subset \mathbf{P}^N$ is a subvariety parametrizing a family of generically irreducible curves. Then the characteristic numbers $N_{\alpha,\beta}$ for R will count curves with a nondecreasing multiplicity as β increases.*

Proof. Let $H_1, ..., H_\alpha$ and $M_1, ..., M_\beta$ be general point and line conditions. We know that the points in $R \cap H_1 \cap .. \cap H_\alpha \cap M_1 \cap .. \cap M_\beta$ counted by $N_{\alpha,\beta}$ all appear with the same multiplicity m. If we remove one of the point conditions, then by (5.2) the components containing the above mentioned points will also have multiplicity m. When these components are intersected with a line condition we see that all the points in the new intersection must have multiplicity at least m. ☐

Now suppose that R consists of singular curves and the general curve has exactly one singularity. Denote by R^l the curves singular on a given line l, and R^p those with singularity at a given point p.

Definition. *Suppose $\alpha + \beta = r - 1$. Define the following numbers associated to the family R:*
$\Gamma^l_{\alpha,\beta}$ = *the total characteristic numbers for R^l*
$N^l_{\alpha,\beta}$ = *the characteristic numbers for R^l*
Suppose $\alpha + \beta = r - 2$. Then define:
$N^p_{\alpha,\beta}$ = *the characteristic numbers for R^p*

6. The variety of complete j-curves. We will now construct the variety of complete j-curves parallel to the construction of complete conics in [11].

In section 4 we calculated the dual map $d\colon \mathbf{P}^8 \to \check{\mathbf{P}}^8$ associating to each reduced curve C (with $j = 0$) its dual $\check{C}$. Let $B \subset \mathbf{P}^8 \times \check{\mathbf{P}}^8$ be the closure of the graph of the dual map. Then B is the blow-up of $\mathbf{P}^8$ in the ideal $(bc+gi, ac+eh, ..., ag+ef)$. But this is the ideal of L (rk$H = 1$), so B is the blow-up of $\mathbf{P}^8$ along L. We call B the variety of complete j-curves.

Let $E \subseteq B$ be the exeptional divisor, and denote by $\tilde{H}$ and $\tilde{M}$ the strict transforms of point and line conditions on $\mathbf{P}^8$. $\tilde{H}$ and $\tilde{M}$ will be called point and line conditions on B.

Proposition 6.1. *The intersection of all the line conditions on B is empty.*

Proof. Let $(a, b, .., i) \times (\bar{a}, \bar{b}, .., \bar{i})$ be coordinates on $\mathbf{P}^8 \times \check{\mathbf{P}}^8$, and let $\pi : B \to \mathbf{P}^8$ be the projection on the first factor. Let $M \subset \mathbf{P}^8$ be the line condition $bc + gi = 0$ (corresponding to the line $x = 0$), and let $N \subset B$ be given by $\bar{a} = 0$ (the inverse image of the pointcondition in $\check{\mathbf{P}}^8$ dual to the line condition $M \subset \mathbf{P}^8$). On $B \backslash E$ we have: $\bar{a} = 0 \Leftrightarrow bc + gi = 0$ so $\pi^{-1}(M \backslash L) \subset N$. Then $\tilde{M} = \overline{\pi^{-1}(M \backslash L)} \subset N$ since N is closed. N is the inverse image of a point condition on $\check{\mathbf{P}}^8$, and the intersection of these is empty. Since $\tilde{M} \subset N$ it follows that the intersection of the line conditions on B is also empty. ☐

Proposition 6.2. *Suppose $R \subset \mathbf{P}^8$ is a subvariety parametrizing a family of generically irreducible curves. Then the total characteristic numbers for R are given by*

$$\Gamma_{\alpha,\beta} = \int_B [\tilde{R}][\tilde{H}]^\alpha[\tilde{M}]^\beta \quad \textit{with } \alpha+\beta = r = dimR$$

Proof. Let $H_1, ..., H_\alpha$ and $M_1, ..., M_\beta$ be general point and line conditions on $\mathbf{P}^8$. Since $\pi : B \longrightarrow \mathbf{P}^8$ restricts to an isomorphism $\pi\,|_{B\backslash E} : B\backslash E \tilde{\longrightarrow} \mathbf{P}^8\backslash L$ it will be sufficient to show that $\tilde{R} \cap \tilde{H}_1 \cap .. \cap \tilde{H}_\alpha \cap \tilde{M}_1 \cap .. \cap \tilde{M}_\beta$ does not intersect E. Since the general curve in R is reduced we can assume that $\dim(\tilde{R} \cap E) \le r-1$. The result follows if we can show that a general $\tilde{M}$ intersects a given irreducible subvariety $V \subset B$ properly. (The case with the point conditions $\tilde{H}$ is simpler.) The set $\{l \in \check{\mathbf{P}}^2 : V \subset \tilde{M}_l\}$ is closed, and by (6.1) this set is not all of $\check{\mathbf{P}}^2$. It follows that the general line condition does not contain V, so the intersection is proper. $\square$

7. An intersection formula. This section contains the key formula for calculating the characteristic numbers.

Proposition 7.1. *Let $L \subset \mathbf{P}^n$ be a nonsingular variety of dimension l, denote by i the inclusion, N its normal bundle, and $\pi : B \longrightarrow \mathbf{P}^n$ the blow up of $\mathbf{P}^n$ along L. Let E be the exceptional divisor, and let $H \subset \mathbf{P}^n$ be a hyperplane not containing L. Then:*

$$\int_B [\tilde{H}]^\alpha[E]^\beta = \int_L s(N)(i^*[H])^\alpha(-1)^{\beta-1}$$

where $\alpha+\beta = n$ and $\beta \ge 1$

Proof. Denote by j the inclusion of E in B, and let p be the restriction of π to E. By definition of Segre classes [5, section 3.1] we have $s_i(N) = p_*(c_1(\mathcal{O}_E(1))^{n+i-l-1})$, and since the ideal sheaf of E in B is $\mathcal{O}_B(1)$ it follows that $j^*[E] = c_1(\mathcal{O}_E(-1))$. Also, since $L \not\subseteq H$ we have $\pi^*[H] = [\tilde{H}]$ so that $j^*[\tilde{H}] = j^*\pi^*[H] = p^*i^*[H]$.

$$\begin{aligned}
\int_B [\tilde{H}]^\alpha[E]^\beta &= \int_E j^*([\tilde{H}]^\alpha[E]^{\beta-1}) = \int_E (j^*[\tilde{H}])^\alpha(j^*[E])^{\beta-1} \\
&= \int_L p_*((j^*[\tilde{H}])^\alpha(j^*[E])^{\beta-1}) = \int_L p_*((p^*i^*[H])^\alpha(j^*[E])^{\beta-1}) \\
&= \int_L (i^*[H])^\alpha p_*((j^*[E])^{\beta-1}) \quad \text{by the projection formula} \\
&= \int_L (i^*[H])^\alpha p_*(c_1(\mathcal{O}_E(-1))^{\beta-1}) = \int_L i^*[H]^\alpha p_*(c_1(\mathcal{O}_E(1))^{\beta-1})(-1)^{\beta-1} \\
&= \int_L (i^*[H])^\alpha s_{\beta+l-n}(N)(-1)^{\beta-1} = \int_L s(N)(i^*[H])^\alpha(-1)^{\beta-1} \quad \square
\end{aligned}$$

8. Characteristic numbers for j-curves. By the results of section 6 we know that the characteristic numbers for j-curves are given by $N_{\alpha,\beta} = \int_B [\tilde{H}]^\alpha [\tilde{M}]^\beta$ with $\alpha + \beta = 8$. First we calculate the degree of M. Without loss of generality we can assume that l is the line given by $x = 0$. A curve C given by $ax^3 + by^3 + ... + iyz^2 = 0$ is tangent to l exactly when $by^3 + gy^2z + iyz^2 + cz^3$ has multiple roots, and that happens exactly when the discriminant $bc + gi$ is zero. So M_l is given by $bc + gi = 0$ and has degree 2. Also note that the singular locus of M_l is given by $b = c = g = i = 0$, that is, the cubics having l as a component. In particular M_l is nonsingular along L.

Lemma 8.1. *In the intersection ring $A(B)$ we have: $[\tilde{M}] = 2[\tilde{H}] - [E]$*

Proof. We know that deg$M = 2$, L is nonsingular and is contained in M with multiplicity 1. Hence $[\tilde{M}] = \pi^*[M] - [E] = \pi^*[2H] - [E] = 2[\tilde{H}] - [E]$.
In the last equality we have used that $L \not\subseteq H$ so that $\tilde{H} = \pi^{-1}H$. $\square$

This reduces our problem to calculate the numbers $\int_B [\tilde{H}]^\alpha [E]^\beta$ with $\alpha + \beta = 8$.
By (7.1) we have

$$\int_B [\tilde{H}]^\alpha [E]^\beta = \int_L s(N)(i^*[H])^\alpha (-1)^{\beta-1}$$

where N is the normal bundle of the Segre embedding $r\colon L \simeq \mathbf{P}^2 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^8$ and $\beta \geq 1$. The Segre class of this normal bundle is not hard to calculate. By [5], Example 3.2.15, we find that[1]

$$\begin{aligned} s(N) &= \frac{(1+h_1)^3(1+h_2)^3}{(1+h_1+h_2)^{10}} \\ &= 1 - 6h_1 - 6h_2 + 21h_1^2 + 21h_2^2 + 45h_1h_2 - 189h_1^2h_2 - 189h_1h_2^2 + 927h_1^2h_2^2 \end{aligned}$$

where $h_i = c_1(pr_i^* \mathcal{O}_{\mathbf{P}^2}(1))$ and $pr_i\colon \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^2$ is the projection on the i^{th} factor.

Proposition 8.2. *Let $I_\beta = \int_B [\tilde{H}]^\alpha [E]^\beta$. Then*

$I_0 = 1$	$I_3 = 0$	$I_6 = -132$
$I_1 = 0$	$I_4 = -6$	$I_7 = -378$
$I_2 = 0$	$I_5 = -36$	$I_8 = -927$

Proof. $I_0 = \int_B [\tilde{H}]^8 = N_{8,0}$, which is clearly 1.
For $\beta \geq 1$ we have

$$I_\beta = \int_L s(N)(i^*[H])^\alpha (-1)^{\beta-1} = \int_L (1 - 6h_1 - 6h_2 + ... + 927h_1^2h_2^2)(h_1+h_2)^\alpha (-1)^{\beta-1}$$

[1] Note the mistake on top of page 61 in [5]. $c(N)$ should be $s(N)$ or the formula has to be inverted.

The rest is simple calculation. For example

$$I_6 = -\int_L (1 - 6h_1 - 6h_2 + 21h_1^2 + 21h_2^2 + 45h_1h_2)(h_1^2 + h_2^2 + 2h_1h_2)$$
$$= -\int_L (21 + 21 + 90)h_1^2h_2^2 = -132$$

We have used $h_i^3 = 0$ and $\int_L h_1^2h_2^2 = 1$ $\square$

Proposition 8.3. *The characteristic numbers for j-curves are:*

$N_{8,0} = 1$ $N_{5,3} = 8$ $N_{2,6} = 4$
$N_{7,1} = 2$ $N_{4,4} = 10$ $N_{1.7} = 2$
$N_{6,2} = 4$ $N_{3,5} = 8$ $N_{0,8} = 1$

and all the numbers above count curves with multiplicity 1.

Proof. We use (8.1) and the numbers from (8.2). For example:

$$N_{3,5} = \int_B [\tilde{H}]^3[\tilde{M}]^5 = \int_B [\tilde{H}]^3(2[\tilde{H}] - [E])^5$$
$$= \int_B 32[\tilde{H}]^8 + 10[\tilde{H}]^4[E]^4 - [\tilde{H}]^3[E]^5 = 32 + 10 \cdot (-6) - (-36) = 8$$

Since the last number, $N_{0,8}$, clearly counts curves with multiplicity 1, it follows from (5.3) that all the other numbers will also count curves with multiplicity 1. $\square$

9. Cubics with cusp at a given point. Let $P \subset \mathbf{P}^8$ parametrize curves with cusp at a given point. If this point is $(0, 0, 1)$ we see that P is given by $c = h = i = 0$, so $P \simeq \mathbf{P}^5$. We now wish to calculate the characteristic numbers for P:

$$N^p_{\alpha,\beta} = \int_B [\tilde{P}][\tilde{H}]^\alpha[\tilde{M}]^\beta = \int_{\tilde{P}} [\tilde{H}]^\alpha[\tilde{M}]^\beta \quad \text{with } \alpha + \beta = 5$$

(In the last expression H and M are conditions on P.)

Proposition 9.1. *$\tilde{P}$ is the blow up of P along $L' = L \cap P$, and L' is the image of the Segre embedding $r\colon \mathbf{P}^2 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^5 \simeq P$.*

Proof. The first assertion follows from the theory of blow-ups. Suppose P is given by $c = h = i = 0$ as above. Then $L' \subset \mathbf{P}^8$ is given by

$$c = h = i = 0 \text{ and } rk\begin{pmatrix} a & f & h \\ d & b & i \end{pmatrix} = 1$$

So $L' \subset \mathbf{P}^5$ is a determinantal variety and equals the image of the Segre embedding $r\colon \mathbf{P}^2 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^5$. $\square$

The rest of this section is analogous to the calculation of the characteristic numbers for j-curves. We just give the results:

Let N be the normal bundle of $r\colon \mathbf{P}^2 \times \mathbf{P}^1 \longrightarrow \mathbf{P}^5$. Then:
$s(N) = 1 - 3h_1 - 4h_2 + 6h_1^2 + 18h_1h_2 - 48h_1^2h_2$.

Let $T_\beta = \int_{\tilde{P}}[\tilde{H}]^\alpha[E]^\beta$. As in (8.2) we find that:

$T_0 = 1 \quad T_3 = -10$
$T_1 = 0 \quad T_4 = -24$
$T_2 = -3 \quad T_5 = -48$

As in (8.1) we have $[\tilde{M}] = 2[\tilde{H}] - [E]$ in $A(\tilde{P})$.

Proposition 9.2. *The characteristic numbers for P are:*

$N_{5,0} = 1 \quad N_{2,3} = 0$
$N_{4,1} = 2 \quad N_{1,4} = 0$
$N_{3,2} = 1 \quad N_{0,5} = 0$ □

10. Characteristic numbers for cuspidal cubics. Our aim in this section is to make a complete list of the numbers $\Gamma_{\alpha,\beta}$, $N_{\alpha,\beta}$, $\Gamma^l_{\alpha,\beta}$, $N^l_{\alpha,\beta}$ and $N^p_{\alpha,\beta}$ associated to the cuspidal cubics.

Let $K \subseteq \mathbf{P}^8$ parametrize the cuspidal cubics. From (2.4) we know that K is given by $\det H = 0$ and has degree 3. By computing the partial derivatives of $\det H$ we see that the singular locus of K is exactly L. Let m be the multiplicity of K along L. As in (8.1), we see that $[\tilde{K}] = 3[\tilde{H}] - m[E]$ in $A(B)$, where $\tilde{K}$ is the strict transform of K. Also, by direct computation or using Macaulay we find that $\tilde{K}$ is nonsingular.

Lemma 10.1. *The multiplicity m of K along L is 2.*

Proof. No cuspidal cubic can be tangent to 5 given lines in general position (each tangent must contain either the cusp or the flex). It follows that $\Gamma_{2,5} = 0$. But

$$\begin{aligned}\Gamma_{2,5} &= \int_B [\tilde{K}][\tilde{H}]^2[\tilde{M}]^5 = \int_B (3[\tilde{H}] - m[E])[\tilde{H}]^2(2[\tilde{H}] - [E])^5 \\ &= 96I_0 + (30 + 40m)I_4 - (3 + 10m)I_5 + mI_6 = 24 - 12m\end{aligned}$$

We have used the numbers I_β from (8.2). It follows that $m = 2$. □

Proposition 10.2. *The total characteristic numbers for cuspidal cubics are:*

$\Gamma_{7,0} = 3 \quad \Gamma_{3,4} = 6$
$\Gamma_{6,1} = 6 \quad \Gamma_{2,5} = 0$
$\Gamma_{5,2} = 12 \quad \Gamma_{1,6} = 0$
$\Gamma_{4,3} = 12 \quad \Gamma_{0,7} = 0$

Proof. The calculation goes as in the lemma. For example

$$\Gamma_{4,3} = \int_B [\tilde{K}][\tilde{H}]^4[\tilde{M}]^3 = \int_B (3[\tilde{H}] - m[E])[\tilde{H}]^4(2[\tilde{H}] - [E])^3 = 24I_0 + 2I_4 = 12 \quad \square$$

Let M be a line condition on K. Then M has two components, F and G, where F parametrizes the curves properly tangent to the line and G the curves with the cusp on the line.

Lemma 10.3. *In the intersection ring $A(\tilde{K})$ we have:*
(1) $[\tilde{G}]^3 = [\tilde{F}]^3 = 0$
(2) $[\tilde{P}] = [\tilde{G}]^2$
(3) $[\tilde{M}] = [\tilde{F}] + [\tilde{G}]$

Proof. (1) The intersection of three general $\tilde{G}'s$ is obviously empty outside E. Since the general line condition on $\tilde{K}$, and in particular $\tilde{G}$, intersects every subvariety properly (see the proof of 6.2), there cannot be any intersection inside E (the dimension will be too small). This shows that $[\tilde{G}]^3 = 0$, and similarily we have $[\tilde{F}]^3 = 0$.
(2) $[\tilde{P}] = [\tilde{G}]^2$ follows if we can show that two $\tilde{G}'s$ intersect transversally outside E (as above the intersection cannot have any components inside E), or equivalently that two $G's$ intersect transversally outside L. But for this it is sufficient to show that $M_1 \cap M_2 \cap K$ is transversal for points in $G_1 \cap G_2$ ($G_i \subseteq M_i \cap K$) where M_1 and M_2 are line conditions on $\mathbf{P}^8$ corresponding to lines l_1 and l_2. Without loss of generality we can assume that l_1 and l_2 are given by $x = 0$ and $y = 0$ respectively. As in section 8 we find that M_1 and M_2 are given by $bc + gi = 0$ and $ac + eh = 0$. It is now easy to calculate the tangent planes for points in $G_1 \cap G_2$ outside L (these points satisfy $c = h = i = 0$), and we find these planes to intersect transversally, so we have $[\tilde{P}] = [\tilde{G}]^2$.
(3) $[\tilde{M}] = [\tilde{F}] + [\tilde{G}]$ follows since $M = F \cup G$ in K. $\square$

Lemma 10.4. *The following relations hold:*
$\Gamma_{\alpha,\beta} = N_{\alpha,\beta} + \beta N^l_{\alpha,\beta-1} + \binom{\beta}{2} N^p_{\alpha,\beta-2}$
$\Gamma^l_{\alpha,\beta} = N^l_{\alpha,\beta} + \beta N^p_{\alpha,\beta-1}$

Proof. This is simple calculation using (10.3) and that $\tilde{F}$ is the condition of being properly tangent. For example:

$$\begin{aligned}\Gamma_{\alpha,\beta} &= \int_B [\tilde{K}][\tilde{H}]^\alpha[\tilde{M}]^\beta = \int_{\tilde{K}} [\tilde{H}]^\alpha[\tilde{M}]^\beta = \int_{\tilde{K}} [\tilde{H}]^\alpha([\tilde{F}] + [\tilde{G}])^\beta \\ &= \int_{\tilde{K}} [\tilde{H}]^\alpha([\tilde{F}]^\beta + \beta[\tilde{F}]^{\beta-1}[\tilde{G}] + \binom{\beta}{2}[\tilde{F}]^{\beta-2}[\tilde{G}]^2) \\ &= N_{\alpha,\beta} + \beta N^l_{\alpha,\beta-1} + \binom{\beta}{2} N^p_{\alpha,\beta-2} \quad \square\end{aligned}$$

Now look at $\Gamma^l_{\alpha,\beta}$ which is defined by

$$\Gamma^l_{\alpha,\beta} = \sum_{P \in \mathbf{P}^8 \setminus L} i(P, G \cdot H_1 \cdots H_\alpha \cdot M_1 \cdots M_\beta) \ \ \textit{with} \ \ \alpha + \beta = 6$$

When $\alpha \geq 5$ only reduced curves will appear in the intersection and we can use Bezout:

$$\Gamma_{6,0} = 3 \quad \Gamma_{5,1} = 6$$

Since $[\tilde{F}]^3 = [\tilde{G}]^3 = 0$ we have that

$N_{\alpha,\beta} = 0$ when $\beta \geq 3$
$\Gamma^l_{\alpha,\beta} = 0$ when $\beta \geq 4$
$N^l_{\alpha,\beta} = 0$ when $\beta \geq 3$

We now have enough information to fill in the table below. Start by filling in the numbers $\Gamma_{\alpha,\beta}$ and $N^p_{\alpha,\beta}$ already calculated. Then add all the zeros and the two numbers $\Gamma_{6,0}$ and $\Gamma_{5,1}$ from above. The table is then completed by repeatedly using (10.4).

Proposition 10.5. *The following table summarizes the results of this section, and all the numbers count curves with multiplicity 1.*

α,β	$\Gamma_{\alpha,\beta}$	$N_{\alpha,\beta}$	$\Gamma^l_{\alpha,\beta-1}$	$N^l_{\alpha,\beta-1}$	$N^p_{\alpha,\beta-2}$
7,0	3	3			
6,1	6	3	3	3	
5,2	12	1	6	5	1
4,3	12	0	6	2	2
3,4	6	0	3	0	1
2,5	0	0	0	0	0
1,6	0	0	0	0	0
0,7	0	0	0	0	0

Proof. We only need to show the last part. By (5.3) this holds for $N_{\alpha,\beta}$ and $N^p_{\alpha,\beta}$. If we can show that $N^l_{4,2}$ count two different curves the proposition will be true for $N^l_{\alpha,\beta}$ and then by (10.4) also for $\Gamma_{\alpha,\beta}$ and $\Gamma^l_{\alpha,\beta}$.

$N^l_{4,2}$ counts the curves passing through 4 given points, with the flex at another given point p (the intersection of the two lines) and with cusp on a given line l. The curves counted by $N^p_{4,1}$ have the same description with cusp and flex interchanged. Suppose $p = (0, 0, 1)$ and l is given by $z = 0$. Let C be a curve counted by $N^p_{4,1}$. Let H be its matrix and let C^t be the cuspidal cubic given by H^t. Since $\mathrm{cof}(H^t) = (\mathrm{cof}\, H)^t$ it follows from (2.5) that C^t has a flex at p and a cusp on l. So C^t is counted by $N^l_{4,2}$. Since $N^p_{4,1}$ counts different curves then also $N^l_{4,2}$ must count different curves. □

References

1. P.Aluffi, *The enumerative geometry of plane cubics I: Smooth cubics*, Trans.AMS **317** (1990), 501-539.
2. P.Aluffi, *The enumerative geometry of plane cubics II: Nodal and cuspidal cubics*, Math.Ann. **289** (1991), 543-572.
3. E.Briescorn, H.Knörrer, *Ebene algebraische Kurven*, Birkhäuser (1981).
4. W.Fulton, S.Kleiman, R.MacPherson, *About the Enumeration of Contacts*, Lecture Notes in Mathematics **997** (1983), 156-196.
5. W.Fulton, *Intersection Theory*, Springer (1984).

6. R.Hartshorne, *Algebraic Geometry*, Springer (1977).
7. S.Kleiman, R.Speiser, *Enumerative geometry of cuspidal plane cubics*, Canad.Math.Soc.Conf. Proc. **6** (1986), 227-268.
8. S.Kleiman, R.Speiser, *Enumerative geometry of nodal plane cubics*, Lecture Notes in Mathematics **1311** (1988), 156-196.
9. S.Kleiman, R.Speiser, *Enumerative geometry of nonsingular plane cubics*, Cont.Math. **116** (1991), 85-113.
10. D.Laksov, *Wronskians and Plücker formulas*, Ann.Ec.Norm.Sup. **17** (1984), 45-66.
11. I.Vainsencher, *Conics in Characteristic 2*, Compositio Mathematica **36** (1978), 101-112.