$K_4(\mathbb{Z})$ IS THE TRIVIAL GROUP

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ABSTRACT. The fourth higher algebraic K-group of the rational integers is the trivial group. We prove this using a convergence result for the spectrum level rank filtration in algebraic K-theory, combined with previous calculations of the third stage of this filtration.

1. INTRODUCTION

We prove the following

Theorem. $K_4(\mathbb{Z}) = 0$.

This is in agreement with the value predicted by the Lichtenbaum–Quillen conjectures [Li], [Qu3], as extended to the two–primary part [Dw–Fr], [Mit3]. Previously the group was known to be a finite two– and three–torsion group, with three–torsion 0 or $\mathbb{Z}/3$ [Qu2], [Le–Sz2], [Sou]. Our result supplements the known computations $K_1(\mathbb{Z}) \cong \mathbb{Z}/2$, $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$ [Mil], and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ [Le–Sz1].

Our argument relies on two preceding papers by the author. In [Ro1] we constructed a spectrum level rank filtration $\{F_k K(R)\}_k$ of spectra approximating the algebraic K-theory spectrum $K(R)$ of a ring R, for suitable R. Roughly, the kth stage of the filtration is the prespectrum built from the category of free finitely generated R-modules of rank less than or equal to k.

In [Ro2] (Theorem 1.1) we calculated the spectrum homology of the third stage of this filtration in the case $R = \mathbb{Z}$, obtaining

$$H_*^{spec}(F_3 K(\mathbb{Z})) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z} \oplus (\#|4), \dots),$$

where $(\#|4)$ is a group of order dividing four.

In [Ro1] (Conjecture 12.3) we conjectured that each inclusion $F_{k-1}K(R) \to F_k K(R)$ is $(2k-3)$-connected for suitable R. In particular $F_3 K(\mathbb{Z}) \to K(\mathbb{Z})$ would be five–connected, and the spectrum homology computation would imply $K_4(\mathbb{Z}) = 0$ and $K_5(\mathbb{Z}) \cong \mathbb{Z} \oplus (\#|8)$, where $(\#|8)$ is a group of order dividing eight. In Section 4 of this paper we prove enough of this connectivity conjecture to conclude that $F_3 K(\mathbb{Z}) \to K(\mathbb{Z})$ is four–connected. Hence the spectrum homology of $K(\mathbb{Z})$ begins $(\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \dots)$.

A complete calculation of $K_*(\mathbb{Z})$ through degree four then follows using the Atiyah–Hirzebruch spectral sequence for stable homotopy. The inputs needed to determine the differentials and extensions in this spectral sequence through total

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degree four are the fact that the image of the J-homomorphism injects into $K_*(\mathbb{Z})$ in low degrees [Qu4], [Mit2], and a comparison with Bökstedt's $JK(\mathbb{Z})$ [Bö]. The argument is described in Section 2 of this paper.

In Section 3 we recall some needed constructions from [Ro1]. In Section 5 we indicate some applications.

We hope that our result will assist a sharper evaluation of the plausibility of the two–primary Lichtenbaum–Quillen conjectures on the values of the higher algebraic K-groups of rings of integers in a number field.

2. K-THEORY OF THE INTEGERS

In this section we show how to prove the theorem $K_4(\mathbb{Z}) = 0$ by following the outline from the introduction. First we recall the rank filtration and stable buildings from [Ro1], and formulate the connectivity conjecture. Next we show how Propositions 1, 2, and 3 of Section 4 about the connectivity of stable buildings for the rational numbers imply vanishing results in the spectrum homology spectral sequence associated to the rank filtration of $K(\mathbb{Z})$. The outcome is that the inclusion $F_3K(\mathbb{Z}) \to K(\mathbb{Z})$ is at least four–connected. Next we recall from [Ro2] the computation of $H_*^{spec}(F_3K(\mathbb{Z}))$, which determines the spectrum homology of $K(\mathbb{Z})$ through degree four. Finally we set up the Atiyah–Hirzebruch spectral sequence in stable homotopy for $K(\mathbb{Z})$, and evaluate it through total degree four by comparison with the analogous spectral sequence for Bökstedt's $JK(\mathbb{Z})$.

Recall from [Ro1], Section 3 that the spectrum level *rank filtration* $\{F_kK(R)\}_k$ is defined for rings R satisfying the *strong invariant dimension property* [Mit1] that R^i split injects into R^j only if $i \leq j$. This filtration exhausts the free K-theory spectrum of R, i.e. that built from the category of free finitely generated R-modules. We will denote this spectrum $K(R)$ here, suppressing any extra contribution in $K_0(R)$ from projective modules, which is not present for $R = \mathbb{Z}$.

Roughly, using Waldhausen's $S_\bullet$-construction [Wa] we can take a model for the spectrum $K(R)$ with nth space the realization of a simplicial category with objects certain n-dimensional cubical diagrams of free R-modules. Then the kth prespectrum $F_kK(\mathbb{Z})$ is the subspectrum of $K(\mathbb{Z})$ with nth space the realization of the full simplicial subcategory of diagrams involving only modules of rank k or less.

In [Ro1], Proposition 3.8 we proved that the kth subquotient $F_kK(R)/F_{k-1}K(R)$ is equivalent to the suspension spectrum on the reduced homotopy orbit space of a $GL_k(R)$-space $D(R^k)$ called the rank k *stable building*. In particular, $F_{k-1}K(R) \to F_kK(R)$ is at least as highly connected as $D(R^k)$. We will recall a precise definition of the stable building in Section 3 of this paper.

By [Ro1], Theorem 12.1, the homology of $D(R^k)$ is concentrated in degrees 0 through $(2k-2)$, and the connectivity Conjecture 12.3 asserts that $D(R^k)$ is $(2k-3)$-connected for R local or a Euclidean domain, hence of the homotopy type of a wedge of $(2k-2)$-spheres.

We now turn to applying the results of Section 4 of this paper. They provide connectivity results for the stable buildings for fields, which we will use in the case $R = \mathbb{Q}$, the rational numbers.

As a corollary to Proposition 1 we will note that $D(R^k)$ is $(k-2)$-connected for rings R such that the Tits building $B(R^k)$ is homotopy equivalent to a wedge of $(k-2)$-spheres. This places the spectrum homology spectral sequence associated to the rank filtration

$$E^1_{s,t} = H^{spec}_{s+t}(F_{s+1}K(R)/F_sK(R)) \Longrightarrow H^{spec}_{s+t}(K(R))$$

in the first quadrant.

In Proposition 2 we will show that if R is a field, and $D(R^k)$ is $(k+c-2)$-connected for some $1 \le c \le k$, then $D(R^{k+1})$ is $(k+c-1)$-connected. The proof exploits the Solomon–Tits decomposition of $B(R^k)$ into a wedge of spheres, and [Ro1], Proposition 11.12. This appears as rows of zero groups extending to the right in the spectrum homology spectral sequence.

Lastly we will prove in Proposition 3 that the connectivity conjecture holds for R a field, when $k = 1, 2$ or 3. The argument in the rank three case uses an explicit chain complex from [Ro1], Lemma 15.3 for computing the homology of $D(R^3)$.

Now note that extension of scalars induces a homeomorphism $D(\mathcal{O}_F^k) \cong D(F^k)$ when $\mathcal{O}_F$ is the ring of algebraic integers in a number field F. This will be clear once we are given the precise definition of the stable building, as extension of scalars induces a bijection between direct sum decompositions into free summands for $\mathcal{O}_F^k$ and F^k.

Thus the conclusions that $D(\mathbb{Q}^2)$ is one–connected and $D(\mathbb{Q}^k)$ is k-connected for $k \ge 3$ imply the same connectivity results for $D(\mathbb{Z}^k)$, and hence for the inclusions $F_{k-1}K(\mathbb{Z}) \to F_kK(\mathbb{Z})$.

In [Ro2], Proof of Theorem 1.1, we computed the spectrum homology of the first three subquotients of the rank filtration of $K(\mathbb{Z})$, and the d^1-differentials terminating below total degree five in the associated spectral sequence. Combining this with our current vanishing results gives the following E^1-term, where the d^1-differentials originating in bidegrees $(2,4)$ and $(2,5)$ are unknown.

5	$\mathbb{Z}/2$	$\leftarrow (\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	?	?	?
4	0	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	?	?	?
3	$\mathbb{Z}/2$	$\leftarrow (\mathbb{Z}/2)^2$	$\leftarrow \mathbb{Z}$	?	?	?
2	0	$\mathbb{Z}/6$	$\leftarrow \mathbb{Z}/3$	?	?	?
1	$\mathbb{Z}/2$	$\leftarrow \mathbb{Z}/2$	0	0	0	0
0	$\mathbb{Z}$	0	0	0	0	0
	0	1	2	3	4	5

The group $\mathbb{Z}/2$ in bidegree $(2,5)$ was not given in the abovementioned proof, but follows immediately from the diagram preceeding Lemma 6.10 of [Ro2], and that lemma.

Hence $H^{spec}_*(K(\mathbb{Z}))$ begins $(\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}\oplus ?, \dots)$ in view of Borel's rational result [Bor]. Therefore the Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H^{spec}_s(K(\mathbb{Z}); \pi^S_t) \Longrightarrow K_{s+t}(\mathbb{Z})$$

looks like :

6	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?
5	0	0	0	0	0	0
4	0	0	0	0	0	0
3	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?
2	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?
1	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	?
0	$\mathbb{Z}$	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}\oplus$?
	0	1	2	3	4	5

The terms in the zeroth column through degree five survive to E^∞ as the image of J injects into $K_*(\mathbb{Z})$ below degree eight [Qu4], [Mit2].

To establish the first nonzero differential $d^2 \colon H_5^{spec}(K(\mathbb{Z})) \to H_3^{spec}(K(\mathbb{Z});\pi_1^S)$ we compare $K(\mathbb{Z})$ with Bökstedt's spectrum $JK(\mathbb{Z})$. The non–triviality of the extension in $K_3(\mathbb{Z})$ follows as a by-product. At the prime two, $JK(\mathbb{Z})$ is defined in [Bö] as the homotopy fiber of the composite

$$kO \xrightarrow{\psi^3 - 1} bspin \xrightarrow{c} bsu$$

where ψ^3 is the Adams operation and c is complexification. One easily finds $JK_*(\mathbb{Z}) \cong (\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/16, 0, \mathbb{Z}, \dots)$ modulo odd torsion, and $H_*^{spec}(JK(\mathbb{Z})) \cong (\mathbb{Z}, 0, 0, \mathbb{Z}/2, 0, \mathbb{Z}, \dots)$.

There is a canonical map $\Phi \colon K(\mathbb{Z})_2^\wedge \to JK(\mathbb{Z})_2^\wedge$ of two–completed [Bo–Ka] spectra, and Bökstedt constructed a section of two–completed looped underlying spaces

$$f \colon \Omega JK(\mathbb{Z})_2^\wedge \to \Omega K(\mathbb{Z})_2^\wedge$$

with $\Omega\Phi \circ f \simeq 1$. Hence $JK_3(\mathbb{Z}) \cong \mathbb{Z}/16$ splits off $K_3(\mathbb{Z})$ and we recover the result $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ of [Le–Sz1]. Thus Φ is at least four–connected.

Let η denote the Hopf map $S^3 \to S^2$ or its stable class in π_1^S, and let λ denote a generator of $K_3(\mathbb{Z})$ or its image under the Hurewicz homomorphism in $H_3^{spec}(K(\mathbb{Z}))$. Then $\lambda' = \pi_3(\Phi)(\lambda)$ generates $JK_3(\mathbb{Z})$, and $\eta \cdot \lambda' = 0$. As multiplication by η is well defined on π_2, we obtain $\eta \cdot \lambda = \eta \cdot \pi_2(f)(\lambda') = \pi_3(f)(\eta \cdot \lambda') = 0$. Hence the class $\eta \cdot \lambda \in E_{3,1}^2$ cannot survive to E^∞.

Let us remark that Φ thus is an isomorphism on homotopy through degree four, and split surjective in degree five, hence is at least five–connected.

This concludes the proof of our theorem.

3. The stable buildings

In this section we shall review the stable buildings and standard apartments from [Ro1], as well as the poset (partially ordered set) filtration on these spaces.

We recall the standard apartment first. Let $[q] = \{0 < 1 < \cdots < q\}$ be a linearly ordered set with $(q+1)$ elements, taken as an object of the simplex category Δ.

Equivalently we view $[q]$ as a category with $(q+1)$ objects and a unique morphism from i to j whenever $i \leq j$. Let $[q]^n$ be its n-fold cartesian product with itself. This category can be depicted as an n-dimensional cubical diagram, with sides of length q. We also view $[q]^n$ as a partially ordered set (poset), with the product ordering.

We have chosen to name the objects of $[q]^n$ *sites*, calling the maximal object $(q,\ldots,q)$ the *top site*, and the set of sites $(i_1,\ldots,i_n)$ with some $i_s = 0$ the *bottom faces*.

Then the *standard apartment* A^n_k (denoted $A_{n,k}$ in [Ro1]) is the simplicial set with q-simplices the ordered k-tuples of sites $(p_1,\ldots,p_k)$ in $[q]^n$, identified to a base point if some p_s lies in the bottom faces. Such a simplex σ is identified with the diagram on $[q]^n$ of subsets of $\{1,\ldots,k\}$ (i.e. a functor from $[q]^n$ to the category of subsets of $\{1,\ldots,k\}$ and inclusions) which to each site $p = (i_1,\ldots,i_n)$ associates the set of s in $\{1,\ldots,k\}$ with $p_s \leq p$ in the product ordering on $[q]^n$.

Necessarily this diagram has empty sets along the bottom faces, and the full set $\{1,\ldots,k\}$ at the top site. Note that not all diagrams on $[q]^n$ of subsets of $\{1,\ldots,k\}$ appear as q-simplices — only those which qualify as n-fold iterated sequences of cofibrations in the category of finite sets, in the terminology of [Wa]. Those diagrams that do appear characterize the k-tuple $(p_1,\ldots,p_k)$ they come from, as p_s will be the minimal site in $[q]^n$ for which s is an element of the associated subset of $\{1,\ldots,k\}$.

We call $(p_1,\ldots,p_k)$ the *pick sites* of the simplex, and the partial ordering this set inherits as a subset of $[q]^n$ is the poset associated to this simplex. When given this numbering of the pick sites, the associated poset can be viewed as a partial ordering on the abstract set $\{1,\ldots,k\}$, and is then denoted ω_σ.

We think of a poset as a category with objects the underlying set, and at most one morphism between any two objects. In particular we do not insist on reflexivity, i.e. there may be distinct elements in the poset, each less than or equal to the other.

The ith simplicial face map d_i of A^n_k deletes from the diagram all those sites where some coordinate is equal to i, when $i > 0$. The zeroth face map is special, but is characterized by sending nondegenerate simplices to the base point, together with the simplicial identities. The poset associated to $d_i\sigma$ is at least as strong as that associated to σ, i.e. can be obtained from ω_σ by possibly adjoining some new relations.

The jth simplicial degeneracy map duplicates all those sites where some coordinate is equal to j. Phrased differently, in each pick site $p_s = (i_1,\ldots,i_k)$ of a simplex $(p_1,\ldots,p_k)$ every coordinate greater than j is incremented by one. A degeneracy does not alter the poset associated to a simplex.

We can filter the standard apartment by the posets associated to the simplices. Given a partial ordering ω on $\{1,\ldots,k\}$, let $F_\omega A^n_k \subseteq A^n_k$ be the subcomplex of simplices whose associated poset is at least as strong as ω. This is the *poset filtration* on the standard apartment.

We now give a description of the n-dimensional building $D^n(R^k)$, which stably approximates the n-fold suspension of the stable building $D(R^k)$.

The *n-dimensional building* $D^n(R^k)$ is a simplicial set with q-simplices certain n-dimensional cubical diagrams of side length q in the category of free submodules of

R^k and inclusions. We view such a diagram as a functor from $[q]^n$ to this category.

The diagrams which occur are the ones appearing through the following construction : Let $g \in GL_k(R)$ be an invertible matrix, with columns $(g_1, \dots, g_k)$ forming an R-basis for R^k. The matrix g determines a functor from subsets of $\{1, \dots, k\}$ to submodules of R^k by taking a subset $I \subseteq \{1, \dots, k\}$ to the free R-module $g \cdot R^I \subseteq R^k$ generated by the $g_s \in R^k$ with $s \in I$. Left composition with this functor maps a q-simplex σ of the standard apartment A^n_k to a q-simplex $g\sigma$ of $D^n(R^k)$.

Note that for a simplex in $D^n(R^k)$, the R-module occurring at any site is a free direct summand of R^k, with free complementary summand. Also note that the choice of a common set of basis elements $(g_1, \dots, g_k)$ for the submodules of R^k occurring in a simplex is not part of the structure.

The simplicial face and degeneracy maps in $D^n(R^k)$ are defined by deletions and repetitions, as for the standard apartment. A simplex will have zero modules along the bottom faces, and a copy of R^k at the top site.

The reader can equate the definition we have given here with Definition 3.9 of [Ro1], using Lemma 5.7 and Proposition 6.4 of that paper.

The pick sites of a simplex σ in $D^n(R^k)$ are the same as those of any simplex in A^n_k mapping to it, but now as an unordered set. This is because the ordering of a common R-basis for the modules of σ is not well defined. If we are given a choice of such an ordered basis $(g_1, \dots, g_k)$, the sth pick site p_s is the minimal site where g_s is an element of the submodule appearing there.

Chosing a numbering of the k pick sites of a simplex σ, the partial ordering they inherit from $[q]^n$ induces a partial ordering on $\{1, \dots, k\}$, which is well defined up to isomorphism, i.e. up to permutation of the underlying set. We denote this isomorphism class $[\omega_\sigma]$.

There is a *poset filtration* on the n-dimensional building, now indexed by isomorphism classes of partial orderings on $\{1, \dots, k\}$. Let $F_{[\omega]}D^n(R^k) \subseteq D^n(R^k)$ be the subcomplex of simplices with associated poset at least as strong as ω, up to isomorphism.

We now turn to the stable buildings.

The standard apartment A^n_k is included in $D^n(R^k)$ by the embedding induced by the identity matrix in $GL_k(R)$, i.e. taking a subset of $\{1, \dots, k\}$ to the free R-module it generates, viewed as a sum of coordinate axes in R^k. We call such modules *axial* submodules. The translated subcomplexes $g \cdot A^n_k$ of A^n_k in $D^n(R^k)$, where $g \in GL_k(R)$ acts through the induced action on submodules of R^k, are called the *apartments* in the n-dimensional building. It is obvious from our definition that the apartments cover $D^n(R^k)$.

By analyzing the connectivity of the iterated intersections in this covering, we proved in [Ro1], Sections 10 and 14, that the n-dimensional building is stably and naturally homotopy equivalent to the join of an n-sphere and the following complex :

Let $\Sigma^{-1}D(R^k)$ be the complex with q-simplices the $(q+1)$-tuples $\{M_0, \dots, M_q\}$ of proper nontrivial submodules of R^k having a common R-basis, i.e. there exists an R-basis $\mathcal{B}$ for R^k for which each M_s has as subset of $\mathcal{B}$ as R-basis.

We define the *stable building* of rank k, $D(R^k)$ to be the suspension of the complex above, i.e. the join with a zero–sphere. Up to suspensions, the stable building should be viewed as the nerve of the covering of $D^n(R^k)$ by apartments.

We can associate a poset up to isomorphism to each simplex of $\Sigma^{-1}D(R^k)$ as follows : Let $\{M_0, \dots, M_q\}$ be a simplex of $\Sigma^{-1}D(R^k)$, with $M_s = g \cdot R^{I_s}$ for some $I_s \subset \{1, \dots, k\}$ for all s, and a fixed $g \in GL_k(R)$. There is then a unique strongest partial ordering ω on $\{1, \dots, k\}$ for which these subsets I_s are *convex*, i.e. closed under passing to predecessors. Its isomorphism class is well defined irrespective of the choice of basis, and is the poset associated to $\{M_0, \dots, M_q\}$.

There is also a *stable apartment* of rank k, A_k, which is the suspension of the subcomplex of $\Sigma^{-1}D(R^k)$ of simplices $\{M_0, \dots, M_q\}$ where all the M_s are axial.

As above we get a *poset filtration* $\{F_{[\omega]}D(R^k)\}$ of $D(R^k)$, indexed by isomorphism classes of partial orderings on $\{1, \dots, k\}$, and a *poset filtration* $\{F_\omega A_k\}_k$ of A_k. These filtrations are compatible with the poset filtrations on A^n_k and $D^n(R^k)$, as $F_{[\omega]}D(R^k)$ may be viewed as the nerve of the covering of $F_{[\omega]}D^n(R^k)$ by $GL_k(R)$-translates of $F_\omega A$, up to suspensions.

For a fixed rank k, the standard apartments $\{A^n_k\}_n$ and buildings $\{D^n(R^k)\}_n$ assemble into prespectra ([Ro1], Lemma 3.3), which are stably homotopy equivalent to the suspension spectra on A_k and $D(R^k)$ respectively ([Ro1], Theorem 10.9). The spectrum structure maps are compatible with all the poset filtrations. Finally, both the inclusion of the standard apartment into the n-dimensional building, and the stable homotopy equivalences just mentioned, are compatible with the poset filtrations.

This ends our review of terminology from [Ro1].

4. Connectivity results

In this section we shall introduce a coarser filtration on the stable building than the poset filtration, by only counting the number of components of the posets appearing. We relate it to Tits buildings to prove Propositions 1, 2 and 3 below, which were used in Section 2.

Let ${}_cD^n(R^k)$ be the union of the $F_{[\omega]}D^n(R^k)$ over the posets ω with at most c components, i.e. whose realization viewed as a category has at most c path components. Let ${}^cD^n(R^k) = {}_cD^n(R^k)/{}_{c-1}D^n(R^k)$ be the filtration subquotient of simplices with associated poset having precisely c components.

Similarly, write ${}^cD(R^k) = {}_cD(R^k)/{}_{c-1}D(R^k)$ for the filtration subquotient of simplices in $D(R^k)$ whose associated poset has exactly c components, and ${}^cA^n_k = {}_cA^n_k/{}_{c-1}A^n_k$ and ${}^cA_k = {}_cA_k/{}_{c-1}A_k$ for the corresponding apartment filtration subquotients.

Two objects in a poset are said to be *comparable* if there is a morphism from one to the other. We say that a poset is it linear if any two of its objects are comparable, and a poset is *componentwise linear* if each component forms a linear subposet. Note that among the partial orderings on $\{1, 2, \dots, k\}$ with precisely c components, the componentwise linear ones form a convex subset, when ordered by decreasing strength.

Assume R is such that the Tits building $B(R^k)$ is homotopy equivalent to a wedge $\vee S^{2k-2}$ of $(2k-2)$-spheres. This hypothesis is satisfied if R is a field [Sol], or of the form $\mathbb{Z}/p^n$ [Ro3], or a ring of integers in a number field.

Let $\Sigma B(R^k)$ denote the suspended Tits building, obtained as the simplicial set with q-simplices $0 \subset V_0 \subseteq V_1 \subseteq \cdots \subseteq V_q \subseteq R^k$, identified to the base point when $V_q \neq R^k$. We will use similar notation when replacing R^k with another finitely generated free R-module.

Proposition 1. *There is a natural chain of $2n$-connected maps inducing a $2n$-equivalence*

$$ {}^cD^n(R^k) \simeq_{2n} \bigvee \Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge {}^cA^n_c $$

where the wedge sum is over unordered sets $\{V_1, \dots, V_c\}$ of free nontrivial submodules of R^k with $\oplus_s V_s = R^k$. Hence stably

$$ {}^cD(R^k) \simeq \bigvee \Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge {}^cA_c $$

with $\Sigma B(V_s) \simeq \vee S^{k_s - 1}$ where $k_s = \mathrm{rank}_R(V_s)$, and ${}^cA_c \simeq \vee S^{2c-2}$.

Corollary 1. *The homology of ${}^cD(R^k)$ is concentrated in degree $k+c-2$, for $1 \leq c \leq k$, and the homology of $D(R^k)$ is concentrated in degrees $k-1$ through $2k-2$.*

Let $\mathrm{St}(V) = H_{k-2}(B(V))$ denote the Steinberg module for $V \cong R^k$, and let $W_k = H_{2k-2}({}^kA_k)$ be the integral Σ_k-representation from [Ro1], Definition 11.10.

Corollary 2. *There is a complex $\{Z_q, d_q\}$ whose homology computes $\widetilde{H}_*(D(R^k))$, with*

$$ Z_{k+c-2} = \bigoplus \mathrm{St}(V_1) \otimes \cdots \otimes \mathrm{St}(V_c) \otimes W_c $$

where the sum is over all unordered decompositions $\oplus_s V_s = R^k$, and $1 \leq c \leq k$.

Proof of Proposition 1. Consider the subcomplex $X \subseteq {}^cD^n(R^k)$ of simplices with poset consisting of precisely c linear components. By [Ro1] Lemma 9.8 and Proposition 11.12, the inclusion is $2n$-connected and stably an equivalence.

The pick sites of a non base–point simplex σ in X lie in c linear chains, each *unrelated* (not comparable) to the others, in the n-dimensional indexing cube $[q]^n$. Such a simplex is determined by the c flags of submodules of R^k appearing at these pick sites, together with the locations of these pick sites. The former amounts to a simplex in $\Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c)$, where $\{V_1, \dots, V_c\}$ are the maximal submodules in the c flags. The latter is homotopy equivalent to the simplicial set of c unrelated pick sites in $[q]^n$, i.e. ${}^cA^n_c$.

To prove the latter assertion, compare with a bisimplicial construction, or follow the argument from [Ro1], Lemma 11.3. □

Now assume R is a field. The hypothesis of the following proposition is only expected to hold for $1 \leq c < k$, but the proof goes through also with $c = k$.

Proposition 2. *Suppose $D(R^k)$ is $(k+c-2)$-connected, for some $1 \leq c \leq k$. Then $D(R^{k+1})$ is $(k+c-1)$-connected.*

Proof. The reader may wish to think of ${}_cD(R^k)$ as the $(k+c-2)$-skeleton of a complex to keep track of connectivities. By hypothesis and induction the inclusions

$$ {}_cD(R^k) \longrightarrow D(R^k) \qquad \text{and} \qquad {}_{c-1}D(R^{k+1}) \longrightarrow D(R^{k+1}) $$

are null homotopic. Recall from [Qu2] the description of the homotopy type of $B(R^k)$ as a wedge of suspended copies of $B(R^{k-1})$, one for each *line* (rank one summand) $L \subset R^k$ transverse to $R^{k-1} \subset R^k$. By induction, each $B(R^{k-1})$ is homotopy equivalent to a wedge of spheres, and $B(R^k)$ is a wedge of spheres, one for each sphere in $B(R^{k-1})$ and each such line L. We refer to the wedge summands corresponding to L as arising by suspension in the L-direction

We obtain a decomposition of ${}^cD(R^{k+1})$ into a wedge of spheres by the argument above, Proposition 1, and [Ro1], Proposition 11.12. Fix any $(k+c-1)$-sphere in this decomposition of some summand

$$\Sigma B(V_1) \wedge \cdots \wedge \Sigma B(V_c) \wedge {}^cA^n_c$$

with $\oplus_s V_s = R^{k+1}$. As $c \le k$ some V_s has rank greater than one, say $V_1 = U_1 \oplus L$ where $\operatorname{rank}_R L = 1$, and let $U_s = V_s$ for $s > 1$. Our sphere is then the suspension in the L-direction of a $(k+c-2)$-sphere in ${}^cD(U)$ for some decomposition $R^{k+1} = U \oplus L$, with $U = \oplus_s U_s$.

Because $D(U)$ is assumed $(k+c-2)$-connected, such an inclusion $S^{k+c-2} \to {}^cD(U)$ lifts over ${}_cD(U) \to {}^cD(U)$, and extends null homotopically into $D(U)$. Precisely, we have a lifting of the left vertical map to the middle vertical map in the diagram below

$$\begin{tikzcd}
S^{k+c-2} \arrow[d] & S^{k+c-2} \arrow[l, "\simeq"'] \arrow[d] \arrow[dr] & \\
{}^cD(U) & {}_cD(U) \arrow[l] \arrow[r, "\simeq *"] & D(U)
\end{tikzcd}$$

where the top horizontal map collapses a hemisphere to a point.

Suspending these chosen null homotopies in the L-direction, we obtain a lifting and null homotopy of the original sphere summand in ${}^cD(R^{k+1})$:

$$\begin{tikzcd}
S^{k+c-1} \arrow[d] & S^{k+c-1} \arrow[l, "\simeq"'] \arrow[d] \arrow[dr] & \\
\Sigma({}^cD(U)) \arrow[d] & \Sigma({}_cD(U)) \arrow[l] \arrow[r, "\simeq *"] \arrow[d] & \Sigma D(U) \arrow[d] \\
{}^cD(R^{k+1}) & {}_cD(R^{k+1}) \arrow[l] \arrow[r] & D(R^{k+1})
\end{tikzcd}$$

Since this applies to every $(k+c-1)$-sphere summand of ${}^cD(R^{k+1})$, the inclusion ${}_cD(R^{k+1}) \to D(R^{k+1})$ is null homotopic, and hence $D(R^{k+1})$ is $(k+c-1)$-connected. $\square$

We are still assuming R is a field.

Proposition 3. *$D(R^k) \simeq \bigvee S^{2k-2}$ for $k = 1$, 2 and 3.*

Proof. The cases $k = 1$ and $k = 2$ are trivial (cf. [Ro1], Lemma 15.2), and $D(R^3)$ is two–connected by Proposition 2. By Corollary 2 it suffices to show that the complex

$$Z_4 \xrightarrow{d_4} Z_3 \xrightarrow{d_3} Z_2 \longrightarrow 0 \longrightarrow 0$$

is exact at Z_3.

Here $Z_2 = \mathrm{St}(R^3)$ is the Steinberg module, $Z_3 = \oplus \mathrm{St}(V)$ where the sum is over all splittings $R^3 = V \oplus L$ of R^3 into submodules V and L of rank two and one respectively, and $Z_4 = \oplus W_3$ where the sum is over all unordered splittings $R^3 = L_1 \oplus L_2 \oplus L_3$ of R^3 into lines.

Consider the diagram

$$\begin{tikzcd}
\bigoplus\limits_{\{L_1,L_2,L_3\}} W_3 \arrow[r, "d_4"] & \bigoplus\limits_{V\oplus L=R^3} \mathrm{St}(V) \arrow[r, "d_3"] & \mathrm{St}(R^3) \\
\mathbb{Z}GL_3(R)/T_3 \arrow[u, "w_1"'] & \mathbb{Z}GL_3(R)/T_3 \arrow[u, "g"'] \arrow[r, equal] & \mathbb{Z}GL_3(R)/T_3 \arrow[u, "f"'] \\
 & & \mathbb{Z}U_3R \arrow[u, "i"']
\end{tikzcd}$$

where $T_3 \subset GL_3(R)$ is the diagonal torus, $U_3R \subset GL_3(R)$ is the group of strictly upper triangular matrices, and

$$f(L_1, L_2, L_3) = \sum_{\pi\in\Sigma_3} \mathrm{sgn}(\pi) \cdot (L_{\pi(1)} \subset L_{\pi(1)} \oplus L_{\pi(2)})$$
$$g(L_1, L_2, L_3) = +(L_1 \subset L_1 \oplus L_2, L_3) - (L_2 \subset L_1 \oplus L_2, L_3).$$

The reader may wish to draw a diagram in $\mathbb{R}P^2$ to visualize these and the following formulas.

The map i is induced by the inclusion of U_3R into $GL_3(R)$. A coset in $GL_3(R)/T_3$ is interpreted as a triple of lines (L_1, L_2, L_3), and the homomorphism w_1 takes a coset xT_3 in $GL_3(R)/T_3$ to $xT_3 \otimes w_1$ in $\mathbb{Z}GL_3(R)/T_3 \otimes_{\Sigma_3} W_3 = Z_4$. A point L_1 in $B(V)$ corresponding to a decomposition $V \oplus L_3 = R^3$ is denoted $(L_1 \subset V, L_3)$, and a one–simplex in $B(R^3)$ is denoted by a flag $L_1 \subset V$.

The differentials d^4 and d^3 are given by [Ro1], Lemma 15.3 as

$$d_3(L_1 \subset V, L_3) = +(L_1 \subset V) - (L_1 \subset L_1 \oplus L_3) + (L_3 \subset L_1 \oplus L_3)$$
$$\begin{aligned} d_4 w_1(L_1, L_2, L_3) = &-(L_1 \subset L_1 \oplus L_2, L_3) + (L_2 \subset L_1 \oplus L_2, L_3) \\ &-(L_1 \subset L_1 \oplus L_3, L_2) + (L_3 \subset L_1 \oplus L_3, L_2). \end{aligned}$$

In particular $d_3 g = f$. By definition $g(L_1, L_2, L_3) = -g(L_2, L_1, L_3)$, and in view of $d_4 w_1(L_1, L_2, L_3) = -g(L_1, L_2, L_3) + g(L_3, L_1, L_2)$ we see that $g(L_1, L_2, L_3) \sim \operatorname{sgn}(\pi) \cdot g(L_{\pi(1)}, L_{\pi(2)}, L_{\pi(3)})$ modulo $\operatorname{im}(d_4)$ for any $\pi \in \Sigma_3$.

By the Solomon–Tits theorem the composite fi is an isomorphism, whence $(gi)(fi)^{-1}$ is a section for d_3. The image of gi is the summand of Z_3 corresponding to $V = R^2$. The remainder of Z_3 is generated by $g(L_1, L_2, L_3)$ with $V = L_1 \oplus L_2 \neq R^2$, $L_1 \not\subset R^2$, $L_2 = R^2 \cap V$, $L_3 \not\subset R^2$, and $V \oplus L_3 = R^3$. Motivated by Quillen's proof [Qu2] of the Solomon–Tits theorem we observe that :

$$d_3 g(L_1, L_2, L_3) = f(R^1, M, L_1) - f(R^1, L_2, L_1) - f(R^1, M, L_3) + f(R^1, L_2, L_3)$$

with $M = (L_1 \oplus L_3) \cap R^2$, where the right hand side is lying in the image of fi. Hence the kernel of d_3 is generated by expressions

$$\begin{aligned} g(L_1, L_2, L_3) - (g(R^1, M, L_1) - g(R^1, L_2, L_1) - g(R^1, M, L_3) + g(R^1, L_2, L_3)) \\ = g(L_1, L_2, L_3) - g(L_2, M, L_1) + g(L_2, M, L_3) \end{aligned}$$

for L_1, L_2, L_3, M as above. Using $L_1 \oplus M = L_1 \oplus L_3 = L_3 \oplus M$, we compute

$$d_4 w_1(M, L_3, L_2) - d_4 w_1(M, L_1, L_2) = g(L_3, L_1, L_2) - g(L_2, M, L_1) + g(L_2, M, L_3)$$

which exhibits each generator of the kernel of d_3 as lying in the image of d_4. This proves exactness at Z_3, and $D(R^3)$ is three–connected. □

5. Remarks

$\Phi \colon K(\mathbb{Z})^\wedge_2 \to JK(\mathbb{Z})^\wedge_2$ is five–connected, so $H_5^{spec}(\Phi)$ is onto, and we can choose a $\beta \in H_5^{spec}(K(\mathbb{Z}))$ mapping to the integral generator β' of $H_5^{spec}(JK(\mathbb{Z}))$. Hence there are differentials $d^2(\beta) = \eta \cdot \lambda$, $d^2(\eta \cdot \beta) = \eta^2 \cdot \lambda$, $d^2_{5,2} = 0$, and $d^5(\eta^2 \cdot \beta) = \nu^2$ in the Atiyah–Hirzebruch spectral sequence for $K_*(\mathbb{Z})$ of Section 2, as is seen by comparison with the corresponding spectral sequence for $JK(\mathbb{Z})$.

By the graded algebra structure on $K_*(\mathbb{Z})$, there must be differentials hitting $\nu \cdot \lambda \in E^2_{3,3}$ and $\nu^2 \cdot \lambda \in E^2_{3,6}$, as $\nu \cdot \lambda = 2\lambda^2 = 0$ in $K_6(\mathbb{Z})$. In particular $H_*^{spec}(K(\mathbb{Z}))$ must contain some two–torsion for some $5 \leq * \leq 7$.

Conjecturally $K_*(\mathbb{Z})$ begins $(\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/48, 0, \mathbb{Z}, 0, \mathbb{Z}/240, 0, \mathbb{Z} \oplus \mathbb{Z}/2, \dots)$; see [Mit3].

Because $D(\mathbb{Z}^4)$ is four–connected, the $GL_4(\mathbb{Z})$ poset spectral sequence from [Ro1] and [Ro2] describes how the homology of $GL_4(\mathbb{Z})$ is generated through degree four from its parabolic subgroups, and similarly for all $GL_k(\mathcal{O}_F)$ through degree k for $k \geq 3$. In general, the stable buildings are candidates for quite highly connected finite $GL_k(R)$-complexes, from which some of the group homology of $GL_k(R)$ can be determined from that of the stabilizer subgroups which appear.

To prove the full connectivity conjecture, it would suffice to prove exactness of the complex in Corollary 2 in degree $(2k-3)$ for every k. This would be feasible given a description of the differentials extending Lemma 15.3 of [Ro1].

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