The Picard boundary value problem for a third order stochastic difference equation

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Abstract

It is considered the multidimensional third order stochastic difference equation

$$\Delta^3 X_{n-2} = f(X_n) + \xi_n \quad , \quad n \in \{2,\dots,N-1\},\ N \geq 5$$

where $X_i \in \mathbb{R}^d, d \geq 1$, and $\{\xi_i\}$ is a sequence of d – dimensional independent random vectors, with the Picard boundary condition

$$X_0 = a_0 \ , \ X_1 = a_1 \ , \ X_N = a_N \ , \qquad a_i \in \mathbb{R}^d \ , \ i = 0, 1, N.$$

We first prove that the boundary value problem admits a unique solution if f is a monotone application. Moreover we are able to compute the density of the law of the solution if the random vectors $\{\xi_i\}$ are absolutely continuous. Thanks to this explicit computation, in the scalar case we prove that the process $\{(X_i, \Delta X_i, \Delta^2 X_i) : i = 0,\dots,N-2\}$ is a Markov chain if and only if f is affine and we provide a simple counterexample to show that a similar strong condition does not hold in the multidimensional case.

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1 Introduction

Recently some authors have studied different types of stochastic differential– and difference equations with boundary conditions (see e.g. [1], [5], [7]). Among those, the one dimensional second order *stochastic differential equation* (SDE) with Dirichlet *boundary condition* (BC)

$$(1.1) \qquad \begin{cases} \dfrac{d^2X_t}{dt^2} + f\left(X_t, \dfrac{dX_t}{dt}\right) = \dfrac{dW_t}{dt}, & t \in [0,1] \\ X_0 = a \ , \ X_1 = b \end{cases}$$

has been studied by Nualart, Pardoux [8]. At the same time the discretized problem, equivalent to (1.1), i.e. the one dimensional second order *stochastic difference equation* (SdE) with Dirichlet *boundary condition* (BC)

$$(1.2) \qquad \begin{cases} \Delta^2 X_{n-1} + f(X_n) = \xi_n \ , & 1 \leq n \leq N-1 \\ X_0 = 0 \ , \ X_N = 0 \end{cases}$$

has been considered by Donati-Martin in [3] and, with a different technique, by Alabert, Nualart in [1].

The result, common to (1.1) and (1.2), is the following: under suitable conditions (usually monotonicity and regularity) over f, that ensure existence and uniqueness, the solution is a Markov process (chain) if and only if f is an affine mapping.

To the best of our knowledge, the study of higher order SDE's and SdE's with BC is still completely open and the present paper can be considered as a first step in the investigation of these problems. We shall consider the third order SdE with Picard BC

$$(1.3) \qquad \begin{cases} \Delta^3 X_{n-2} = f(X_n) + \xi_n \ , & 2 \leq n \leq N-1 \\ X_0 = a_0 \ , \ X_1 = a_1 \ , \ X_N = a_N \end{cases}$$

where $X_i \in I\!R^d$ and $\{\xi_i\}$ is a sequence of d - dimensional independent random vectors. The particular choice of the BC will be justified in the following Remark 2.2. We shall prove in Section 2 an existence and uniqueness result under monotonicity conditions over f. In the third section we shall assume that the random vector $(\xi_2, \ldots, \xi_{N-1})$ is absolutely continuous and we will be able to compute explicitly the density of the law of the solution $(X_2, \ldots, X_{N-1})$. Thanks to this computation, in Section 4 we shall prove easily that in the scalar case the solution comply with a suitable Markov condition (see Definition 4.2) if and only if f is an affine mapping. Furthermore we shall prove that a similar strong dichotomy does not hold in the multidimensional case.

Although the result is not surprising and has been obtained for other classes of similar problems, the existence and uniqueness part involves new arguments. Furthermore the technique that we use to study the Markov property of the solution, developed in [5], seams more direct and simpler than those used in the previous papers on the second order equations [1] and [3].

The extension of the present results to higher order SdE with BC appears really difficult.

2 Existence and uniqueness

Let us consider the following third order SdE

$$\Delta^3 X_{n-2} = f(X_n) + \xi_n \ , \quad 2 \le n \le N-1, \ N > 3 \tag{2.1}$$

where $\Delta^3 X_{n-2} \stackrel{def}{=} X_{n+1} - 3X_n + 3X_{n-1} - X_{n-2}$ is the third order difference operator, $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a continuous application and $\{\xi_n\}_{n=2,\ldots,N-1}$ is a sequence of d – dimensional independent random vectors. Instead of the customary initial condition

$$X_0 = \alpha \ , \ \Delta X_0 = \beta \ , \ \Delta^2 X_0 = \gamma \quad , \quad \alpha, \beta, \gamma \in \mathbb{R}^d \tag{2.2}$$

we shall consider in the present paper the Picard BC

$$X_0 = a_0 \ , \ X_1 = a_1 \ , \ X_N = a_N \quad , \quad a_0, a_1, a_N \in \mathbb{R}^d. \tag{2.3}$$

Remark 2.1 *Since in the difference case we have that* $\Delta X_0 = X_1 - X_0$ *and* $\Delta^2 X_0 = X_2 - 2X_1 + X_0$*, condition (2.2) is equivalent to fix the value of* X_0*,* X_1 *and* X_2*.*

Let $M_{m,n}$ denote the set of the $m \times n$ real matrices and let $M_n = M_{n,n}$. In the sequel we shall say that a matrix $A \in M_n$ is positive definite if $x^T A \, x > 0$ for every $x \in \mathbb{R}^n \backslash \{0\}$, even if A is not symmetric, and that is negative definite if $-A$ is positive definite. Trivially a positive (negative) definite matrix is non singular.

A simple computation shows that the problem of finding a sequence $\{X_0, \ldots, X_N\}$ satisfying (2.1) – (2.3) is equivalent to determine a $(N-2)\,d$ – dimensional vector $X = \Big(X_2, \ldots, X_{N-1}\Big)$ verifying

$$\mathcal{A}\, X + a = F(X) + \xi \tag{2.4}$$

where $\mathcal{A} \in M_{(N-2)\ d}$ is the matrix:

$$
(2.5) \qquad \mathcal{A} = \begin{bmatrix}
-3\,\mathrm{I} & \mathrm{I} & O & O & \cdots & O & O \\
3\,\mathrm{I} & -3\,\mathrm{I} & \mathrm{I} & O & \cdots & O & O \\
-\mathrm{I} & 3\,\mathrm{I} & -3\mathrm{I} & \mathrm{I} & \cdots & O & O \\
O & -\mathrm{I} & 3\,\mathrm{I} & -3\,\mathrm{I} & \cdots & O & O \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
O & O & O & O & \cdots & -3\,\mathrm{I} & \mathrm{I} \\
O & O & O & O & \cdots & 3\,\mathrm{I} & -3\,\mathrm{I}
\end{bmatrix}
$$

where $\mathrm{I},\ O \in M_d$ are the identity and zero matrices, respectively, a is the $(N-2)\ d$ – dimensional vector $\left(3a_1 - a_0, -a_1, 0,\ \ldots\ , 0, a_N\right)$, where 0 is the d – dimensional zero vector, $F : \mathbb{R}^{(N-2)\ d} \longrightarrow \mathbb{R}^{(N-2)\ d}$ is defined by $F(X) = \left(f(X_2), f(X_3), \ldots, f(X_{N-1})\right)$ and $\xi = \left(\xi_2, \xi_3, \ldots, \xi_{N-1}\right)$. If we denote by $\mathcal{B}$ the symmetric part of the matrix $\mathcal{A}$, i.e. $\mathcal{B} = \frac{1}{2}\left(\mathcal{A} + \mathcal{A}^T\right)$, a simple computation gives

$$
(2.6) \qquad -2\,\mathcal{B} = \begin{bmatrix}
6\,\mathrm{I} & -4\,\mathrm{I} & \mathrm{I} & O & \cdots & O & O \\
-4\,\mathrm{I} & 6\,\mathrm{I} & -4\,\mathrm{I} & \mathrm{I} & \cdots & O & O \\
\mathrm{I} & -4\,\mathrm{I} & 6\,\mathrm{I} & -4\,\mathrm{I} & \cdots & O & O \\
O & \mathrm{I} & -4\,\mathrm{I} & 6\,\mathrm{I} & \cdots & O & O \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
O & O & O & O & \cdots & 6\,\mathrm{I} & -4\,\mathrm{I} \\
O & O & O & O & \cdots & -4\,\mathrm{I} & 6\,\mathrm{I}
\end{bmatrix}.
$$

It is easy to see that the matrix $-2\,\mathcal{B}$ is positive definite. In fact we have that $-2\,\mathcal{B}$ can be factorized as the product $\mathcal{W}\,\mathcal{W}^T$, where $\mathcal{W} \in M_{(N-2)\ d}$ is the following triangular matrix:

$$\begin{bmatrix} \mathrm{I} & -2\,\mathrm{I} & \mathrm{I} & O & \cdots & O & O \\ O & \mathrm{I} & -2\,\mathrm{I} & \mathrm{I} & \cdots & O & O \\ O & O & \mathrm{I} & -2\,\mathrm{I} & \cdots & O & O \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ O & O & O & O & \cdots & \mathrm{I} & -2\,\mathrm{I} \\ O & O & O & O & \cdots & O & \mathrm{I} \end{bmatrix}$$

Since $\det \mathcal{W} = 1$, $-2\,\mathcal{B}$ is positive definite and therefore $\mathcal{A}$ is negative definite.

We shall now prove an existence and uniqueness theorem for equation (2.4) under more general assumptions and derive the result for (2.1) – (2.3) as an immediate corollary.

Let $p \in \mathbb{N}$, $a, \xi \in \mathbb{R}^p$, and let us consider the following set of hypotheses:

$$\text{(H.1)} \quad \begin{cases} \mathcal{A} \in M_p \text{ is negative definite,} \\ F \;:\; \mathbb{R}^p \longrightarrow \mathbb{R}^p \;\text{ is a continuous and monotone map} \end{cases}$$

(let us recall that a mapping F is said to be monotone if

$$\langle F(x) - F(y)\;,\; x - y \rangle \;\geq\; 0, \qquad \forall\, x, y \in \mathbb{R}^p$$

where $\langle \cdot\;,\;\cdot \rangle$ denotes the scalar product in $\mathbb{R}^p$).

The following result holds:

Theorem 2.1 *Under (H.1), equation (2.4) admits a unique solution.*

Proof:

Existence: Following the same lines of the proof of Lemma 3.1 in [1], we shall prove that, for every $\xi \in \mathbb{R}^p$, there exists a vector $X \in \mathbb{R}^p$ verifying equation (2.4). Let us fix $\xi \in \mathbb{R}^p$ and define $\psi_\xi(\cdot) : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ by

$$\psi_\xi(X) \;=\; \xi - \mathcal{A}X + F(X) - a;$$

it will be sufficient to prove that there exists X_ξ such that $\psi_\xi(X_\xi) \;=\; 0$.

From the assumptions over F and over the matrix $\mathcal{A}$ we obtain:

$$
\begin{aligned}
\langle \psi_\xi(X) \, , \, X \rangle \; &= \; \langle \xi - a \, , \, X \rangle \; + \; \langle -\mathcal{A}X \, , \, X \rangle \; + \; \langle F(X) \, , \, X \rangle \\
&= \; \langle \xi - a + F(0) \, , \, X \rangle \; + \; \langle -\frac{1}{2} \left(\mathcal{A} + \mathcal{A}^T \right) X \, , \, X \rangle \; + \; \langle F(X) - F(0) \, , \, X \rangle \qquad (2.7) \\
&\geq \; - \left\| \xi - a + F(0) \right\| \, \left\| X \right\| \; + \; \lambda \, \left\| X \right\|^2
\end{aligned}
$$

where $\lambda > 0$ is the smallest eigenvalue of the matrix $-\frac{1}{2} \left(\mathcal{A} + \mathcal{A}^T \right)$. From (2.7), we obtain that there exists $\delta > 0$ such that

$$\langle \psi_\xi(X) \, , \, X \rangle \; \geq \; 0 \quad , \quad \forall \, X \; , \; \|X\| \; = \; \delta$$

and an immediate application of Lemma 4.3, pag. 54 in Lions [6] ensures that there exists X_ξ such that $\psi_\xi(X_\xi) \; = \; 0$.

Uniqueness: Let X and Y be two solutions of (2.4). We have

$$\mathcal{A}(X - Y) \; - \; F(X) - F(Y) \; = 0 \qquad (2.8)$$

and, by (H.1),

$$\langle \mathcal{A}(X - Y) \, , \, X - Y \rangle \; - \; \langle F(X) - F(Y) \, , \, X - Y \rangle \; < \; 0 \qquad (2.9)$$

if $X \neq Y$. This clearly implies that $X \equiv Y$ and the theorem is proved.

□

From Theorem 2.1 it is immediate to obtain the following result for the SdE with Picard BC (2.1) – (2.3)

Corollary 2.1 *If the map f in (2.1) is monotone, then (2.1) – (2.3) admits a unique solution.*

Remark 2.2 *It is not difficult to see that a result similar to Corollary 2.1 holds considering equation (2.1) with the Picard BC*

$$X_0 \; = \; a_0 \; , \; X_{N-1} \; = \; a_{N-1} \; , \; X_N \; = \; a_N \quad , \quad a_0, a_{N-1}, a_N \in I\!R^d. \qquad (2.10)$$

We obtain in this case that (2.1) - (2.10) is equivalent to (2.4), where $X = \left(X_1, \ldots, X_{N-2}\right)$, $\mathcal{A}$ *is substituted by*

$$
\mathcal{A}' = \begin{bmatrix} 3\,I & -3\,I & I & O & \cdots & O & O \\ -\,I & 3\,I & -3\,I & I & \cdots & O & O \\ O & -\,I & 3\,I & -3\,I & \cdots & O & O \\ O & O & -\,I & 3\,I & \cdots & O & O \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ O & O & O & O & \cdots & 3\,I & -3\,I \\ O & O & O & O & \cdots & -\,I & 3\,I \end{bmatrix} \tag{2.11}
$$

and the vector a *is substituted by* $a' = \left(-a_0, 0, \ \ldots \ , 0, a_{N-1}, a_N - 3a_{N-1}\right)$. *From* $\left(\mathcal{A} + \mathcal{A}^T\right) = -\left(\mathcal{A}' + \mathcal{A}'^T\right)$ *we deduce that (2.1) – (2.10) admits a unique solution if* $-f$ *is monotone.*

On the other hand, if we consider the generic Picard BC

$$
X_0 = a_0 \ , \ X_i = a_i \ , \ X_N = a_N \quad , \quad a_0, a_i, a_N \in I\!R^d, \ 1 < i < N-1. \tag{2.12}
$$

it is not difficult to prove that (2.1) – (2.12) is equivalent to (2.4), with $X = \left(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{N-1}\right)$ *and the matrix* $\mathcal{A}$ *replaced by*

$$
\mathcal{A}'' = \begin{bmatrix} B_1 & O \\ O^T & B_2 \end{bmatrix}
$$

where $B_1 \in M_{(i-1)\ d}$ *is the submatrix of (2.11) formed by the first* $(i-1)\ d$ *rows and columns,* $B_2 \in M_{(N-i-1)\ d}$ *is the submatrix of (2.5) formed by the first* $(N-i-1)\ d$ *rows and columns and* $O \in M_{(i-1)\ d, (N-i-1)\ d}$ *is the zero matrix. In this case it is impossible to prove a result similar to Theorem 2.1 and monotonicity conditions over* f *do not ensure uniqueness, even in the linear case, where an explicit computation can be carried out.*

To conclude notice that the same kind of restrictions in the Picard BC are present in many papers on difference equations of order greater then 2 (see e.g. Peterson [9]).

Remark 2.3 *Previous Theorem 2.1 and Corollary 2.1 provide an existence and uniqueness result also for the forth order SdE with Picard BC*

$$
\begin{cases} \Delta^4 \, X_{n-2} = f(X_n) + \xi_n \ , & 2 \le n \le N-2 \\ X_0 = a_0 \ , \ X_1 = a_1 \ , \ X_{N-1} = a_{N-1} \ , \ X_N = a_N & \end{cases} \tag{2.13}
$$

where $\Delta^4 X_{n-2} \overset{def}{=} X_{n+2} - 4X_{n+1} + 6X_n - 4X_{n-1} + X_{n-2}$. *In fact it is not difficult to prove that (2.13) is equivalent to (2.4) with* $\mathcal{A} \in M_{(N-3)d}$ *equal to the matrix* $-2\mathcal{B}$ *defined in (2.6),* $a = \big(a_0 - 4a_1, a_1, 0, \ \ldots \ , 0, a_{N-1}, a_N - 4a_{N-1}\big)$ *and* $F(X) = \big(f(X_2), f(X_3), \ldots, f(X_{N-2})\big)$. *Therefore it will be sufficient to assume that* $-f$ *is a monotone map.*

3 Absolute continuity

In this section we shall assume that the random vectors ξ_i are absolutely continuous. Thanks to this assumption, we shall prove that the law of the solution to (2.1) – (2.3) is itself absolutely continuous and we shall compute explicitly its density. Again we shall consider first the problem (2.4) and derive as a corollary the result for (2.1) – (2.3).

Let $p \in I\!N$ and, when (2.4) admits a unique solution $X(\xi)$ for each ξ fixed, let us denote by $\Phi : I\!R^p \longrightarrow I\!R^p$ the map $\xi \longmapsto X(\xi)$.

Lemma 3.1 *Under (H.1), assuming that* ξ *is an absolutely continuous* $p-$ *dimensional random vector with density* $\lambda(\cdot)$ *and* $F \in C^1(I\!R^p)$ *, the unique solution* X *of (2.4) is an absolutely continuous random vector with density*

$$\rho_X(x) \ = \ \lambda\left(\Phi^{-1}(x)\right) \ \Big| \ \det\left(\mathcal{A} \ - \ \nabla F(x)\right) \ \Big| \ > \ 0 \ , \qquad\qquad \mu - a.s.$$

Proof: It is sufficient to prove that Φ is a C^1 global diffeomorphism onto $I\!R^p$. From the monotonicity of F, we obtain that ∇F is non negative definite and therefore, by the assumption on $\mathcal{A}$, that $\det\left(\mathcal{A} \ - \ \nabla F(x)\right) \neq 0$. This implies that Φ^{-1}, defined by

$$\Phi^{-1}(x) \ = \ \mathcal{A}x \ - \ F(x) \ + \ a \ .$$

is a C^1 local diffeomorphism. It is immediate to check that Φ is a bijection form $I\!R^p$ into itself and the result is therefore proved.

□

Let us now derive the result for the model (2.1) – (2.3) as a corollary of previous Lemma 3.1. We shall denote here by Φ the application from $I\!R^{(N-2)d}$ into itself that maps $\xi = (\xi_2, \ldots, \xi_{N-1})$ into the unique solution to (2.1) – (2.3) and we shall make the following assumption:

$$\text{(H.2)} \qquad \begin{cases} \{\xi_2, \ldots, \xi_{N-1}\} \ \text{ are independent } d-\text{dim. absolutely continuous r.v.'s} \\ \\ \text{with densities } \ \lambda_i(\cdot) \ > \ 0 \ \text{ a.e. }, \ \ 2 \leq i \leq N-1, \text{ respectively.} \end{cases}$$

Corollary 3.1 *If $f \in C^1(I\!R^d)$ is monotone and $\{\xi_2, \ldots, \xi_{N-1}\}$ satisfy (H.2), then the unique solution of (2.1) - (2.3), $X = (X_2, \ldots, X_{N-1})$, is an absolutely continuous random vector with a.e. strictly positive density*

$$
\begin{aligned}
\rho_X(x_2, \ldots, x_{N-1}) &= \prod_{i=2}^{N-1} \lambda_i\Big(x_{i+1} - 3x_i + 3x_{i-1} - x_{i-2} - f(x_i)\Big) \\
&\quad \times \prod_{i=2}^{N-1} |\det B_i(x_i, \ldots, x_{N-1})| \quad , \qquad (3.1)
\end{aligned}
$$

$(x_0 = a_0, x_1 = a_1, x_N = a_N)$ where, putting $D(x) = -3I - \nabla f(x)$, the matrix - valued maps B_i's are recursively defined by:

$$
(3.2) \quad \begin{cases}
B_{N-1}(x_{N-1}) = D(x_{N-1}) \\
B_{N-2}(x_{N-2}, x_{N-1}) = D(x_{N-2}) - 3\, B_{N-1}^{-1}(x_{N-1}) \\
B_i(x_i, \ldots, x_{N-1}) = D(x_i) - \Big(3I + B_{i+2}^{-1}(x_{i+2}, \ldots, x_{N-1})\Big) \\
\qquad \times B_{i+1}^{-1}(x_{i+1}, \ldots, x_{N-1}), \qquad i = 2, \ldots, N-3
\end{cases}
$$

Proof: The only nontrivial part is the computation of $\det(\mathcal{A} - \nabla F(x))$, where here $\mathcal{A}$ is the matrix defined in (2.5) and $F(x_2, \ldots, x_{N-1}) = \Big(f(x_2), \ldots, f(x_{N-1})\Big)$. We have

$$
\mathcal{A} - \nabla F(x) = \begin{bmatrix}
D(x_2) & \mathrm{I} & O & O & \cdots & O & O \\
3\,\mathrm{I} & D(x_3) & \mathrm{I} & O & \cdots & O & O \\
-\mathrm{I} & 3\,\mathrm{I} & D(x_4) & \mathrm{I} & \cdots & O & O \\
O & -\mathrm{I} & 3\,\mathrm{I} & D(x_5) & \cdots & O & O \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
O & O & O & O & \cdots & D(x_{N-2}) & \mathrm{I} \\
O & O & O & O & \cdots & 3\,\mathrm{I} & D(x_{N-1})
\end{bmatrix}
$$

and, by the assumption on $\mathcal{A}$ and f, that $\det(\mathcal{A} - \nabla F(x)) \neq 0$. Applying a standard procedure to compute explicitly the determinant of the matrix $\mathcal{A} - \nabla F(x)$ (see e.g. [2]), we obtain easily that it is equal to $\prod_{i=2}^{N-1} \det B_i(x_i, \ldots, x_{N-1})$, where the matrices B_i's are recursively defined by (3.2). Note that $\det B_i \neq 0$ because $\mathcal{A} - \nabla F(x)$ is non singular.

□

4 Markov property

In the present section we want to investigate the Markov property of the solution to the Picard boundary value problem (2.1) – (2.3). We first need to define the two Markov properties which are relevant in our framework.

Definition 4.1 *We shall say that a sequence of random vectors* $\left\{X_0, \ldots, X_M\right\}$ *is a* Markov chain (Mc) *if for every* $0 < m < M$, *the* $\sigma-$ *fields* $\sigma(X_0, \ldots, X_{m-1})$ *and* $\sigma(X_{m+1}, \ldots, X_M)$ *are conditionally independent given* $\sigma(X_m)$.

Definition 4.2 *We shall say that a sequence of random vectors* $\left\{X_0, \ldots, X_M\right\}$ *is a* third-order Markov chain ($3^{rd}-$ Mc) *if the process* $\{(X_i, \Delta X_i, \Delta^2 X_i) : i = 0, \ldots, M-2\}$ *is a Markov chain.*

Let us recall an easy characterization of the Markov property in terms of a factorization property.

Lemma 4.1 *Let* $\left\{X_0, \ldots, X_M\right\}$ *be a sequence of r.v. and let* $X = (X_0, \ldots, X_M)$ *have an absolutely continuous law with density* $\rho_0(x_0, \ldots, x_M)$. *Then* $\left\{X_0, \ldots, X_M\right\}$ *is a Mc if and only if, for every* $0 < m < M$, *there exist two measurable functions* $g_1(x_0, \ldots, x_m)$ *and* $g_2(x_m, \ldots, x_M)$ *such that*

$$\rho_0(x_0, \ldots, x_M) = g_1(x_0, \ldots, x_m)\, g_2(x_m, \ldots, x_M) \qquad a.e.$$

Let us consider, for a while, the initial value problem (2.1) – (2.2). It is immediate to prove that, for each continuous application f, (2.1) – (2.2) admits a unique solution $X = \left(X_3, \ldots, X_N\right)$ and, under (H.2), that X is absolutely continuous with density

$$\rho_X(x_3, \ldots, x_N) = \prod_{i=2}^{N-1} \lambda_i\Big(x_{i+1} - 3x_i + 3x_{i-1} - x_{i-2} - f(x_i) \Big)$$

($x_0 = \alpha,\ x_1 = \alpha + \beta,\ x_2 = \alpha + 2\beta + \gamma$). A natural question for the present problem is whether or not the solution $\left\{X_3, \ldots, X_N\right\}$ is a $3^{rd}-$ Mc. Since $\{(X_i, \Delta X_i, \Delta^2 X_i) : i = 0, \ldots, N-2\}$ is a Mc if and only if $\{(X_i, X_{i+1}, X_{i+2}) : i = 0, \ldots, N-2\}$ is a Mc, an easy application of Lemma (4.1) gives that the solution to (2.1) – (2.2) is a $3^{rd}-$ Mc for every continuous map f.

Is the same result true in the case of the Picard boundary value problem (2.1) – (2.3) ? The answer is negative in most cases and the reason lies in the fact that in general the determinant that appears in (3.1) does not factorize. Anyway in the scalar case (i.e. $d = 1$) we can give a complete characterization of the problems, whose solution is a 3^{rd}– Mc. In fact we shall prove that in this case the process solution to (2.1) – (2.3) is a 3^{rd}– Mc if and only if the application f is affine. Conversely in the multidimensional case (i.e. $d > 1$) a similar strong dichotomy does not hold, as it happens for other classes of SDE and SdE with BC already considered in the literature (see e.g. [4]). This will be proved by means of a counterexample at the end of this section.

Let us consider from now on (2.1) – (2.3) with $d = 1$. If f is monotone and (H.2) holds, we have in this case that the unique solution $X = \left(X_2, \ldots, X_{N-1}\right)$ is an absolutely continuous r.v. with a.e. strictly positive density (3.1), where the B_i's, defined by (3.2), are here real - valued maps. We are able now to prove the main result of the present paper:

Theorem 4.1 *Let $N \geq 7$, $f \in C^3(I\!R)$, $f'(x) \geq 0$ for every $x \in I\!R$ and assume that $\{\xi_2, \ldots, \xi_{N-1}\}$ satisfy (H.2). Denoting by $\left\{X_2, \ldots, X_{N-1}\right\}$ the unique solution to (2.1) – (2.3), $\left\{X_2, \ldots, X_{N-1}\right\}$ is a 3^{rd}– Mc if and only if f is an affine map.*

Proof: Thanks to hypotheses (H.2) and Lemma 4.1, $\left\{X_2, \ldots, X_{N-1}\right\}$ comply with (M) if and only if, for each $2 < m < N-3$, there exist two measurable functions g_1, g_2 such that

$$\rho_X(x_2, \ldots, x_{N-1}) = g_1(x_2, \ldots, x_{m+2})\, g_2(x_m, \ldots, x_{N-1}) \quad \text{a.e.} \tag{4.1}$$

Let us first assume that f is affine; from (3.1) - (3.2) we obtain that there exists a constant K such that:

$$\rho_X(x_2, \ldots, x_{N-1}) = K \prod_{i=2}^{N-1} \lambda_i\Big(x_{i+1} - 3x_i + 3x_{i-1} - x_{i-2} - f(x_i)\Big)$$

and (4.1) is satisfied.

Let us now assume that (4.1) holds and fix $m = 3$. Since the λ_i's are strictly positive a.e. and the B_i's in (3.1) are nonzero, we have that there exists two measurable functions h_1, h_2 such that

$$B_2(x_2, \ldots, x_{N-1}) = h_1(x_2, \ldots, x_5)\, h_2(x_3, \ldots, x_{N-1}) \quad \text{a.e.} \tag{4.2}$$

Since

$$B_2(x_2, \ldots, x_{N-1}) = D(x_2) - \Big(3 + B_4^{-1}(x_4, \ldots, x_{N-1})\Big)\, B_3^{-1}(x_3, \ldots, x_{N-1})$$

and $D(\cdot)$ is a strictly negative function, form (4.2) we obtain that

$$\begin{aligned} &1 - D^{-1}(x_2)\Big(3 + B_4^{-1}(x_4, \ldots, x_{N-1})\Big)\, B_3^{-1}(x_3, \ldots, x_{N-1}) \\ &\quad = \tilde{h}_1(x_2, \ldots, x_5)\, h_2(x_3, \ldots, x_{N-1}) \quad \text{a.e.} \end{aligned} \tag{4.3}$$

where $\tilde{h}_1 = h_1 \, D^{-1}$. It is easy to prove (see [1] and [5]) that (4.3), joint with the regularity of the function f, implies the following analytical property

$$\frac{d}{dx_2}\Big[\, D^{-1}(x_2) \,\Big] \, \frac{d}{dx_6}\Big[\, (3 + B_4^{-1}(x_4, \ldots, x_{N-1})) \; B_3^{-1}(x_3, \ldots, x_{N-1}) \,\Big] \;\equiv\; 0. \tag{4.4}$$

Let us proceed by contradiction and assume that there exists $\overline{x} \in I\!R$ such that $f''(\overline{x}) \neq 0$. Without loss of generality we can assume that $f''(\overline{x}) > 0$ and that f is strictly increasing on an open neighbourhood U of $\overline{x}$. From (4.4) and choosing $x_2 = \overline{x}$, we obtain

$$\frac{d}{dx_6}\Big[\, (3 + B_4^{-1}(x_4, \ldots, x_{N-1})) \; B_3^{-1}(x_3, \ldots, x_{N-1}) \,\Big] \;=\; 0$$

for each $(x_3, \ldots, x_{N-1}) \in I\!R^{N-3}$. This implies that:

$$\begin{aligned} 0 \;&=\; \frac{d}{dx_3}\left[\, \frac{d}{dx_6}\Big[\, \Big(3 + B_4^{-1}(x_4, \ldots, x_{N-1})\Big) \; B_3^{-1}(x_3, \ldots, x_{N-1}) \,\Big] \right] \\ &=\; \frac{d}{dx_6}\left[\, \frac{f''(x_3)}{B_3^2} \, \Big(3 + B_4^{-1}(x_4, \ldots, x_{N-1})\Big) \right] \\ &=\; f''(x_3) \, \Big(3 + B_4^{-1}(x_4, \ldots, x_{N-1})\Big) \, \frac{d}{dx_6} B_3^{-2}. \end{aligned} \tag{4.5}$$

Choosing now $x_3 = \overline{x}$, from (4.5) we obtain

$$\Big(3 + B_4^{-1}(x_4, \ldots, x_{N-1})\Big) \, \frac{d}{dx_6} B_3^{-2}(\overline{x}, x_4, \ldots, x_{N-1}) = 0 \;, \qquad \forall \; (x_4, \ldots, x_{N-1}) \in U^{N-4}. \tag{4.6}$$

If $B_4^{-1} \equiv -3$ on U^{N-4}, we deduce that $f''(x_4) \equiv 0$, $\forall \; x_4 \in U$, which leads to a contradiction. By the regularity of B_4 we can therefore assume that there exist open subsets $V_4, \ldots, V_{N-1}$ of U such that $B_4^{-1} \neq -3$ on $V_4 \times \cdots \times V_{N-1}$. From (4.6) we deduce

$$\frac{d}{dx_6} B_3^{-2}(\overline{x}, x_4, \ldots, x_{N-1}) \;=\; 0 \qquad \forall \; (x_4, \ldots, x_{N-1}) \in V_4 \times \cdots \times V_{N-1}.$$

A simple computation shows, denoting $c_i \;=\; \Big(3 + B_i^{-1}\Big)$, that

$$\frac{d}{dx_6} B_3^{-2} \;=\; \frac{2\, f''(x_6)}{B_3^3 \; B_4 \; B_5 \; B_6^2} \left[\frac{c_5}{B_4} \;+\; \frac{c_7}{B_5}\Big(1 + \frac{c_5 \, c_6}{B_4}\Big) \right].$$

Again, since $x_6 \in V_6 \subseteq U$, we have

$$\frac{c_5}{B_4} \left(1 \;+\; \frac{c_6 \, c_7}{B_5}\right) \;+\; \frac{c_7}{B_5} \;=\; 0$$

and, differentiating with respect to x_4, we obtain

$$\frac{c_5}{B_4^2\, B_5}\, f''(x_4)\left(B_5 \;+\; c_6\, c_7\right) \;=\; 0 \qquad \forall\,(x_4,\ldots,x_{N-1}) \in V_4\times\cdots\times V_{N-1}.$$

As before we can assume that there exist open subsets $W_i \subseteq V_i$ for $i = 5,\ldots,N-1$ such that

$$B_5 \;+\; c_6\, c_7 \;=\; 0 \qquad \forall\,(x_5,\ldots,x_{N-1}) \in W_5\times\cdots\times W_{N-1}.$$

Differentiating now with respect to x_5, we conclude that $f''(x_5)\equiv 0$ for $x_5 \in W_5 \subseteq U$, which clearly leads to a contradiction.

$\square$

Remark 4.1 *Notice that, for each $d \geq 1$, if the application f is affine, then the solution to (2.1) - (2.3) is a 3^{rd}– Mc. In fact, in this case, the matrix - value function $D(x) = -3I - \nabla f(x)$ is constant and therefore all the B_i's, defined in (3.2), are constants.*

A simple generalization of the trivial sufficient condition of Remark 4.1 is that given by the triangular case. Let us recall the definition of a triangular map (see [4]):

Definition 4.3 *We say that a map f from $I\!R^d$ into itself is* triangular *if, for each $i \in \{1,\ldots,d\}$, $f_i(x_1,\ldots,x_d)$ depend only on the first i variables.*

Let us now assume that the map f in (2.1) is triangular and belongs to C^1. It is immediate to see that in this case the Jacobian matrix $\nabla f(x)$ is a lower triangular matrix (this property justifies the name). Since the set of the lower triangular matrices is a ring, we obtain that the matrices B_i's, defined in (3.2), are lower triangular and $\det\, B_i(x_i,\ldots,x_{N-1})$ depends only on $\frac{\partial}{\partial x_m} f_m$, for each $m \in \{i,\ldots,d\}$. Therefore, if every f_i is linear in the last variable, i.e. $f_i(x_1,\ldots,x_i) \;=\; \alpha_i(x_1,\ldots,x_{i-1}) \;+\; \beta_i\, x_i$ $(\alpha_1 \equiv 0)$, and $\beta_i \geq 0$, then (2.1) - (2.3) admits a unique solution which trivially is a 3^{rd}– Mc, being $\prod_{i=2}^{N-1} |\det\, B_i| \equiv \text{const}$. Since in this case the functions α_i's are completely free of constraint, this clearly implies that it is impossible to have in the multidimensional case a strong dichotomy similar to that of the scalar case.

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