Positive projections of von Neumann algebras onto JW-algebras

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1 Introduction

Let N be a von Neumann algebra and $E : N \to N$ a positive linear unital map. We say E is a *projection* (or positive projection) if E is idempotent, $E = E^2$. If E is faithful and normal the image of E is a Jordan algebra [3], in particular its self-adjoint part $A = E(N_{\text{sa}})$ is a JW-subalgebra of N_{sa} with the usual Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. It was shown in [1] that E is completely positive if and only if $E(N)$ is a von Neumann algebra, and it was shown in [7] that E is decomposable, i.e. the sum of a completely positive and co-positive map, if and only if A is a reversible JW-algebra. Recall that A is called reversible if $A = R(A)_{\text{sa}}$, where $R(A)$ denotes the weakly closed real $*$-algebra generated by A. Let M denote the von Neumann algebra generated by A, or equivalently by $E(N)$. Then it is natural to ask 1) whether there exists a faithful normal conditional expectation of N onto M, and 2) if it does, will E factor through M, i.e. if there exists a faithful normal conditional expectation $F : N \to M$ and a (possibly canonical) projection $P : M \to A + i\,A$ such that $E = P \circ F$.

In the present paper we shall present answers to the above questions, the results varying with the kind of JW-algebra A is. We shall also in the last section prove a theorem on the existence of positive projections, the result being an extension of Takesaki's existence theorem for conditional expectations [9] to Jordan algebras.

We shall mainly concentrate our attention to faithful projections. There are two technical reasons for this. The first is that then $A = E(N_{\text{sa}})$ is a JW-subalgebra of N_{sa}. Secondly, we can always restrict attention to this situation. Indeed, let e be the support of E in N. By [3, Lem. 1.2] $e \in A' \cup N$, and from the proof of [7, Lem. 1.2] the map $E_e : N_e \to N_e$ defined by

$$E_e(exe) = \lambda^{-1}E(exe)e, \qquad x \in N, \;\; \lambda = E(e) \in A \cup A',$$

is a faithful normal projection onto $E(N)e$. (We should remark that in [7] A is assumed to be a JW-factor, but the result extends easily to the general case by a modification of the proof of Proposition 3.1 below).

We refer the reader to the book [5] for the theory of JW-algebras.

2 Projections from the enveloping von Neumann algebra

In this section we study the existence problem for positive normal projections of the enveloping von Neumann algebra onto the JW-algebra. To be specific let A be a JW-algebra and $M = A''$ the von Neumann algebra generated by A. From the structure theory of JW-algebras, see [5] there exist projections e, f, g, h in the center $Z(A)$ of A with sum 1 such that the following hold:

(i) $eA = eM_{\text{sa}}$,

(ii) $(f+g)A$ is reversible, $R(A) + i\,R(A) = M$, $R(A) \cap i\,R(A) = \{0\}$. The map $\alpha(x+iy) = x^*+iy^*$, $x, y \in R(A)$ is an involutive $*$-antiautomorphism of M such that $A = \{x \in (f+g)M_{\text{sa}} : \alpha(x) = x\}$, $R(A) = \{x \in (f+g)M : \alpha(x) = x^*\}$. fA and gA have the following further properties:

(iia) There exist two projections p, g in the center $Z(M)$ of M with $p + g = f$ such that $\alpha(p) = q$. $pA = pM_{\text{sa}}$, $qA = qM_{\text{sa}}$.

(iib) $Z(gA) = Z(gM)_{\text{sa}}$

(iii) hA is of type I_2.

Note that a positive projection P of M_{sa} onto A leaves the projections e, f, g, h invariant, hence the different cases (i)–(iii) invariant, so they can be studied separately. For simplicity of notation we shall say P is a projection of M onto A instead of M_{sa} onto A. Then in case (i) the identity map is a projection of M onto A. In case (ii) the map $P(x) = \frac{1}{2}(x + \alpha(x))$ is a projection of M onto A which we shall call the *canonical projection.* Thus the existence problem is reduced to the I_2-case. For a discussion of JW-algebras of type I_2 see [5, §6.3], and in particular the definition of JW-algebra of type $I_{2,k}$, $k \in \mathbb{N}$. For us all we need to know is that such a JW-algebra is of the form $C(X, V_k)$, where $Z(A) \cong C(X)$, X compact Hausdorff, and V_k is the spin factor generated by a spin system of k symmetries [5, Prop. 6.3.13].

Theorem 2.1 Let A be a JW-algebra of type I_2 and M the von Neumann algebra generated by A. Then there exists a faithful normal projection P of M onto A if and only if M is finite. If P exists and τ is a normal trace on A then $\tau \circ P$ is a trace on M. If A has no direct summand of type $I_{2,k}$ with k an odd integer then P is unique.

The proof will be divided into some lemmas. The necessity part of the theorem follows from the following more general result. For a discussion of traces on JW-algebras see [6].

Lemma 2.2 Let N be a von Neumann algebra, A a JW-subalgebra and $E : N \to A$ a faithful normal projection. Suppose τ is a faithful normal semifinite trace on A such that $\tau \circ E$ is a semifinite weight on N. Then there exists a faithful normal conditional expectation F of N onto the centralizer $N_{\tau\circ E}$ of $\tau \circ E$ in N such that $E = E/N_{\tau\circ E} \circ F$. Furthermore, if M denotes the von Neumann algebra generated by A, then $M \subset N_{\tau\circ E}$, so in particular $\tau \circ E$ restricts to a trace on M.

Proof If s is a symmetry in A and $x \in N$ then by [7, Lem. 4.1] $E(sxs) = sE(x)s$, hence

$$\tau \circ E(sxs) = \tau(sE(x)s) = \tau(E(x)).$$

Replacing x by xs we obtain $\tau\circ E(sx) = \tau\circ E(xs)$. Since the symmetries span a dense subset of A, $A \subset N_{\tau\circ E}$. Since $N_{\tau\circ E}$ is a von Neumann subalgebra of N, and $A \subset N_{\tau\circ E}$, $M \subset N_{\tau\circ E}$. Since τ is semifinite on A, $\tau \circ E$ is semifinite on M, hence $\tau \circ E$ restricts to a semifinite trace on M.

Let $a \in A$ and p be a finite projection in A, i.e. $\tau(p) < \infty$. Then for each finite projection q in A, $p \vee q$ is finite, and the restriction of τ to $p \vee q\, A\, p \vee q$

is a finite trace. From the identity $\tau(yxy) = \tau(y^2 \circ x)$ for a $x, y \in p \vee q \, A \, p \vee q$ [6], it follows that

$$\tau(p\,qaq\,p) = \tau(p \circ qaq)\,.$$

Since the functional $x \longrightarrow \tau(p \circ x)$ is normal, letting $q \longrightarrow 1$ we obtain the identity

$$(*) \qquad \tau(pap) = \tau(p \circ a)\,, \qquad a \in A\,.$$

Note that the states $\rho(a) = \tau(h \circ a)$ with $h \in A^+$, $\tau(h) = 1$ form a separating family of states on A. Indeed, if $a = a^+ - a^-$, $a^+a^- = 0$, $a^+, a^- \in A^+$, and $\tau(h \circ a) = 0$ for all h as above, then if p is a finite projection in A with $p \leq \text{support}(a^+)$ then by $(*)$

$$\tau(pa^+p) = \tau(pap) = \tau(p \circ a) = 0\,.$$

Since τ is faithful $pa^+p = 0$. Letting $p \nearrow \text{support}(a^+)$ we obtain $a^+ = 0$, and similarly $a^- = 0$. Thus $a = 0$.

Let σ_t denote the modular group of the weight $\tau \circ E$ on N, and let $\rho(a) = \tau(h \circ a)$ be a state as above. Then for $x \in N$

$$\begin{aligned}
\rho \circ E(\sigma_t(x)) &= \tau(h \circ E(\sigma_t(x))) \\
&= \tau(E(h \circ \sigma_t(x))) \quad \text{by[7, lem.4.1]} \\
&= \tau(E(\sigma_t(hx))) \qquad \text{since } h \in N_{\tau\circ E} \\
&= \tau \circ E(h \circ x) \\
&= \rho(E(x)).
\end{aligned}$$

By the previous paragraph $E(\sigma_t(x)) = E(x)$ for all $t \in \mathbf{R}$, hence E factors through $N_{\tau\circ E}$. QED

Lemma 2.3. Let A be a spin factor and B the C^*-algebra generated by A. Then there exists a positive projection of $E : B \longrightarrow A$. E is unique if $A \cong V_k$ with k even or ∞. If $A \cong V_k$ with k odd then there is a 1-parameter family of positive projections of B onto A.

Proof From [3] there exists a positive projection $E : B \longrightarrow A$. Let τ denote the trace on A see [5, 6.1.7]. By the argument of Lemma 2.2, $\mathrm{Tr} = \tau \circ E$ is a trace on B. By [3] E is the orthogonal projection of B onto A with respect to the inner product $\langle x, y\rangle = \mathrm{Tr}(xy) = \mathrm{Tr}(x \circ y)$. Let $\mathcal{A}$ denote the

CAR-algebra. Then by [5, 6.2.2] we have

$$B \cong \begin{cases} M_{2^{n-1}}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C}) & \text{if } k = 2n-1 \\ M_{2^n}(\mathbb{C}) & \text{if } k = 2n \\ \mathcal{A} & \text{if } k = \infty . \end{cases}$$

If $k = 2n$ or ∞ there exists a unique trace on B, so $\mathrm{Tr} = \tau \circ E$ determines E uniquely. If k is odd there is a 1-parameter family of positive projections of B onto A, as each trace Tr on B defines a projection by the formula $\mathrm{Tr}(E(x)y) = \mathrm{Tr}(xy)$ for $x \in B$, $y \in A$. QED

Lemma 2.4 Let A be a JW-algebra and M the von Neumann algebra generated by A. If M is finite there exists a faithful normal projection $P : M \to A$. If moreover $Z(A) = Z(M)$ then P is unique.

Proof Cutting down by central projections if necessary we may assume M has a faithful normal tracial state tr. As for von Neumann algebras for each $x \in M_{\mathrm{sa}}$ there is $P(x) \in A$ such that

$$\mathrm{tr}(x \circ a) = \mathrm{tr}(xa) = \mathrm{tr}(P(x)a) = \mathrm{tr}(P(x) \circ a), \qquad a \in A$$

P so defined is a faithful normal projection of M onto A.

Assume $Z(A) = Z(M)$, and let $\psi : M \to Z(A)$ be the unique center valued trace on M with $\psi(1) = 1$. Let $\Phi = \psi|_A \circ P$. If $z \in Z(A)$ then for $x \in M$, $\Phi(zx) = \psi(P(zx)) = \psi(zP(x)) = z\psi P(x) = z\Phi(x)$, so Φ is also a faithful normal center valued trace, hence $\Phi = \psi$. If Q is another faithful normal projection $M \to A$ then similarly $\psi|_A \circ Q = \psi$, hence

$$\psi|_A(P(x) - Q(x)) = 0\,, \qquad x \in M\,.$$

If $a \in A$ then

$$0 = \psi|_A(P(a \circ x) - Q(a \circ x)) = \psi|_A(a \circ (P(x) - Q(x)))\,.$$

In particular this holds when x is self-adjoint and $a = P(x) - Q(x)$, hence by faithfulness of ψ, $P(x) = Q(x)$. Thus P is unique. QED

Proof of Theorem 2.1

Assume A is of type I_2 and M is finite. By Lemma 2.4 there exists a faithful normal projection $P : M \to A$ and if P exists then M is finite by Lemma 2.2. Since by [5, 6.3.14] A is a direct sum of JW-algebras of type $I_{2,k}$, and if A is of type $I_{2,k}$ then $M \cong C(X, V_k)$ with $Z(A) \cong C(X)$, so the uniqueness statement follows from Lemma 2.4 and Lemma 2.3.

3 Conditional expectations onto the generated von Neumann algebra

In this section we study the following problem. Suppose N is a von Neumann algebra, A a JW-subalgebra, and M the von Neumann algebra generated by A. Suppose $E : N \to A$ is a faithful normal projection. Then

(i) Does there exist a faithful normal conditional expectation $F : N \to M$?

(ii) If F exists can it be chosen so that $E = E|_M \circ F$?

Note that if A has a faithful normal semifinite trace τ such that $\tau \circ E$ is semifinite, then the answer to both questions is affirmative by Lemma 2.2.

The following proposition is used in the proof of [8, Thm]. However, in that proof we refer to [7, Lem. 4.2], which is only proved for JW-factors. For completeness we include a proof. We use the notation N_p for the von Neumann algebra $\{pxp : x \in N\}$ when p is a projection in N.

Proposition 3.1 Let N be a von Neumann algebra, A a JW-subalgebra and $E : N \to A$ a faithful normal projection. In the notation of §2 assume A is of type (iia) with $p + q = 1$. Then there exist faithful normal conditional expectations $F_p : N_p \to pA = pM_{\mathrm{sa}}$ and $F_q : N_q \to qM_{\mathrm{sa}}$ such that

$$F(x) = F_p(pxp) + F_p(qxq), \qquad x \in N\,,$$

defines a faithful normal conditional expection $N \to M$.

Proof For $a \in A^+$ and e a central projection in M, by [7, Lem. 4.1] $a \circ E(e) = E(a \circ e) = E(ae) \geq 0$, hence by [7, Lem. 3.1] $E(e) \in Z(A)$. In particular if $0 \neq e \in Z(A)$ then $ep \neq 0$, hence $E(p)e = E(pe) \neq 0$. By spectral theory there is a largest projection $e_n \in Z(A)$ such that $e_n E(p) \geq \frac{1}{n} e_n$ for each $n \in \mathbb{N}$. Then $e_n \geq e_m$ if $n \geq m$, so the sequence (e_n) is increasing and converges by the above strongly to 1. Let $a_n \in A$ be the inverse of the operator $e_n E(p)$ considered as acting on $e_n H$, where H is the underlying Hilbert space. Define

$$E_n : N_p \to A\, e_n p$$

by

$$E_n(pxp) = a_n E(pxp) e_n p\,.$$

Clearly E_n is normal and positive. Furthermore, if $x \in N^+$ then

$$E_n(pe_nxe_np) = a_nE(pe_nxe_np)e_np = a_nE(pxp)e_np\,.$$

Thus if $E_n(pe_nxe_np) = 0$ then $0 = E(pxp)e_np = E(pe_nxe_np)e_n$, so $E(pe_nxe_np) = 0$. Since E is faithful, $pe_nxe_np = 0$. Thus the restriction $E_n|_{N_{pe_n}}$ is faithful. If $a \in A$ then

$$\begin{aligned} E_n(p(e_na)p) &= a_nE(pe_nae_np)e_np \\ &= a_n(e_naE(p)e_np \\ &= ae_np\,. \end{aligned}$$

Thus $E_n|_{N_{pe_n}}$ is a projection of N_{pe_n} onto Ae_np. Since $a_ne_m = a_m$ if $n \geq m$ a straightforward computation shows

$$E_n|_{N_{e_mp}} = E_m|_{N_{e_mp}}, \qquad n \geq m\,.$$

We also find

$$E_n(pxp)e_m = E_m(pxp)\,.$$

Thus for $x \in N^+$ the sequence $(E_n(pxp))$ is increasing and bounded in norm by $\|pxp\|$. Let $F_p(pxp)$ be its strong limit. Then

$$F_p(pxp)e_n = E_n(pxp), \qquad n \in \mathbb{N}\,.$$

Thus $F_p : N_p \longrightarrow Ap = Mp$ is positive, $F_p(p) = p$, and if $a \in A$, $F_p(pap) = pap$. Since we have $1 = e_1 + \sum_1^\infty(e_{n+1} - e_n)$,

$$\begin{aligned} F_p(pxp) &= F_p(pxp)e_1 + \sum_1^\infty F_p(pxp)(e_{n+1} - e_n) \\ &= E_1(pxp) + \sum_1^\infty E_{n+1}(pxp)(e_{n+1} - e_n)\,, \end{aligned}$$

is an orthogonal sum of normal maps, so is normal. Thus $F_p : N_p \longrightarrow M_p$ is a positive normal conditional expectation. Finally, if $x \in N^+$ and $F_p(pxp) = 0$ then $E_n(pe_nxe_np) = 0$ for all n, hence $pe_nxe_np = 0$ for all n, and so $pxp = 0$. Thus F_p is also faithful.

Similarly we can define $F_q : N_q \longrightarrow M_q$ and show it is a faithful normal conditional expectation. Thus the map $F : N \longrightarrow M$ defined by

$$F(x) = F_p(pxp) + F_q(qxq)$$

is a faithful normal conditional expections. QED

In the above situation F is not necessarily unique, see [7, Prop. 6.4].

In [8] it was shown that if N is a von Neumann algebra, A a reversible JW-subalgebra and E a faithful normal projection of N onto A such that $\alpha \circ E = E$ for an involution α of N, then there exists a faithful normal conditional expectation F of M onto A, where as before M is the von Neumann algebra generated by A. We now show that we can get rid of the hypothesis on the existence of α and thus answer questions (i) and (ii) affirmatively when A is of type (iib) in §2.

Theorem 3.2 Let N be a von Neumann algebra and A a reversible JW-subalgebra such that $R(A) \cap i\,R(A) = (0)$, and $Z(A) = Z(M)_{\mathrm{sa}}$, where $M = R(A) + i\,R(A)$ is the von Neumann algebra generated by A. Suppose $E : N \to A$ is a faithful normal projection. Then there exists a unique conditional expectation $F : N \to M$ such that if $P : M \to A$ is the canonical projection, then $E = P \circ F$.

Proof Let α be the canonical involution of M, $\alpha(x+iy) = x^* + iy^*$. Denote by N^{op} the opposite algebra of N, and put

$$\widetilde{N} = N \oplus N^{\mathrm{op}}.$$

N is imbedded in $\widetilde{N}$ by $x \to (x, 0)$. We define an involution σ of $\widetilde{N}$ by

$$\sigma(x, y) = (y, x).$$

Let

$$\widetilde{M} = \{(x, \alpha(x)) : x \in M\},$$

and imbed M in $\widetilde{M}$ by $x \to (x, 0)$. Define an involution $\tilde{\alpha}$ on $\widetilde{M}$ by

$$\tilde{\alpha}(x, \alpha(x)) = (\alpha(x), x) = (\alpha(x), \alpha(\alpha(x))).$$

Then $\tilde{\alpha} = \sigma|_{\widetilde{M}}$. Let

$$\widetilde{A} = \{(x, x) : x = \alpha(x) \in A\}$$

and imbed A $\widetilde{A}$ by $x \to (x, 0)$. The canonical projection $P : M \to A$ satisfies $P(x) = \frac{1}{2}(x + \alpha(x))$. Define

$$\widetilde{P} : \widetilde{M} \to \widetilde{A}$$

by $\widetilde{P}(x, \alpha(x)) = \left(\frac{1}{2}(x + \alpha(x)), \frac{1}{2}(x + \alpha(x)) = (P(x), P(x)\right)$. Define

$$\widetilde{E} : \widetilde{N} \to \widetilde{A}$$

by $\widetilde{E}(x,y) = \left(\frac{1}{2}E(x+y), \frac{1}{2}E(x+y)\right)$. Then $\widetilde{E}$ is a faithful normal projection, and

$$\widetilde{E} \circ \sigma = \sigma \circ \widetilde{E} = \widetilde{E}.$$

From the definition of α it follows that $\widetilde{M}$ is the von Neumann algebra generated by $\widetilde{A}$. Thus by [8, Thm. and comments following it] there exists a faithful normal conditional expectation $\widetilde{F} : \widetilde{N} \to \widetilde{M}$ such that

$$\widetilde{E} = \widetilde{E}|_{\widetilde{M}} \circ \widetilde{F}.$$

If $x \in M$ then

$$\begin{aligned}
\widetilde{E}(x, \alpha(x)) &= \left(\frac{1}{2}E(x+\alpha(x)), \frac{1}{2}E(x+\alpha(x))\right) = (EP(x), EP(x)) \\
&= (P(x), P(x)) = \widetilde{P}(x, \alpha(x)).
\end{aligned}$$

Thus $\widetilde{E} = \widetilde{P} \circ \widetilde{F}$.

Define $F_i : N \to M$, $i = 1, 2$, by

$$\begin{aligned}
\widetilde{F}(x, 0) &= (F_1(x), \alpha F_1(x)), \qquad x \in N. \\
\widetilde{F}(0, y) &= (\alpha F_2(y), F_2(y)), \qquad y \in N.
\end{aligned}$$

Since $\widetilde{F}$ is a conditional expectation, if $z \in M$, $x \in N$,

$$\begin{aligned}
(zF_1(x), \alpha(zF_1(x)) &= (z, \alpha(z))(F_1(x), \alpha F_1(x)) \\
&= (z, \alpha(z))\widetilde{F}(x, 0) \\
&= \widetilde{F}((z, \alpha(z))(x, 0)) \\
&= \widetilde{F}(zx, 0) \\
&= (F_1(zx), \alpha F_1(zx)).
\end{aligned}$$

Thus $zF_1(x) = F_1(zx)$, and by symmetry $F_1(xz) = F_1(x)z$. In particular $F_1(z) = zF_1(1) = F_1(1)z$, so $F_1(1) \in Z(M) = Z(A)$.

Similarly $F_2(1) \in Z(M) = Z(A)$, and $F_2(zx) = zF_2(x)$, $F_2(xz) = F_2(x)z$.

If $x \in N$ then

$$\begin{aligned}
\widetilde{E}(x, 0) &= \widetilde{P}\widetilde{F}(x, 0) = \widetilde{P}(F_1(x), \alpha F_1(x)) \\
&= \left(\frac{1}{2}(F_1(x) + \alpha F_1(x)), \frac{1}{2}(F_1(x) + \alpha F_1(x))\right)
\end{aligned}$$

However, $\widetilde{E}(x, 0) = \left(\frac{1}{2}E(x), \frac{1}{2}E(x)\right)$. Therefore we have

$$F_1(x) + \alpha F_1(x) = E(x).$$

In particular since $F_1(1) \in Z(A)$,

$$2F_1(1) = F_1(1) + \alpha F_1(1) = E(1) = 1\,.$$

Thus $F_1(1) = \frac{1}{2}1$, so from the above $F = 2F_1$ is a conditional expectation of N onto M. Furthermore if $x \in N$, $P \circ F(x) = \alpha P \circ F(x)$, so that

$$\begin{aligned} E(x) &= F_1(x) + \alpha F_1(x) = \\ &= 2P(F_1(x)) \\ &= P \circ F(x)\,. \end{aligned}$$

Similarly we obtain $E = P \circ 2F_2$.

It remains to show uniqueness, hence in particular $F_1 = F_2$. Suppose $G : N \to M$ is a conditional expectation such that $P \circ G = E$. Let $x \in N_{\mathrm{sa}}$. Then we have

$$\begin{aligned} P((F-G)(x)^2) &= P(F(x)^2 - F(x)G(x) - G(x)F(x) + G(x)^2) \\ &= P(F(xF(x)) - F(xG(x)) - F(G(x)x) + G(xG(x))) \\ &= E(xF(x) - xG(x) - G(x)x + xG(x)) \\ &= P \circ G(xF(x)) - P \circ F(G(x)x) \\ &= P(G(x)F(x) - G(x)F(x)) \\ &= 0\,. \end{aligned}$$

Since P is faithful $F(x) = G(x)$, so $F = G$. QED

Corollary 3.3 Let A be a reversible JW-algebra and M the von Neumann algebra generated by A. If $Z(A) = Z(M)_{\mathrm{sa}}$ then there exists a unique faithful normal projection of M onto A.

Proof If $A = M_{\mathrm{sa}}$ the result is obvious. Otherwise it suffices to look at the case $M = R(A) + iR(A)$, $R(A) \cap iR(A) = (0)$. If $Z(A) = Z(M)_{\mathrm{sa}}$ then by Theorem 3.2 applied to $N = M$, it follows that every faithful normal projection of M onto A must be equal to the canonical projection P.

4 The Jordan analogue of Takesaki's theorem

In the present section we shall study the existence problem for faithful normal projections of a von Neumann algebra N, or more generally JW-algebra, onto a JW-subalgebra. The theorem will be a generalization of Takesaki's theorem for von Neumann algebras [9], which in the case of states says that if $M \subset N$ are von Neumann algebras, and φ is a faithful normal state on N with modular group σ_t^φ, then there exists a φ-invariant faithful normal conditional expectation of N onto M if and only if $\sigma_t^\varphi(M) = M$ for all $t \in \mathbb{R}$. In the JW-algebra case σ_t^φ is replaced by a 1-parameter family (ρ_t^φ) of operators on N, which in the von Neumann algebra case are given by $\rho_t^\varphi(a) = \frac{1}{2}\big(\sigma_t^\varphi(a) + \sigma_{-t}^\varphi(a)\big)$. The extension of the Tomita-Takesaki theorem to JW-algebras, or rather JBW-algebras is as follows [4, Thm. 3.3].

Theorem 4.1 (Haagerup and Hanche-Olsen) Let N be a JBW-algebra with a faithful normal state φ. Then there is a unique 1-parameter family $(\rho_t^\varphi)_{t\in\mathbb{R}}$ of operators on N, satisfying

(i) The map $t \longrightarrow \rho_t^\varphi(x)$ in w^*-continuous for all $x \in N$.

(ii) Each ρ_t^φ is unital, positive, normal.

(iii) $\rho_0^\varphi = \mathrm{id}_N$, $\rho_s^\varphi \rho_t^\varphi = \frac{1}{2}(\rho_{s+t}^\varphi + \rho_{s-t}^\varphi)$, $s, t \in \mathbb{R}$.

(iv) $\varphi(\rho_t^\varphi(a) \circ b) = \varphi(a \circ \rho_t^\varphi(b))$, $a, b \in N$.

(v) The bilinear form on N defined by $s_\varphi(a,b) = \int_{-\infty}^{\infty} \varphi(\rho_t^\varphi(a) \circ b) \cosh(\pi t)^{-1} dt$, $a, b \in N$, is a self-polar form on N.

We can now state our generalization of Takesaki's theorem. The result also extends [2].

Theorem 4.2 Let N be a JBW-algebra and $A \subset N$ a JBW-subalgebra. Suppose ψ is a faithful normal state on N, and let $\varphi = \psi|_A$. Then the following three conditions are equivalent:

(i) There exists a faithful normal projection $E : N \longrightarrow A$ such that $\varphi \circ E = \psi$.

(ii) $s_\varphi = s_\psi|_{A \times A}$.

(iii) $\rho_t^\varphi(a) = \rho_t^\psi(a)$, $a \in A$, $t \in \mathbb{R}$.

Proof We shall show (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

(i)⇒(ii) Put $s_1(x,y)=s_\psi(E(x),E(y))$, $x,y\in N$. Then $s_1(x,x)\geq 0$, $x\in N$, and $s_1(x,y)\geq 0$ if $x,y\in N^+$. In the notation of [4], if s is a bilinear form on $N\times N$ then $s^*: N\to N^*$ is given by $(s^*(x),y)=s(x,y)$. Thus we have

$$\begin{aligned}(s_1^*(1),y) &= s_\psi(1,E(y))=\psi(E(y))=\varphi\circ E(y)=\\ &= \psi(y)=(s_\psi^*(1),y)\,.\end{aligned}$$

Therefore $s_1^*(1)=s_\psi^*(1)$. By [10, Thm. 1.1]

$$s_1(x,x)\leq s_\psi(x,x), \qquad x\in N\,,$$

or

$$s_\psi(E(x),E(x))\leq s_\psi(x,x)\,.$$

Therefore E can be extended to a contractive idempotent $\widetilde{E}$ on the real Hilbert space obtained by completing N in the norm induced by the inner product s_ψ. But contractive idempotents on a Hilbert space are automatically self-adjoint, i.e. $\widetilde{E}=\widetilde{E}^*=\widetilde{E}^*\widetilde{E}$. Therefore

$$s_\psi(E(x),E(y))=s_\psi(E(x),y)=s_\psi(x,E(y))\,.$$

for all $x,y\in N$. In particular we have

$$s_\psi(E(x),x)=s_\psi(x,E(x))=s_\psi(E(x),E(x))\leq s_\psi(x,x), \qquad x\in N\,.$$

Let

$$s_2=s_\psi|_{A\times A}\,.$$

Then $s_2(x,y)=s_\psi(E(x),E(y))$, $x,y\in A$. We assert that s_2 is a self-polar form on $A\times A$. The only nontrivial property to be shown is that

$$s_2^*([0,1])=[0,s_2^*(1)],$$

where $[0,1]=\{x\in A: 0\leq x\leq 1\}$, $[0,s_2^*(1)]=\{\omega\in A^*: 0\leq\omega\leq s_2^*(1)\}$. Indeed, let $0\leq x\leq 1$ in A. Then for $y\in A^+$,

$$\begin{aligned}(s_2^*(x),y) &= s_2(x,y)=s_\psi(E(x),E(y))\\ &= s_\psi(x,E(y))\\ &\leq s_\psi(1,E(y))\\ &= \psi(E(y))\\ &= (s_2^*(1),y)\end{aligned}$$

Thus $s_2^*(x)\in[0,s_2^*(1)]$.

Suppose $\rho \in A^*$, $0 \leq \rho \leq s_2^*(1)$. Then $0 \leq \rho \circ E \leq s_\psi^*(1)$, because if $y \in N^+$

$$\begin{aligned} \rho \circ E(y) &\leq (s_2^*(1), E(y)) \\ &= s_\psi(1, E(y)) \\ &= s_\psi(E(1), y) \\ &= (s_\psi^*(1), y) . \end{aligned}$$

Since s_ψ is a self-polar form $s_\psi^*([0,1]) = [0, s_\psi^*(1)]$, hence there exists $x \in N$, $0 \leq x \leq 1$, such that for $y \in N$,

$$\begin{aligned} \rho \circ E(y) &= (s_\psi^*(x), E(y)) \\ &= s_\psi(x, E(y)) \\ &= s_\psi(E(x), y) \\ &= (s_2(E(x)), E(y)) . \end{aligned}$$

In particular, if $y \in A$, then $\rho(y) = (s_2^*(E(x)), y)$. Since $s_2^*(E(x)) \in [0, s_2^*(1)]$, we have shown that $[0, s_2^*(1)] \subset s_2^*([0,1])$, hence they are equal, and s_2 is a self-polar form on $A \times A$ as asserted. If $y \in A$ we have

$$(s_2^*(1), y) = s_\psi(1, E(x)) = \psi(E(y)) = \varphi(y) = (s_\varphi^*(1), y) .$$

Thus by [10, Thm. 1.2], $s_2 = s_\varphi$, i.e. $s_\varphi = s_\psi|_{A \times A}$, proving (ii).

(ii)$\Rightarrow$(i) Let $x \in N$, $0 \leq x \leq 1$. The function

$$a \to s_\psi(a, x), \qquad a \in A ,$$

defines a functional φ_x on A such that $0 \leq \varphi_x \leq \psi|_A = \varphi$. Since s_φ is a self-polar form $s_\varphi^*([0,1] = [0, s_\varphi^*(1)]$, hence there is $y \in A$, $0 \leq y \leq 1$, such that

$$\varphi_x(a) = s_\varphi(a, y) .$$

y is unique since s_φ is an inner product on A, φ being faithful. Put $E(x) = y$. We thus get a map

$$\{x \in N : 0 \leq x \leq 1\} \to \{y \in A : 0 \leq y \leq 1\} .$$

By definition of y

$$s_\psi(a, x) = s_\varphi(a, E(x)), \qquad a \in A, \;\; x \in N, \;\; 0 \leq i \leq 1 .$$

As $N = \text{span}\{x \in N : 0 \le x \le 1\}$, E has a unique extension to a linear map $N \to A$ such that $s_\psi(a,x) = s_\varphi(a, E(x))$ for all $a \in A$, $x \in N$. By (ii) it follows that for $x \in A$

$$s_\varphi(a,x) = s_\psi(a,x) = s_\varphi(a, E(x)), \qquad a \in A.$$

Thus $E(x) = x$, and $E : N \to A$ is a positive projection. Furthermore, for $x \in N$,

$$\varphi(E(x)) = s_\varphi(1, E(x)) = s_\psi(1,x) = \psi(x).$$

Thus (i) follows, since the identity $\varphi \circ E = \psi$ shows that E is normal and faithful.

(ii)$\Rightarrow$(iii). Since (i)$\Leftrightarrow$(ii) there is a faithful normal projection $E : N \to A$ such that $\varphi \circ E = \psi$, and $s_\varphi = s_\psi|_{A\times A}$. Let $H_\varphi^\#$ denote the completion of A with respect to the norm $\|x\|_\varphi^\# = \varphi(x \circ x)^{1/2}$. Similarly define $H_\psi^\#$. Then there is a natural inclusion $H_\varphi^\# \subset H_\psi^\#$.

We assert that the orthogonal projection $p : H_\psi^\# \to H_\varphi^\#$ is an extension of E. For this we must show that for $x, y \in N$, with obvious notation,

$$(E(x), y)_\psi^\# = (x, E(y)_\psi^\# = (E(x), E(y))_\varphi^\#.$$

But, by an application of [7, Lem 4.1] we have

$$\begin{aligned}
(E(x), y)_\psi^\# &= \psi(E(x) \circ y) = \\
&= \psi(E(E(x) \circ y)) \\
&= \psi(E(x) \circ E(y))) \\
&= \varphi(E(x) \circ E(y)) \\
&= (E(x), E(y))_\varphi^\#,
\end{aligned}$$

and similarly for $(x, E(y))_\psi^\#$. Thus the assertion follows. From the proof of [4, Thm. 3.3] ρ_t^φ extends to a self-adjoint operator u_t on $H_\varphi^\#$ and ρ_t^ψ to a self-adjoint operator v_t on $H_\psi^\#$, satisfying $\|u_t\| \le 1$, $\|v_t\| \le 1$, and

$$u_s u_t = \frac{1}{2}\Big(u_{s+t} + u_{s-t}\Big), \qquad u_0 = 1,$$

and similarly for v_t. Furthermore there exist, possibly unbounded, positive self-adjoint operators D and D' on $H_\varphi^\#$ and $H_\psi^\#$ respectively such that

$$u_s = \cos(sD), \quad v_s = \cos(sD'), \qquad s \in \mathbb{R}.$$

Thus by the proof of [4, Thm. 3.3]

$$s_\varphi(x,y) = \left(\cosh\left(\frac{D}{2}\right)^{-1} x, y\right)^{\#}_\varphi, \qquad x,y \in A.$$
$$s_\psi(x,y) = \left(\cosh\left(\frac{D'}{2}\right)^{-1} x, y\right)^{\#}_\psi, \qquad x,y \in N.$$

Let $C = \cosh\left(\frac{D}{2}\right)^{-1}, C' = \cosh\left(\frac{D'}{2}\right)^{-1}$. Then C and C' are bounded self-adjoint operators. We assert that $C = C'|_{H^{\#}_\varphi}$. For this it suffices to show that for $a \in A$, $y \in N$

$$(Ca, y)^{\#}_\psi = (C'a, y)^{\#}_\psi .$$

However, from the above $p : H^{\#}_\psi \to H^{\#}_\varphi$ extends E, so that

$$\begin{aligned}(Ca, y)^{\#}_\psi &= (p(Ca), y)^{\#}_\psi = (Ca, py)^{\#}_\psi \\ &= (Ca, E(y))^{\#}_\psi = (Ca, E(y))^{\#}_\varphi .\end{aligned}$$

Therefore it remains to be shown that

$$(C'a, y)^{\#}_\psi = (Ca, E(y))^{\#}_\varphi ,$$

or rather

$$s_\psi(a, y) = s_\varphi(a, E(y)) .$$

But this was shown in the proof of (i)⇒(ii). It follows that $H^{\#}_\varphi$ is C'-invariant, and $C = C'|_{H^{\#}_\varphi}$ as asserted.

Now the functions $C \to D \to \cos(sD) \to u_s$, and similarly for $C' \to v_s$, are Borel functions of C and C' respectively. Thus $u_s = v_s|_{H^{\#}_\varphi}$, and we can conclude that $\rho^\varphi_s = \rho^\psi_s|_A$.

(iii)⇒(ii) By Theorem 4.1, for all $x, y \in A$

$$\begin{aligned}s_\varphi(x,y) &= \int_{-\infty}^{\infty} \varphi(\rho^\varphi_t(x) \circ y) \cosh(\pi t)^{-1} dt \\ &= \int_{-\infty}^{\infty} \psi(\rho^\psi_t(x) \circ y) \cosh(\pi t)^{-1} dt \\ &= s_\psi(x,y) ,\end{aligned}$$

proving (ii). This completes the proof of the theorem.

Corollary 4.3 Let N be a von Neumann algebra and A a reversible JW-subalgebra of N_{sa} such that $Z(A) = Z(M)_{\mathrm{sa}}$, where M is the von Neumann algebra generated by A. Suppose ψ is a faithful normal state of N such that

$$\sigma_t^{\psi}(a) + \sigma_{-t}^{\psi}(a) \in A \ \forall t \in \mathbf{R},\ a \in A\,.$$

Then $\sigma_t^{\psi}(M) = M \ \forall t \in \mathbf{R}$.

Proof Since $\rho_t^{\psi}(x) = \frac{1}{2}(\sigma_t^{\psi}(x)+\sigma_{-t}^{\psi}(x))$, $x \in N_{\mathrm{sa}}$, by Theorem 4.2 there exists a faithful normal projection $E : N \to A$ such that $\varphi \circ E = \psi$, where $\varphi = \psi|_A$. From our assumptions on A and the classification of JW-algebras there exist two central projections e and f in A with sum 1 such that $eA = eM_{\mathrm{sa}}$, $(R(A) + iR(A))f = Mf$, $(R(A) \cap iR(A))f = \{0\}$. We have $E(exe) = eE(x)e = E(x)e = eE(x)$ for $x \in N$, and similarly for f. Thus $E(x) = E(exe) + E(fxf)$, so that $E(xe) = E(exe) = E(x)e$. It follows that

$$\psi(xe) = \varphi(E(xe)) = \varphi(E(x)e) = \varphi(eE(x)) = \psi(ex)\,.$$

Thus e and $f \in M_{\psi}$ – the centralizer of ψ. In particular $\sigma_t^{\psi}(e) = e$, $\sigma_t^{\psi}(f) = f$. It thus suffices to consider the two cases $e = 1$ and $f = 1$ separately. If $A = M_{\mathrm{sa}}$ then E is a conditional expectation, so the conclusion follows from Takesaki's theorem [9].

Assume $R(A) \cap iR(A) = \{0\}$ and $Z(A) = Z(M)_{\mathrm{sa}}$. By Theorem 3.2 there exists a faithful normal conditional expectation $F : N \to M$ such that $E = P \circ F$ where $P : M \to A$ is the canonical projection. Since $P = E|_M$, $\varphi \circ P = \psi|_M$. Thus

$$\psi = \varphi \circ E = \varphi \circ P \circ F = \psi|_M \circ F\,,$$

so F is ψ-invariant. Again it follows from Takesaki's theorem that
$\sigma_t^{\psi}(M) = M$, $t \in \mathbf{R}$. QED

References

[1] Choi, M-D., and Effros, E., Injectivity and operator spaces, *J. Functional Anal.* **24** (1974), 156–209.

[2] Edwards, C. M., Conditional expectations on Jordan algebras, *Fund. Aspects of Quantum Theory*, Nato ASI Series, Plenum Press (1985), 75–81.

[3] Effros, E., and Størmer, E., Positive projections and Jordan structure in operator algebras, *Math. Scand.* **45** (1979), 127–138.

[4] Haagerup, U., and Hanche-Olsen, H., Tomita-Takesaki theory for Jordan algebras, *J. Operator Theory* **11** (1984), 343–364.

[5] Hanche-Olsen, H., and Størmer, E., Jordan opereator algebras, Monographs and Studies in Math. 21, Pitman 1984.

[6] Pedersen, G. K., and Størmer, E., Traces on Jordan algebras, *Can. J. Math.* **34** (1982), 370–373.

[7] Størmer, E., Decomposition of positive projections on C^*-algebras, *Math. Ann.* **247** (1980), 21–41.

[8] Størmer, E., Positive projections onto Jordan algebras and their enveloping von Neumann algebras. Ideas and Methods in Math. Anal., Stochastics and Applic., Cambridge Univ. Press (1992), 389–393.

[9] Takesaki, M., Conditional expectation in von Neumann algebra, *J. Functional Anal.* **9** (1972), 306–321.

[10] Woronowicz, S. L., Self-polar forms and their applications to the C^*-algebra theory, *Rep. Math. Phys.* **6** (1974), 487–495.