Non commutative deformations of modules

by

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Introduction

In this paper I shall describe, or rather sketch, a non commutative deformation theory, and show its relationship to the notion of quivers, and other combinatorial invariants, in the theory of representations of Artin k-algebras, and to the notion of almost split sequences in general.

I claim that the formal or local moduli in this non commutative context will be of interest in many situations, for example in the study of singularities in algebraic geometry.

The idea is very simple. Let k be a field and let $\boldsymbol{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R, for which $R/\boldsymbol{r} \simeq k^r$ where $\boldsymbol{r}$ is the radical of R. For $r = 1$, there is an obvious inclusion of categories

$$\boldsymbol{l} \subseteq \boldsymbol{a}_1$$

where $\boldsymbol{l}$, as usual, denotes the category of commutative local artinian k-algebras with residue field k.

Fix a not necessarily commutative k-algebra A and consider an A-module M. The ordinary deformation functor

$$\mathrm{Def}_M : \boldsymbol{l} \to \mathrm{Sets}$$

is then defined. Assuming $\mathrm{Ext}^i_A(M,M)$ has finite k-dimension for $i = 1,2$, it is well known, see [S] or [La2], that Def_M has a noetherian prorepresenting hull H, *the formal moduli of M*. Moreover H can be computed in terms of $\mathrm{Ext}^i_A(M,M)$, $i = 1,2$ and their matric Massey products, see [La2].

Notice that when A is a commutative local k-algebra with residue field $A/\boldsymbol{m} = k$, then the completion $\hat{A}$ is the formal moduli of the residue field k considered as an A-module, see [La2].

In particular $\hat{A}$ may be reconstructed from $\mathbf{Ext}^i_A(k,k)$, $i = 1,2$ and their matric Massey products.

Consider, in the general case, a family $V = \{V_i\}_{i=1}^r$ of A-modules. In §2 we shall define a deformation functor.

$$\mathrm{Def}_V : \boldsymbol{a}_r \to \mathrm{Sets}$$

generalizing the functor Def_M above, retaining its main properties, such that:

(i) There exists a pro-object $H(V) = H$ of $\boldsymbol{a}_r$, a prorepresenting hull of the functor Def_V, which we shall refer to as the formal moduli of V.

(ii) H can be computed in terms of the system of Ext-spaces

$$\mathrm{Ext}^l_A(V_i, V_j),\ l = 1,2,\ i,j = 1,\cdots,r,$$

and their matric Massey products.

(iii) There exists a formal versal family of $H\hat{\otimes}\hat{A}$-modules $\widetilde{V}$ which, when algebraic, describes the system of deformations of the individual V_i's and their incidences, etc.

(iv) If A is an object of $\boldsymbol{a}_r$, and if $V = \{k_i\}_{i=1}^r$ is the family of simple A-modules, i.e. such that $A/\boldsymbol{r} \simeq \prod_{i=1}^{r} k_i$, then $A \simeq H(V)$.

Applying this deformation theory to the case where A is hereditary, we observe that quite a few of the known results on such algebras can be deduced rather easily from general principles. This holds in particular for the classification of algebras of finite representation type, their Gabriel quiver structure and the properties of the corresponding quadratic forms, see e.g. [G], and §1 and §2, below.

In general, the Auslander-Reiten quiver relates to the tangent structure of $H(V)$ in the following way, see §3:

There exists in the non commutative deformation theory an obvious analogy to the notion of prorepresenting (modular) substratum $\boldsymbol{H}_0$ of the formal moduli $\boldsymbol{H}$, on which the construction of the local moduli suite in the commutative case, is based, see [La-Pf]. The tangent space $\boldsymbol{t}_0$ of H_0 is determined by a family of subspaces

$$\boldsymbol{t}_0(i,j) =: \mathrm{Ext}_0^1(V_i, V_j) \subseteq \mathrm{Ext}_A^1(V_i, V_j), \qquad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to V, and by a subspace,

$$\boldsymbol{t}_0(\Delta) \subseteq \prod_i \mathrm{Ext}_A^1(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\{V_i\}_{i=1}^r$. If $V = \{V_i\}_{i=1}^r$ is the set of all indecomposables of some "natural" category of A-modules, say maximal Cohen-Macaulay, and if

(i) $\mathrm{End}_A(V_i)$ is a local k-algebra for $i = 1, \cdots, r$

(ii) $\dim_k \mathrm{Ext}_A^1(V_i, V_j) < \infty$ for $i, j = 1, \cdots, r$.

then we show that the above notion of "almost split sequence" coincides with that of Auslander.

The notion of A.-R. quiver then turns out to correspond to an incidence diagram for modular deformations. Observe that, in general, the k-algebra H_0 and its corresponding modular family $\widetilde{V}_0$ contains much more information than what may be deduced from the tangent level.

This paper is a corrected and slightly extended version of a preliminary manuscript written and circulated in 1988. The ideas of that manuscript have been the basis for a couple of Master theses at the Mathematics Department of the University of Oslo.

In particular Runar Ile has in his Masters theses, Oslo 1990, computed the non-commutative deformations for some classes of Maximal Cohen-Macaulay-modules over the simple singularities, restricting to one-member families $\{V\}$, and Arvid Siqveland has in his Master theses, Oslo 1990, done nice calculations on the formal moduli, in the commutative situation, for the MCM-modules for E_6 (curve case). These results will, hopefully, occur shortly in these Preprint Series.

1 Preliminaries on deformations of modules

1.1 Formal moduli

Let k be an algebraically closed field of characteristic 0. Given a k-algebra A and an A-module M, we may consider the deformation functor

$$\mathrm{Def}_M : \boldsymbol{l} \to \mathbf{Sets}$$

where $\boldsymbol{l}$ is the category of local artinian k-algebras s with residue field k, i.e. such that $\dim_k s$ is finite. In [La2] we prove that, when

$$\dim_K \mathrm{Ext}^i_A(M,M) < \infty, \qquad i = 1,2$$

there exists a complete local k-algebra $H^\wedge$ determined by a family of partially defined matric Massey products

$$\bigotimes^n \mathrm{Ext}^1_A(M,M) \cdot \to \mathrm{Ext}^2_A(M,M),$$

which is a prorepresenting hull for Def_M, i.e. such that there is a surjective smooth morphism of functors

$$\rho : \mathrm{Mor}(H^\wedge, -) \to \mathrm{Def}_M$$

inducing an isomorphism on the tangent level, i.e. such that

$$\rho(k[\varepsilon]) : \mathrm{Mor}(H^\wedge, k[\varepsilon]) = \mathrm{Def}_M(k[\varepsilon])$$

Notice that we may identify $\mathrm{Mor}(H^\wedge, k[\varepsilon])$ with the Zariski tangent space $\boldsymbol{t}_H \simeq (\boldsymbol{m}/\boldsymbol{m}^2)^*$ where $\boldsymbol{m}$ is the maximal ideal of $H^\wedge$, and we may identify $\mathrm{Def}_M(k[\varepsilon])$ with $\mathrm{Ext}^1_A(M,M)$. There is no requirement of commutativity in this set-up. Therefore A may very well be a non commutative k-algebra. If it is, we shall agree to consider only right A-modules M.

Observe that corresponding to the identity $1_H \in \mathrm{Mor}_k(H^\wedge, H^\wedge)$ there exists the *formal versal family* $M^\wedge = \{M_n\}_{n\geq 1}$, where each M_n is a deformation of M to $H/\boldsymbol{m}^n$, i.e. an $H/\boldsymbol{m}^n \otimes_k A$-module, flat as an $H/\boldsymbol{m}^n$-module, such that $M \simeq k \otimes_H M_n$.

1.2 The Kodaira-Spencer morphism

Let S be any commutative k-algebra and consider any $S \otimes_k A$-module M_S. There exists a Kodaira-Spencer morphism

$$g : \mathrm{Der}_k(S) \to \mathrm{Ext}^1_{S \otimes_k A}(M_S, M_S)$$

as explained in [La-Pf] §3. Explicitely, g is given as follows. Consider an $S \otimes_k A$-free resolution $\widetilde{L}$ of M_S with differential d. Given any $D \in \mathrm{Der}_k(S)$, we obtain

$$0 = D(d \cdot d) = D(d) \cdot d + d \cdot D(d)$$

It turns out that the element

$$\{(-1)^i D(d_i)\} \in \mathrm{Hom}^1_{S \otimes A}(\widetilde{L}.,\widetilde{L}.)$$

is a cocycle in the Yoneda complex $\mathrm{Hom}^{\cdot}_{S \otimes A}(\widetilde{L}.,\widetilde{L}.)$, which defines an element

$$g(D) \in \mathrm{Ext}^1_{S \otimes A}(M_S, M_S).$$

Obviously, if $g(D) = 0$, then

$$D(d) = d \cdot \eta + (-1)^* \eta \cdot d$$

for some $\eta = \{y_i\} \in \mathrm{Hom}^0_{S \otimes A}(\widetilde{L}.,\widetilde{L}.)$. This η induces an isomorphism

$$id + \eta\varepsilon \in \mathrm{Hom}_{S[\varepsilon] \otimes A}(M_S[\varepsilon], M_S[\varepsilon])$$

which is $id + D\varepsilon : S[\varepsilon] \to S[\varepsilon]$ linear. This, incidentally, proves that

$$D \to g(D) = \{D(d_i)\}$$

is the Kodaira-Spencer map and provides us with a neat way of computing g.

Now, given any two A-modules M and N such that $\mathrm{Ext}^i_A(M, N) = 0$ for $i \gg 0$ and $e_i = \dim_k \mathrm{Ext}^i_A(M, N) \leq \infty$ for all $i \geq 0$. Put

$$\mathcal{X}(M,N) := \sum_{i=1}^{r} (-1)^i e_i$$

Suppose there is a family $V = \{V_i\}_{i=1}^r$ of A-modules such that each pair V_i, V_j satisfy the conditions above, and suppose M is constructed by successive extensions using x_i times the module V_i, then

$$\mathcal{X}(M) := \mathcal{X}(M,M) = \sum_{i,j=1}^{r} x_i \cdot x_j \mathcal{X}(V_i, V_j)$$

Definition In the situation above, we shall denote by

$$q_V = q$$

the quadratic form

$$q_V(M,N) = \mathcal{X}(M,N) + \mathcal{X}(N,M).$$

In particular we find, with the notations above, that $q_V(M,M)=2x_i^2\mathcal{X}(V_i,V_i)+\sum_{i,j=1,i\neq j}^r x_i\cdot x_j\mathcal{X}(V_i,V_j)$. This, type II, form will be called *the quadratic form associated to* V.

1.3 Algebraization of the formal versal family

Going back to the situation of (1.1), we shall say that a pointed k-algebra H of finite type and a $H\otimes_k A$-module $\widetilde{M}$ is *an algebraization* of the formal versal family if for every n there is an isomorphism of projective systems,

$$(H/\boldsymbol{m}_0^n)\otimes_H \widetilde{M}=M_n$$

where $\boldsymbol{m}_0$ is the maximal ideal corresponding to the base point of H. Using Artins approximation theorem we may easily prove the following.

Proposition If A is a graded Noetherian or an Artinian k-algebra, M is an A-module of finite type and of finite projective dimension such that

(i) $\dim_k \mathrm{Ext}_A^i(M,M)<\infty, \quad i=1,2,$

(ii) All sufficiently high order Massey products are zero.

Then there is an algebraization of the formal versal family of M.

Proof: In this case there exists a pointed k-algebra of finite type H, such that $H^\wedge$, the completion of H is the formal moduli, see [La2]. Let $L^\wedge.$ be a finite free resolution of the formal versal family, i.e. such that $L^\wedge. : 0\to (H\hat{\otimes}A)^{n_{p+1}} \xrightarrow[d_p]{} \cdots \xrightarrow[d_0]{} (H^\wedge\otimes A)^{n_0} \xrightarrow[\rho]{} M^\wedge\to 0$ is an exact sequence.

Since the equations of matrices $d_0\cdot d_1=0$ have solutions in $H^\wedge$, there are solutions in the Henselization $\widetilde{H}$ providing us with the resolution $\widetilde{L}.$ of an algebraization $\widetilde{M}$ of $M^\wedge$. QED

Corollary Let A be a finite dimensional hereditary k-algebra and let M be any finite type A-module. Then there is an algebraic miniversal deformation $\widetilde{M}$ of M with base space $\boldsymbol{H}=\mathrm{Spec}(H)$ such that $H=\mathrm{Sym}_k(\mathrm{Ext}_A^1(M,M)^*)\simeq k[t_1,\ldots,t_{e_1}]$.

Proof: Let $0\to L_1\xrightarrow{d} L_0\to M\to 0$ be an A-free resolution of M. Then $\mathrm{Ext}_A^1(M,M)\simeq \mathrm{Hom}_A(L_1,L_0)/(\mathrm{im}d_*+\mathrm{im}d^*)$ where $d_*:\mathrm{Hom}_A(L_1,L_1)\to \mathrm{Hom}_A(L_1,L_0)$ and $d^*:\mathrm{Hom}_A(L_0,L_0)\to\mathrm{Hom}_A(L_1,L_0)$ are the obvious morphisms. Pick a basis $\{t_1^*,\ldots,t_{e_1}^*\}$ of $\mathrm{Ext}_A^1(M,M)$ represented as morphisms $\xi_1,\ldots,\xi_{e_1}$ of $\mathrm{Hom}_A(L_1,L_0)$. Consider the morphism

$$\tilde{d}: H\otimes_k L_1\to H\otimes_k L_0$$

defined by

$$\tilde{d}=\mathrm{id}_H\otimes d+\sum_{i=1}^{e_1} t_i\xi_i\ .$$

By the general theory $\widetilde{H}=\mathrm{coker}\,\tilde{d}$ is a miniversal deformation of M in a neighborhood of $\mathbf{0}\in H$. QED

Remark Let, from now on, $\boldsymbol{H}$ be an affine open subset of this universal base space, containing 0 and on which $\tilde{d}$ has maximal rank. We may assume $H = k[t_1, \dots, t_{e_1}]_{\{s\}}$ for some $s \in k[t_1, \dots, t_{e_1}]$, $s(0) \neq 0$. There is an exact sequence of $H \otimes_k A$-modules

$$0 \to H \otimes_k L_1 \xrightarrow{\tilde{d}} H \otimes_k L_0 \to \widetilde{M} \to 0$$

Since $\tilde{d}$ has maximal rank everywhere on $\boldsymbol{H}$ we find for every $\boldsymbol{t} \in \boldsymbol{H}$ an exact sequence of A-modules.

$$0 \to L_1 \xrightarrow{\tilde{d}(t)} L_0 \to M(\boldsymbol{t}) \to 0$$

In particular $\dim_k \widetilde{M}(\boldsymbol{t}) = \dim_k L_0 - \dim_k L_1$ is constant. There is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 \to & \mathrm{End}_{H\otimes_k A}(\widetilde{M}) & \to & \mathrm{Hom}_{H\otimes_k A}(H\otimes_k L_0, \widetilde{M}) & \to & \mathrm{Hom}_{H\otimes_k A}(H\otimes_k L_1, \widetilde{M}) & \to & \mathrm{Ext}^1_{H\otimes_k A}(\widetilde{M}, \widetilde{M}) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \psi\downarrow & \\
0 \to & \mathrm{End}_A(\widetilde{M}(\boldsymbol{t})) & \to & \mathrm{Hom}_A(H\otimes_k L_0, \widetilde{M}(\boldsymbol{t})) & \to & \mathrm{Hom}_A(H\otimes_k L_1, \widetilde{M}(\boldsymbol{t})) & \to & \mathrm{Ext}^1_A(\widetilde{M}(\boldsymbol{t}), \widetilde{M}(\boldsymbol{t})) & \to 0
\end{array}$$

Since $\mathrm{Ext}^i_A = 0$ for $i \geq 2$, the vertical map ψ is onto and the Snakes lemma produces a long exact sequence

$$0 \to \mathrm{Hom}_{H\otimes_k A}(\widetilde{M}, \boldsymbol{m_t} \cdot \widetilde{M}) \to \mathrm{End}_{H\otimes_k A}(\widetilde{M}) \to \mathrm{End}_A(\widetilde{M}(\boldsymbol{t}))$$

$$\to \mathrm{Ext}^1_{H\otimes_k A}(\widetilde{M}, \boldsymbol{m_t} \cdot \widetilde{M}) \to \mathrm{Ext}^1_{H\otimes_k A}(\widetilde{M}, \widetilde{M}) \to \mathrm{Ext}^1_A(\widetilde{M}(\boldsymbol{t}), \widetilde{M}(\boldsymbol{t})) \to 0$$

from which we read off that

$$\mathcal{X}(\widetilde{M}(\boldsymbol{t})) = \mathcal{X}(M) \qquad \text{for all } \boldsymbol{t} \in \boldsymbol{H}.$$

Recall the following

Definition The A-module M is called rigid if $\mathrm{Ext}^1_A(M, M) = 0$.

1.4 Local moduli and the conditions $(A_1), (A_2)$ and (V')

In [La-Pf] we prove that under the conditions,

(A_1) There exists an algebraization $(H, \widetilde{M})$ of the formal versal family.

(A_2) This (mini-) versal family is formally versal in a neighborhood of the base point.

(V') Essentially saying that $\mathrm{Ext}^2_A(M, M) = 0$, implying that H is nonsingular.

there exists a *local moduli suite* for M, i.e. a family of algebraic spaces

$$\{\boldsymbol{M}_\tau\}_{\tau=0}^{e_1} \quad \text{where} \quad e_1 = \dim_k \mathrm{Ext}_A^1(M, M),$$

the points of which classify the isomorphism classes of the A-modules

$$\{\widetilde{M}(\boldsymbol{t}) = k(\boldsymbol{t}) \otimes_H \widetilde{M} | \boldsymbol{t} \in \mathrm{Spec}(H) = \boldsymbol{H}\}$$

up to a finite – to – one correspondence.

The construction of this moduli suite depends upon the notion of modular, or prorepresentable, substratum of $\boldsymbol{H}^\wedge$. This is a closed subscheme $\boldsymbol{H}_0^\wedge \subseteq \boldsymbol{H}^\wedge$, maximal with respect to the following property: In the commutative diagram of functors on $\boldsymbol{l}$,

$$\begin{tikzcd}
\mathrm{Mor}(H^\wedge, -) \arrow[dr] & \\
& \mathrm{Def}_M \\
\mathrm{Mor}(H_0^\wedge, -) \arrow[uu] \arrow[ur, "\rho_0"'] &
\end{tikzcd}$$

the morphism ρ_0 is injective.

It is obvious that the Lie-algebra $\mathrm{End}_A(M)$ operates on $\mathrm{Ext}_A^1(M, M)$. In [La-Pf] §2, we prove the following.

Proposition The tangent space $\boldsymbol{t}_{H_0}$ of $H_0^\wedge$ coincides with the subspace

$$\mathrm{Ext}_A^1(M, M)^{\mathrm{End}_A(M,M)} = \{\xi \in \mathrm{Ext}_A^1(M, M) \mid \forall \phi \in \mathrm{End}_A(M),\ \phi\xi - \xi\phi = 0\}.$$

of the tangent space $\boldsymbol{t}_H = \mathrm{Ext}_A^1(M, M)$ of $H^\wedge$.

Assume for a moment that there exists an algebraization $(H, \widetilde{M})$ of the formal versal family of M. It is then reasonable to make the following.

Definition M is said to be of *simple deformation type*, if for all $\boldsymbol{t} \in \boldsymbol{H}$, in a neighborhood of the base point, $H_0(\widetilde{M}(\boldsymbol{t})) \simeq k$.

Proposition Suppose M is not rigid, and

(i) $\mathrm{End}_A(M)$ is a local k-algebra

(ii) $\dim_k \mathrm{Ext}_A^1(M, M) < \infty$

Then $\boldsymbol{t}_{H_0} \neq 0$, so M is not of simple deformation type.

Proof: As Lie-algebra $\mathrm{End}_A(M)$ acts nilpotentely on $\mathrm{Ext}_A^1(M, M)$. Therefore by Engels theorem

$$\boldsymbol{t}_{H_0} = \mathrm{Ext}_A^1(M, M)^{\mathrm{End}_A(M)} \neq 0.$$

QED

Proposition Assume the conditions $(A_1), (A_2)$ and (V') above and let $(\boldsymbol{H}, \widetilde{M})$ be an algebraization of the formal versal family of M. Suppose $\mathrm{End}_A(M) = k$. Then

(1) M is indecomposable

(2) $\boldsymbol{H}_0 = \boldsymbol{H}$ in a neighborhood of the base point, and for every $\boldsymbol{t} \in \boldsymbol{H}$, $\widetilde{M}(\boldsymbol{t})$ is indecomposable with $\mathrm{End}_A(\widetilde{M}(\boldsymbol{t})) = k$.

Proof: Since $\mathrm{End}_A(M) = k$ it is easy to see that the natural map $H_*^\wedge = \mathrm{End}_{H_*\hat{\otimes} A}(\widetilde{M}_*^\wedge)$ is an isomorphism, and that $H_{0*}^\wedge = H_*^\wedge$, this follows from Schlessinger [Sch].

Now consider the Kodaira-Spencer map

$$g : \mathrm{Der}_k(H) \to \mathrm{Ext}^1_{H\otimes_k A}(\widetilde{M}, \widetilde{M})$$

and let $\boldsymbol{V} = \ker g$.

Then, by (3.12) [La-Pf], the modular substratum $\boldsymbol{H}_0 \subseteq \boldsymbol{H}$ is the closed subscheme along which $\boldsymbol{V}$ vanish. Since $H_{0*}^\wedge = H_*^\wedge$, $\boldsymbol{H}_0$ contains an open neighborhood of $*$, and we may, as well, assume this to be $\boldsymbol{H}$. Therefore $\boldsymbol{H}$ is modular. But then the morphism

$$\mathrm{End}_{H\otimes A}(\widetilde{M}) \to \mathrm{End}_A(\widetilde{M}(t), \widetilde{M}(t))$$

is surjective for all $\boldsymbol{t} \in \boldsymbol{H}$. Moreover the completion of the natural morphism

$$H \to \mathrm{End}_{H\otimes A}(\widetilde{M})$$

is an isomorphism.

As our assumptions imply that $\mathrm{End}_{H\otimes A}(\widetilde{M})$ is an H-module of finite type, this again implies that for some neighborhood of $*$,

$$\mathrm{End}_{H\otimes A}(\widetilde{M}) = H$$

and the Proposition follows. QED

In [La-Pf], §3, we construct the room $\boldsymbol{M}_\tau$ of the local moduli suite by glueing together local representatives of the $\boldsymbol{H}_0^\wedge$'s corresponding to those $\widetilde{M}(\boldsymbol{t})$, $\boldsymbol{t} \in \boldsymbol{H}$ such that,

$$e_1(\boldsymbol{t}) := \dim_k \mathrm{Ext}^1_A(\widetilde{M}(\boldsymbol{t}),\ \widetilde{M}(\boldsymbol{t})) = \tau.$$

This is possible in the étale topology, but not necessarily in the Zariski topology.

Notice that the following result is slightly different from the above.

Corollary Suppose $(A_1), (A_2)$ and (V') hold. Suppose moreover that $\mathrm{End}_A(M) = k$ and that $\mathrm{Ext}^1_A(M,M) \neq 0$ (i.e. M is not rigid). Then there exists an infinite modular family of deformations of M. In particular there exists an infinite number of isomorphism classes of indecomposable A-modules.

Proof: Since by the above Proposition, $\boldsymbol{H}_0^\wedge = \boldsymbol{H}^\wedge$ for every module in the family $\widetilde{M}(\boldsymbol{t})$, it follows from [La-Pf] §3, that $\dim \boldsymbol{M}_{e_1} = \dim_k \mathrm{Ext}^1_A(M,M) \geq 1$. The rest follows from the:

Proposition Let $(\widetilde{M}, H)$ be a modular family of A-modules, such that all geometric fibers $M(\boldsymbol{t})$ are isomorphic. Then $\boldsymbol{H}$ is finite, i.e. H is artinian.

Proof: Suppose $\boldsymbol{H}$ is not finite, then there exists a valuation ring O and a surjective homomorphism $H \to O$. By modularity the Kodaira-Spencer morphism

$$g : \mathrm{Der}_k(O) \to \mathrm{Ext}_{O \otimes_k A}(\widetilde{M} \otimes_H O, \widetilde{M} \otimes_H O)$$

is injective. Let K be the field of quotients of O and let $k \subseteq \Omega$ be any big enough algebraically closed field extension. Then $(\widetilde{M} \otimes_H O) \otimes_O \Omega = \widetilde{M} \otimes_H \Omega$ is the generic fiber of $\widetilde{M}$, which by assumption is isomorphic to $M \otimes_k \Omega$, M being the fiber of $\widetilde{M}$ at the center of O. But this is a contradiction, since tensorizing g with K on O, we find a commutative diagram

$$\begin{array}{ccc}
g : \mathrm{Der}_k(O) & \longrightarrow & \mathrm{Ext}^1_{O \otimes_k A}(\widetilde{M} \otimes_H O, \widetilde{M} \otimes_H O) \\
\downarrow & & \downarrow \\
g_K : \mathrm{Der}_k(K) & \longrightarrow & \mathrm{Ext}^1_{K \otimes_k A}(\widetilde{M} \otimes_H K, \widetilde{M} \otimes_H K)
\end{array}$$

In which g_K must be injective. However tensorization by Ω on K gives us a morphism of Ω-vectorspaces

$$g_\Omega : \mathrm{Der}_k(K) \otimes_K \Omega \to \mathrm{Ext}^1_{\Omega \otimes_k A}(\widetilde{M} \otimes_H \Omega, \widetilde{M} \otimes_H \Omega)$$

which must be zero since $\widetilde{M} \otimes_H \Omega \simeq M \otimes_k \Omega$ is a "constant" family.

QED

Proposition Let A be an artinian k-algebra. Suppose A is hereditary and that there exists an A-module M with $\mathcal{X}(M) \leq 0$, $(q(M) \leq 0)$ then there exists an infinite family of indecomposable modules.

Proof: Consider the algebraic miniversal family $\widetilde{M}$ of M. There are two cases. Either $\widetilde{M}$ is modular, therefore infinite, or it is not. Suppose $\widetilde{M}$ modular and suppose $M \simeq M_1 \oplus M_2$, $M_1 \neq 0, M_2 \neq 0$. Then $\mathrm{Ext}^1_A(M_i, M_j) = 0$ for $i \neq j$. Otherwise the endomorphisms $(1_{M_1}, 0)$ and $(0, 1_{M_2})$ in $\mathrm{End}_A(M_1 \oplus M_2)$ acts nontrivially on the components $\mathrm{Ext}^1_A(M_i, M_j)$, $i \neq j$ of $\mathrm{Ext}^1_A(M,M)$, which would contradict modularity. Therefore

$$\mathrm{Ext}^1_A(M,M) = \mathrm{Ext}^1_A(M_1,M_1) \oplus \mathrm{Ext}^1_A(M_2,M_2)\,.$$

But then $\widetilde{M} \simeq \widetilde{M}_1 \oplus \widetilde{M}_2$ and we must have $\mathcal{X}(M_i) \leq 0$ for at least one $i=1,2$. Moreover $\dim_k M_i < \dim M$, $i=1,2$. Assume that $\widetilde{M}$ is not modular. Then there is a deformation M_1 of M such that $\dim_k \mathrm{Ext}^1_A(M_1, M_1) = \min\{\dim_k \mathrm{Ext}^1_A(\widetilde{M}(\underset{\smile}{t}), \widetilde{M}(t)) \mid t \in \boldsymbol{H}\}$. Since $\mathcal{X}(M_1) = \mathcal{X}(M) < 0$ the miniversal base $\boldsymbol{H}_1$ of $\widetilde{M}_1$ is nontrivial, and since $\widetilde{M}$ is formally versal on $\boldsymbol{H}$, $\widetilde{M}_1$ is modular. But then either $\widetilde{M}_1$ is indecomposable or we produce one module M_2, $\dim_k M_2 < \dim_k M_1$ with $\mathcal{X}(M_2) \leq 0$, and we keep going. This proves the Proposition. QED

1.5 Examples. Hereditary algebras, Gabriel quivers and the associated quadratic form

Let Λ be any finite category of global dimension ≤ 1, say a quiver in the sense of Gabriel, see [G], or an ordered set. Put $A = k[\Lambda]$, then the category of A-modules is equivalent to the category of presheaves of k-vectorspaces on Λ. Let M be such a presheaf of finite dimensional k-vectorspaces.

The conditions $(A_1), (A_2)$ and (V^1) of (1.4) obviously hold in this case.

In particular let for every $\lambda \in \Lambda$, k_λ be the presheaf defined by $k_\lambda(\lambda') = 0$ if $\lambda' \neq \lambda$, $k_\lambda(\lambda) = k$. Then the family of simple $A = k[\Lambda]$-modules, or the family of irreducible representations of A, is $\{k_\lambda\}_{\lambda \in \Lambda}$.

One checks that the associated quadratic form $\mathcal{X}$ is exactly the quadratic form of the quiver considered in the literature, see [G].

Now suppose M is a rigid indecomposable A-module. Then obviously

$$\begin{aligned} \mathcal{X}(M) &= e_0 - e_1 = 1\,. \\ q_V(M) &= 2\,. \end{aligned}$$

In fact if we pick for every $\lambda \in \Lambda$ a k-vectorspace $L(\lambda)$ then we know from [La1] that the projective system on Λ, i.e. the $k[\Lambda]$-module defined by

$$L_\lambda = \coprod_{\lambda' \geq \lambda} L(\lambda')$$

with obvious inclusion morphisms

$$L_{\lambda_1} \to L_{\lambda_2}$$

if $\lambda_1 \geq \lambda_2$, is a projective object in the abelian category of projective systems (presheves) on Λ. From this follows immediately,

Proposition Let Λ be an ordered set and $\lambda_1, \lambda_2 \in \Lambda$ then

$$\begin{aligned}
\mathrm{Hom}_{k[\Lambda]}(k_{\lambda_1}, k_{\lambda_2}) &= \begin{cases} 0 & \text{if } \lambda_1 \neq \lambda_2 \\ k & \text{if } \lambda_1 = \lambda_2 \end{cases} \\
\mathrm{Ext}^1_{k[\Lambda]}(k_{\lambda_1}, k_{\lambda_2}) &= \begin{cases} k & \text{if } \lambda_1 > \lambda_2 \text{ and minimally such.} \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{Ext}^2_{k[\Lambda]}(k_{\lambda_1}, k_{\lambda_2}) &= \begin{cases} 0 & \text{unless } \Lambda \text{ contains a loop.} \\ k & \text{if } \{\lambda' \in \Lambda | \lambda_1 \geq \lambda' \geq \lambda_2\} \text{ is a simple loop.} \end{cases}
\end{aligned}$$

Remark Given the k-algebra $k[\Lambda]$ we easily recover the ordered structure of Λ, by simply considering the irreducible (simple) $k[\Lambda]$-modules $V_i = k_{\lambda_i}$ and then computing the $\mathrm{Ext}^1_{k[\Lambda]}(V_i, V_j)$. This however presupposes that we know that $k[\Lambda]$ is the k-algebra of some ordered set.

The full characterization, given in (2.4) relies on the Massey product structure of the V_i's.

It is now easy to prove the structure theorem of Gabriel for finite representation type k-algebras of the form $k[\Lambda]$. In fact consider the ordered set Λ:

$$\begin{tikzcd} & \circ_4 & \\ \circ_1 & \circ_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \circ_5 \\ & \circ_2 & \end{tikzcd} \qquad \text{corresponding to the quiver} \qquad \begin{tikzcd} & \circ \arrow[d, dash] & \\ \circ \arrow[r, dash] & \circ \arrow[r, dash] \arrow[d, dash] & \circ \\ & \circ & \end{tikzcd}$$

and the corresponding forms:

$$\mathcal{X}: \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad q: \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$\mathcal{X}$ has a positive zero $(1,1,2,1,1)$ corresponding to the module $M = k_1 \oplus k_2 \oplus k_3^2 \oplus k_4 \oplus k_5$.

By the above **Proposition** this shows that $k[\Lambda]$ does not have finite representation type: In the same way we list the other relevant ordered sets:

$$\Lambda: \quad \begin{tikzcd} \circ_1 \arrow[r] & \circ_2 & \overset{3}{\circ} \arrow[l] \arrow[r] \arrow[d] & \circ_6 & \circ_7 \arrow[l] \\ & & \circ_4 & & \\ & & \circ_5 \arrow[u] & & \end{tikzcd}$$

$$\mathcal{X}: \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$\mathcal{X}$ has a positive zero $(1,2,3,2,1,2,1)$, so $k[\Lambda]$ does not have finite representation type.

$$\Lambda: \quad \begin{tikzcd} \circ \arrow[r] & \circ & \circ \arrow[l] \arrow[r] & \circ & \circ \arrow[l] \arrow[r] & \circ & \circ \arrow[l] \\ & & & \circ \arrow[u] & & & \end{tikzcd}$$

$$\mathcal{X}: \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$\mathcal{X}$ has a positive zero $(1,2,3,4,2,3,2,1)$, so $l[\Lambda]$ does not have finite representation type.

$$E_9 = \Lambda: \qquad \begin{tikzcd} \circ & \circ \arrow[l] \arrow[r] & \circ & \circ \arrow[l] \arrow[r] & \circ & \circ \arrow[l] \arrow[r] & \circ & \circ \arrow[l] \\ & & \circ \arrow[u] & & & & & \end{tikzcd}$$

$$\mathcal{X}: \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$\mathcal{X}$ has a positive zero $(2,3,4,6,5,4,3,2,1)$, so $k[\Lambda]$ does not have finite representation type.

$$E_8 = \Lambda: \qquad \begin{tikzcd} \underset{1}{\circ} \arrow[r, dash] & \overset{3}{\circ} \arrow[r, dash] & \overset{4}{\circ} \arrow[r, dash] \arrow[d, dash] & \overset{5}{\circ} \arrow[r, dash] & \overset{6}{\circ} \arrow[r, dash] & \overset{7}{\circ} \arrow[r, dash] & \overset{8}{\circ} \\ & & \underset{2}{\circ} & & & & \end{tikzcd}$$

$$\mathcal{X}: \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad q: \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

We know that $q = \Gamma_8$ is nondegenerate of type II, see [Serre]. This means of course that q has no positive zeros. From the results above it follows that the $k[\Lambda]$'s corresponding to quivers of the form A_n, D_n or E_6, E_7, E_8 are the only algebras of this form which are hereditary and of finite representation type.

This is Gabriels classification theorem, see [G], or [R].

Given a category $\underline{C}$, see [La2] for the definition of the category $\operatorname{Mor} \underline{C}$ and the complex $D^*(\underline{C}, -)$.

Theorem Let Λ be an ordered set and F a $k[\Lambda]$-bimodule. Then

$$F(\lambda_1, \lambda_2) = \varepsilon_{\lambda_1} \cdot F \cdot \varepsilon_{\lambda_2}$$

is a presheaf $\boldsymbol{F}$ on $\operatorname{Mor}\Lambda$ and we have the canonical isomorphism

$$HH^*(k[\Lambda], F) \simeq H^*(D^*(\Lambda, \boldsymbol{F})) = \varprojlim_{\operatorname{Mor}\Lambda}{}^{(*)} F\,.$$

Proof of Theorem: Consider the Hochschild comples $CH^* = CH^*(k[\Lambda], F)$ and a cocycle $\xi \in CH^p$. Let $\psi_i = (\lambda_i' \geq \lambda_i)$ be the generators of $k[\Lambda]$. Assume $\psi_i \cdot \psi_{i+1} = 0$, i.e. $\lambda_i \neq \lambda_{i+1}'$, and consider the

$$\xi(\psi_1, \dots, \psi_p) = \xi(\psi_1, \dots, \psi_i, \psi_{i+1}, \dots, \psi_p)\,.$$

Let $\varepsilon_i = 1_{\lambda_i} = (\lambda_i \geq \lambda_i)$ and $\varepsilon_i' = 1_{\lambda_i}' = (\lambda_i' \geq \lambda_i')$ and evaluate $d\xi$ on $(\psi_1, \dots, \psi_i, \varepsilon_i, \psi_{i+1}, \dots, \psi_p)$,

$$\begin{aligned} d\xi(\psi_1, \dots, \psi_i, \varepsilon_i, \psi_{i+1}, \dots, \psi_p) &= \psi_1 \xi(\psi_2, \dots, \psi_i, \varepsilon_i, \psi_{i+1}, \dots, \psi_p) \\ &+ \sum_{j=1}^{i-1} (-1)^j \xi(\psi_1, \dots, \psi_j \cdot \psi_{j+1}, \cdots \psi_i, \varepsilon_i, \psi_{i+1}, \dots, \psi_p) \\ &\quad + (-1)^i \xi(\psi_1, \dots, \psi_i, \psi_{i+1}, \dots, \psi_p) \\ &+ \sum_{j=i+1}^{p-1} (-1)^{j+1} \xi(\psi_1, \dots, \psi_i, \varepsilon_i, \psi_{i+1}, \dots, \psi_j \cdot \psi_{j+1}, \dots, \psi_p) \\ &\quad + (-1)^{p+1} \xi(\psi_1, \dots, \psi_i, \varepsilon_i, \psi_{i+1}, \dots, \psi_{p-1}) \psi_p \end{aligned}$$

From this and from the symmetrical formula, i.e. the one we get by considering $d\xi$ evaluated on $(\psi_1, \dots, \psi_i, \varepsilon_{i+1}', \psi_{i+1}, \dots, \psi_p)$, we deduce that if $\psi_i \psi_{i+1} = 0$ we may write

$$\xi(\psi_1, \dots, \psi_i, \psi_{i+1}, \dots, \psi_p)$$

as a sum of ξ evaluated on strings with one more composable identity ε_i.

Continuing, repeating this procedure we see that $\xi(\psi_1, \dots, \psi_i, \psi_{i+1}, \dots, \psi_p)$ is a sum of ξ evaluated at composable strings of elements, and this in such a way that ξ obviously is determined by its values on such composable strings. Moreover we see that $d\xi(\psi_1, \dots, \psi_p, \varepsilon_p') = 0$ implies that $\xi(\psi_1, \dots, \psi_p) = \xi(\psi_1, \dots, \psi_p)\varepsilon_p' +$ a sum of ξ evaluated at strings of the type $(\psi_1', \dots, \psi_{p-1}', \varepsilon_p')$. Since $\xi(\varepsilon_i, \varepsilon_i, \dots, \varepsilon_i) \in \varepsilon_i F \varepsilon_i$, this and the dual evaluation implies that

$$\xi(\psi_1, \dots, \psi_p) \in \varepsilon_1 F \varepsilon_p'\,.$$

But then the inclusion $D^*(\Lambda, \boldsymbol{F}) \subseteq CH^*(k[\Lambda], F)$ induces an isomorphism in cohomology. QED

Corollary If M and N are two modules on $k[\Lambda]$ (left or right) then $\operatorname{Hom}_k(M_{\lambda_1}, N_{\lambda_2})$ defines a presheaf $\mathbf{Hom}_k(M, N)$ on $\operatorname{Mor}(\Lambda)$ and

$$\operatorname{Ext}^*_{k[\Lambda]}(M, N) = \varprojlim_{\operatorname{Mor}\Lambda}{}^{(*)} \mathbf{Hom}_k(M, N)\,.$$

Example Let $\Lambda = A_3: \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ}$

$\operatorname{Mor}\Lambda$ is then Γ:

$$\begin{tikzcd}[column sep=small, row sep=small]
\overset{1}{\circ} \arrow[dr, dash] & & \overset{2}{\circ} \arrow[dl, dash] \arrow[dr, dash] & & \overset{3}{\circ} \arrow[dl, dash] \\
& \circ \arrow[dr, dash] & & \circ \arrow[dl, dash] & \\
& & \circ & &
\end{tikzcd}$$

Consider k_1 and k_2 and let us compute

$$\operatorname{Hom}_k(k_1, k_2)$$

as a presheaf on $\operatorname{Mor}\Lambda = \Gamma$.

$$\operatorname{Hom}_k(k_1, k_2)(i,j) = \begin{cases} 0 & i \neq 1 \text{ or } \; j \neq 2 \\ k & i = 1 \qquad j = 2 \end{cases}$$

so, together with an injective resolution, it looks like

$$R^* : 0 \longrightarrow \begin{smallmatrix} 0 & & 0 & & 0 \\ & k & & 0 & \\ & & 0 & & \end{smallmatrix} \longrightarrow \begin{smallmatrix} k & & k & & 0 \\ & k & & 0 & \\ & & 0 & & \end{smallmatrix} \longrightarrow \begin{smallmatrix} k & & k & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{smallmatrix} \longrightarrow 0$$

Recall

$$\varprojlim_{\Gamma}{}^{(1)} \boldsymbol{F} = \operatorname{Ext}^1_\Gamma(k, \boldsymbol{F}) = H^1(\varprojlim_{\Gamma}(R^{\cdot})) = k \oplus k / \langle (\alpha, \alpha) \rangle \simeq k$$

And obviously

$$\varprojlim_{\Gamma}{}^{(0)} F = \operatorname{Hom}_\Gamma(k, \boldsymbol{F}) = H^\circ(\varprojlim_{\Gamma} R^{\cdot}) = 0\,.$$

as it should.

Corollary Global $\dim k[\Lambda] \leq \dim \operatorname{Mor}\Lambda$.

Proof: Global $\dim k[\Lambda] = \min\{n \,|\, \operatorname{Ext}^{n+1}_{k[\Lambda]} = 0\}$. Since

$$\dim \operatorname{Mor}\Lambda = \min\left\{ m \,\Big|\, \varprojlim_{\operatorname{Mor}\Lambda}{}^{m+1} = 0 \right\}$$

the Corollary follows from the above results. QED

Corollary Suppose $A = k[\Lambda]$ has finite representation type. Let M be an indecomposable A-module, put

$$M = (x_\lambda)_{\lambda \in \Lambda}\,, \text{ where for } \lambda \in \Lambda,\; M(\lambda) = k^{x_\lambda}, \text{ then } q(x_\lambda) = 2\,.$$

Conversely, given a vector $(x_\lambda)_{\lambda \in \Lambda}$ such that $q(x_\lambda) = 2$, then there exists an indecomposable A-module M with vector (x_λ).

Proof: See [G] or observe that the only statement that needs a proof is the last one.

Take $M_0 = \oplus k_\lambda^{x_\lambda}$. Since A is supposed to be of global dimension ≤ 1, and has finite representation type, we may show that there exists a rigidification of M_0. Call this M. Since $\mathcal{X}(M) = \mathcal{X}(M_0) = 1$, $\mathrm{End}_A(M) = k$ and M is indecomposable, obviously with $q(x_\lambda) = 2\mathcal{X}(M) = 2$.

2 Non commutative deformations

2.1 The category a_r, test algebras and liftings of modules

Let $\boldsymbol{a_r}$ be the category of "r-pointed" Artinian k-algebras. An object R of $\boldsymbol{a_r}$ is a pair of morphism of Artinian k-algebras $k^r \to R \to k^r$ such that the composition is the identity and such that

$$R/\boldsymbol{r}(R) \simeq \prod_{j=1}^{r} k_j, \qquad k_j \simeq k$$

where $\boldsymbol{r}(R)$ is the radical of R. A morphism $\phi : R \to S$ of $\boldsymbol{a_r}$ is a morphism of k-algebras inducing the identity on k^r, i.e. such that

$$k^r \simeq R/\boldsymbol{r}(R) \to S/\boldsymbol{r}(S) \simeq k^r,$$

is the identity. Pick idempotents $e_i \in R$ such that

$$\sum_{i=1}^{r} e_i = 1, \qquad e_i e_j = 0 \text{ if } i \neq j.$$

Then, for every (i,j), we shall consider the subspace $R_{ij} = e_i R e_j \subseteq R$, and the pairing

$$R_{ij} \otimes_k R_{jk} \to R_{ik}$$

given in terms of the multiplication in R.

Let

$$R' = (R_{ij})$$

be the matrix algebra, the elements of which are matrices of the form

$$(e_{ij})$$

with $e_{ij} \in R_{ij}$, $i,j = 1, \cdots, r$. There is an obvious homomorphism of k-algebras

$$j : R \to R'$$

defined by

$$j(\alpha) = (e_i \alpha e_j).$$

Since $1 = \sum_{i=1}^{r} e_i$ it is clear that j is an isomorphism.

Now, for any pair (i,j), $i,j = 1,\ldots,r$, consider the symbol ϵ_{ij}, and let's agree to put all products of such symbols equal to zero. Then we define the (i,j)-test algebra $R(i,j)$ as the matrix algebra

$$R(i,j) = \begin{array}{cc} & \quad\ \ j \\ i & \begin{pmatrix} k & \vdots & 0 \\ \cdots & k\cdot\epsilon_{ij} & \cdots \\ 0 & \vdots & k \end{pmatrix} \end{array} \qquad \text{for } i \neq j$$

$$R(i,i) = \begin{array}{cc} & \quad\ \ j \\ i & \begin{pmatrix} k & \vdots & 0 \\ \cdots & k[\epsilon_{ij}] & \cdots \\ 0 & \vdots & k \end{pmatrix} \end{array} \qquad \text{for } i = j$$

Denote by $HH^{\cdot}(A,-)$ the Hochschild cohomology of the k-algebra A. If W is an A-bimodule denote by $\mathrm{Der}_k(A,W)$ the k-vectorspace of derivations of A in W. Thus $\psi \in \mathrm{Der}_k(A,W)$ is a linear map from A to W such that $\psi(a_1 \cdot a_2) = a_1\psi(a_2) + \psi(a_1)a_2$.

In particular, any element $w \in W$ determines a derivation $i(w) \in \mathrm{Der}_k(A,W)$ defined by $i(w)(a) = aw - wa$. There is an exact sequence

$$0 \to HH^0(A,W) \to W \to \mathrm{Der}_k(A,W) \to HH^1(A,W) \to 0$$

If V_i, V_j are right A-modules, then

$$W_{ij} = \mathrm{Hom}_k(V_i,V_j) \simeq V_i^* \otimes_k V_j$$

is an A-bimodule. In fact if $\phi \in W_{ij}$, then $a\phi$ is defined by $(a\phi)(v) = \phi(va)$, and ϕa is defined by $(\phi a)(v) = \phi(v)a$.

Moreover, it is easy to see that

$$HH^0(A, V_i^* \otimes V_j) = \mathrm{Hom}_A(V_i,V_j)$$

$$HH^1(A, V_i^* \otimes V_j) = \mathrm{Ext}^1_A(V_i,V_j).$$

Fix from now on a family $V = \{V_i\}_{i=1}^{r}$ of right A-modules, and consider for every $\psi \in \mathrm{Der}_k(A, V_i^* \otimes V_j)$ the left $R(i,j)$-module and right A-module

$$V_{ij}(\psi) = \begin{array}{cc} & \qquad\qquad\qquad j \\ i & \begin{pmatrix} V_1 & & & \vdots & \\ \cdots & V_i & \cdots & \epsilon_{ij}V_j & \cdots \\ & & & V_j & \\ & & & \vdots & V_r \end{pmatrix} \end{array}$$

defined by

$$(1)\quad \begin{pmatrix} v_1 & & & & \\ & v_i & \epsilon_{ij}v'_j & & \\ & & v_j & & \\ & & & & v_r \end{pmatrix} \cdot a = \begin{pmatrix} v_1 a & & & & \\ & v_i a & & \epsilon_{ij}(\psi(a, v_i) + v'_j a) & \\ & & v_j a & & \\ & & & v_j a & \\ & & & & v_r a \end{pmatrix}$$

and the obvious left $R(i, j)$-action.

The $R(i, j)$- and the A-action commute, therefore we have got a $R(i, j) \otimes A$-module, such that

$$k_\ell \otimes_{R(i,j)} V_{ij}(\psi) \simeq V_\ell .$$

$V_{ij}(\psi)$ is called a lifting of V to $R(i, j)$. It is easy to see that if ψ maps to zero in $HH^1(A, V_i^* \otimes V_j) = \mathrm{Ext}^1_A(V_i, V_j)$ then the lifting $V_{ij}(\psi)$ is trivial, i.e. isomorphic to the trivial one. Conversely, if $V_{ij}(\psi)$ is trivial, then ψ maps to zero in $\mathrm{Ext}^1_A(V_i, V_j)$.

2.2 The non commutative deformation functor

We are now ready to start studying non commutative deformations of the family $V = \{V_i\}_{i=1}^r$. We define the deformation functor

$$\mathrm{Def}_V : \boldsymbol{a}_r \longrightarrow \mathrm{Sets}$$

as follows:

$$\begin{aligned}\mathrm{Def}_V(R) = \{&\text{isoclasses of } R \otimes_k A\text{-modules } \widetilde{V} \text{ together with isomorphisms} \\ &k_i \otimes_R \widetilde{V} = V_i \text{ such that } \widetilde{V} \text{ is "}R\text{-flat" (or, is a lifting)}\}.\end{aligned}$$

Notice that flatness means the following:

$$\widetilde{V} \simeq (R_{ij} \otimes_k V_j)$$

as matrices of k-vectorspaces, where $R_{ij} = e_i R e_j$ as above.

Let $\pi : R \longrightarrow S$ be a morphism of $\boldsymbol{a}_r$, such that $\boldsymbol{r}(R) \cdot \ker \pi = 0$. Morphisms like this will be called *small*. Then, if $\widetilde{V} \in \mathrm{Def}_V(R)$ it is easy to see that $S \otimes_R \widetilde{V} \in \mathrm{Def}_V(S)$ and that $\overline{V} = \ker\{\widetilde{V} \longrightarrow S \otimes_R \widetilde{V}\}$ is, as an R-module, an $R/\boldsymbol{r}(R) = \bigoplus_{i=1}^r k_i$-module. Put $\ker \pi = (K_{ij})$, then $\overline{V} = (\overline{V}_{ij})$ where $\overline{V}_{ij} = K_{ij} \otimes_k V_j$.

Consider now the k-vector spaces

$$E^k_{ij} = \mathrm{Ext}^k_A(V_i, V_j)^* ,$$

i.e. the dual k-vectorspaces of $\mathrm{Ext}^k_A(V_i, V_j)$, and the k-algebra of matrices,

$$T^k_2 = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix} + (\epsilon_{ij} E^k_{ij})$$

where as above, we put all products of the ϵ_{ij}'s equal to zero. Now let for every $i, j = 1, \ldots, r$, and $k = 1, 2$,

$$\left\{t_{ij}^k(\ell)\right\}_{\ell=1}^{e_{ij}^k}$$

be a basis of E_{ij}^k, and let $\{\psi_{ij}^k(\ell)\}_{\ell=1}^{e_{ij}^k}$ be the dual basis. Thus $e_{ij}^k = \dim_k E_{ij}^k$. Consider the k-algebra

$$T^k = \begin{pmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{pmatrix} + (\widetilde{E}_{ij}^k) \tag{2}$$

"freely generated" as matrix algebra by the subspace (E_{ij}^k). An element of $\widetilde{E}_{ij}^k$ is then a matrix where the elements are linear combinations of elements of the form:

$$\begin{aligned}\tau_{ij} = t_{ij_1}^k(\ell_1) \otimes t_{j_1 j_2}^k(\ell_2) \otimes \cdots \otimes t_{j_{m-1} j_m}^k(\ell_m)\,,& \\ j = j_m,\ 1 \leq \ell_s \leq e_{j_{s-1} j_s}^k,\ 1 \leq j_s \leq r,\ m \geq 1\,.&\end{aligned}$$

of $E_{t_{ij}}^k \otimes E_{j_1 j_2}^k \otimes \cdots \otimes E_{j_{m-1} j}^k$.

Obviously

$$T_2^1 = T^1/\boldsymbol{r}(T^1)^2.$$

where $\boldsymbol{r}(T^1)$ is the two-sided ideal of T^1 generated by (E_{ij}^1).

Lemma Let R be an object of $\boldsymbol{a}_r$ and suppose that there exists a surjective homomorphism

$$\phi_2 : T_2^1 \to R/\boldsymbol{r}(R)^2,$$

then there exists a surjective homomorphism

$$\phi : T^1 \to R$$

which lifts ϕ_2.

Definition For every object R of $\boldsymbol{a}_r$, put

$$\boldsymbol{t}_R = (\boldsymbol{r}(R)/\boldsymbol{r}(R)^2)^*$$

and call it the tangent space of R.

Lemma Let $\phi : R \to S$ be a morphism of $\boldsymbol{a}_r$. Assume ϕ induces a surjective homomorphism

$$\phi^1 : \boldsymbol{t}_R^* \to \boldsymbol{t}_S^*$$

(or an injective homomorphism on the tangent space level). Then ϕ is surjective.

Notice that if, in the situation above, we pick any k-vectorspaces F_{ij}, then there is a unique maximal pro-algebra $F = F(F_{ij})$ in $\boldsymbol{a}_r$ with tangent space

$$\boldsymbol{t}_F \simeq (F_{ij}^*)$$

F is defined by the formula (2) above, with E replaced by F.

To prove the existence of a hull for the deformation functor Def_V the basic tool is the obstruction calculus, which in this case is easily established:

Proposition Suppose $R \xrightarrow{\phi} S$ is a surjective small morphism of $\boldsymbol{a}_r$. i.e. suppose $\ker\phi \cdot r(R) = 0$. Put $\ker\phi = (I_{ij})$. Consider any $V_S \in \mathrm{Def}_V(S)$. Then there exists an obstruction

$$o(\phi, V_S) \in (\mathrm{Ext}^2_A(V_i, V_j) \otimes_k I_{ij})$$

which is zero if and only if there exists a lifting $V_R \in \mathrm{Def}_V(R)$ of V_s. The set of isomorphism classes of such liftings is a torsor under

$$(\mathrm{Ext}^1_A(V_i, V_j) \otimes_k I_{ij}).$$

Proof: As a k-vectorspace $V_R = (R_{ij} \otimes V_j)$ maps onto $V_S = (S_{ij} \otimes V_j)$. The action of an element $a \in A$ on V_S is uniquely given in terms of the maps

$$a_{ij} : V_i \longrightarrow S_{ij} \otimes V_j .$$

We may of course lift these to linear maps

$$\sigma(a)_{ij} : V_i \longrightarrow R_{ij} \otimes V_j$$

inducing a lift of the action of A on

$$\bigoplus_{j=1}^{r} S_{ij} \otimes V_j$$

to a k-linear action of A on

$$\bigoplus_{j=1}^{r} R_{ij} \otimes V_j .$$

The obstruction for this to be an A-module structure is as usual the Hochschild 2-cocycle

$$\psi^2(a,b) = \sigma(ab) - \sigma(a)\cdot\sigma(b) \in (\mathrm{End}_k(V_i, V_j) \otimes_k I_{ij}).$$

The fact that this is a 2-cocycle follows from

$$\begin{aligned} \sigma(c)\cdot\psi^2(a,b) &= c\cdot\psi^2(a,b) \\ \psi^2(a,b)\cdot\sigma(c) &= \psi^2(a,b)\cdot c \end{aligned}$$

and the obvious relation

$$\begin{aligned}d\psi^2(a,b,c) &= a\psi^2(b,c) - \psi^2(ab,c) + \psi^2(a,bc) - \psi^2(a,b)\cdot c\\ &= \sigma(a)(\sigma(bc) - \sigma(a)\sigma(c)) - (\sigma(abc) - \sigma(ab)\sigma(c)) + (\sigma(abc) - \sigma(a)\sigma(bc))\\ &\quad -(\sigma(ab) - \sigma(a)\sigma(b))\sigma(c) \equiv 0\,.\end{aligned}$$

Suppose the class of ψ^2 in $(\mathrm{Ext}^2_A(V_i, V_j) \otimes_k I_{ij})$ is zero. This means that $\psi^2 = d\phi$, where $\phi \in \mathrm{Hom}_k(A, (\mathrm{End}_k(V_i, V_j) \otimes I_{ij})$, $\psi^2(a,b) = d\phi(a,b) = a\phi(b) - \phi(ab) + \phi(a)b$. Let $\sigma' = \sigma + \phi$ and consider

$$\sigma'(ab) - \sigma'(a)\sigma'(b) = \sigma(ab) - \sigma(a)\sigma(b) + \phi(ab) - \sigma(a)\phi(b) - \phi(a)\sigma(b) - \phi(a)\phi(b)\,.$$

Since the matrix $\phi(a)\phi(b) = 0$ as $I_{ij}\cdot I_{jk} = 0$, $\forall i,j,k$ and since $\sigma(a)\phi(b) = a\phi(b)$, $\phi(a)\sigma(b) = \phi(a)b$ for the same reasons, we find that $\sigma'(ab) - \sigma'(a)\sigma'(b) = 0$, i.e. there is a lifting of the A-module action to $V_R = (R_{ij} \otimes V_j)$.

If we have given one A-module action σ on V_R lifting the action on V_S, then for any other σ' we may consider the difference

$$\sigma' - \sigma : A \to (\mathrm{Hom}_k(V_i, V_j) \otimes I_{ij})\,.$$

Consider

$$d(\sigma' - \sigma)(a,b) = a(\sigma'(b) - \sigma(b)) - (\sigma'(ab) - \sigma(ab)) + (\sigma'(a) - \sigma(a))b\,.$$

As above we may substitute $\sigma'(a)$ for a and $\sigma(b)$ for b, and expression becomes zero.

Thus $\sigma' - \sigma = \bar{\xi}$ defines a class ξ in

$$(\mathrm{Ext}^1_A(V_i, V_j) \otimes_k I_{ij})\,.$$

If $\xi = 0$, then $\bar{\xi} = d\phi$, $\phi \in (\mathrm{Hom}_k(V_i, V_j) \otimes_k I_{ij})$ such that $\sigma'(a) - \sigma(a) = a\phi - \phi a$.

Let $\phi = (\phi_{ij})$, then ϕ_{ij} defines an isomorphism

$$\begin{aligned}&\bar{\phi} : \bigoplus_j R_{ij} \otimes V_j \to \bigoplus_j R_{ij} \otimes V_j\\ &\bar{\phi} = id + \phi\end{aligned}$$

lifting the identity of $\bigoplus_j S_{ij} \otimes V_j$. Moreover

$$\begin{aligned}\sigma(a)(id + \phi)(v_i) &= (\sigma(a)v_i + a\phi(v_i)\\ &= \sigma'(a)(v_i) + \phi(av_i) = (id + \phi)\sigma'(a)(v_i)\end{aligned}$$

since $\phi(\sigma'(a)v_i) = \phi(av_i)$.

Therefore the A-module structures on

$$V_R = (R_{ij} \otimes V_j)$$

defined by σ and σ' are isomorphic. The rest is clear. QED

Theorem The functor Def_V has a prorepresentable hull $\boldsymbol{H}$ in $\boldsymbol{a}_r$, i.e. there exists a surjective morphism of functors on $\boldsymbol{a}_r$,

$$\rho : \mathrm{Mor}(\boldsymbol{H}, -) \longrightarrow \mathrm{Def}_V$$

such that ρ is smooth and an isomorphism on the tangent level. Moreover, $\boldsymbol{H}$ is uniquely determined by a set of matric Massey products of the form

$$\mathrm{Ext}^1(V_i, V_{j_1}) \otimes \cdots \otimes \mathrm{Ext}^1(V_{j_{n-1}}, V_j) \cdots \longrightarrow \mathrm{Ext}^2(V_i, V_j)\,.$$

Notice first that ρ is an isomorphism at the tangent level means that ρ is an isomorphism for all objects R of $\boldsymbol{a}_r$ for which $\boldsymbol{r}(R)^2 = 0$.

Proof: Word for word we may copy the proof (4.2) of [La2]. In particular $\boldsymbol{H}/\boldsymbol{r}(\boldsymbol{H})^2 \simeq T_2^1$ and $\mathrm{Mor}(\boldsymbol{H}, R(i,j)) \simeq \mathrm{Hom}_k(E_{ij}^1, k) \simeq \mathrm{Ext}_A^1(V_i, V_j) \simeq \mathrm{Def}_V(R(i,j))$.

Notice that the universal lifting of V to T_2^1 is the $T_2^1 \otimes_k A$-module $\widetilde{V}_2$

$$\begin{pmatrix} V_1 & & 0 \\ & \ddots & \\ 0 & & V_2 \end{pmatrix} + (E_{ij}^1 \otimes_k V_j)$$

with the right A-action defined as above (1) and with the obvious T_2^1 left-action. To obtain $\boldsymbol{H}$ we kill obstructions for lifting $\widetilde{V}_2$ successively to $T_3^1 = T^1/\boldsymbol{r}(T^1)^3, T_4^1$ etc. just like in the commutative case. QED

Remark (i) The action of an element $a \in A$ on an element $w = (w_{ij})$ of $\widetilde{V}_2$ is given as follows.

Since $w_{ii} \in V_i \oplus E_{ii}^1 \otimes_k V_i$ and for $i \neq j$, $w_{ij} \in E_{ij}^1 \otimes_k V_j$, we assume that

$$\begin{aligned} w_{ii} &= v_i + t_{ii}^1(\ell_i) \otimes v_{ii}' \\ w_{ij} &= t_{ij}^1(\ell_{ij}) \otimes v_{ij}' \qquad i \neq j \end{aligned}$$

where $v_i \in V_i$, $v_{ij}' \in V_j$, and where $\{t_{ij}^1(\ell)\}_{\ell=1}^{e_{ij}}$ is the chosen basis of E_{ij}^1. Recall that $\{\psi_{ij}^1(\ell)\}_{\ell=1}^{e_{ij}}$, the dual base, consists of elements $\psi_{ij}^1(\ell) \in \mathrm{Ext}_A^1(V_i, V_j)$, which may be represented as elements of $\mathrm{Der}_k(A, V_i^* \otimes_k V_j)$. Then the matrix

$$w \cdot a = ((w \cdot a)_{ij})$$

is given as:

$$(w \cdot a)_{ii} = v_i \cdot a + \sum_{\ell_i} t_{ii}^1(\ell_i) \otimes (\psi_{ii}^1(\ell_i)(a, v_i) + v_{ii}' \cdot a)$$

and

$$(w \cdot a)_{ij} = \sum_{j, \ell_{ij}} t_{ij}^1(\ell_{ij}) \otimes (\psi_{ij}^1(\ell_{ij})(a, v_i) + v_{ij}' \cdot a)\,.$$

(ii) The proof of the existence of a prorepresentable hull of Def_V can, of course, be modeled on the classical proof of M. Schlessinger [S], This has been carried out by Runar Ile in his Masters Thesis, Oslo 1990.

2.3 A general structure theorem for artinian k-algebras

Observe that for every deformation $V' \in \mathrm{Def}_v(R)$ there exists a unique homomorphism

$$\eta_{V'} : A \to R \hat{\otimes} \mathrm{End}(V') := (R_{ij} \otimes V_i^* \otimes V_j)$$

where, as usual $R_{ij} = e_i R e_j$.

This map is given by the following: Let $a \in A$ and consider the element $v \in V_i$, then obviously

$$\begin{array}{c} \\ \bar{v} = \end{array} \begin{array}{c} i \\ \begin{pmatrix} 0 & & \vdots & & 0 \\ & \ddots & \vdots & & \\ & & 1 \otimes v & & \\ & & \vdots & \ddots & \\ 0 & & \vdots & & 0 \end{pmatrix} \end{array} \in V'$$

Since for $1_i \in R$ $1_i \cdot \bar{v} = \bar{v}$, it is clear that

$$1_i(\bar{v} \cdot a) = \bar{v} \cdot a$$

so $(\bar{v} \cdot a)_{k\ell} = 0$ for $k \neq i$, thus $\bar{v} \cdot a$ is a linear combination of elements of the form:

$$\begin{array}{cc} \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ r_{i1} \otimes v_1, & r_{i2} \otimes v_2, & \cdots & r_{ir} \otimes v_r \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ -\, i \\ \\ \\ \end{array} \end{array}$$

and we associate to $a \in A$ the morphism

$$a : V_i \to \begin{array}{cc} \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ R_{i1} \otimes V_1, & R_{i2} \otimes V_2, & \cdots & R_{ir} \otimes V_r \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ -\, i \\ \\ \\ \end{array} \end{array}$$

or the corresponding element of

$$(R_{ij} \otimes_k V_i^* \otimes_k V_j) =: R \hat{\otimes} \mathrm{End}(V)$$

Now $R \hat{\otimes} \mathrm{End}(V)$ is a k-algebra, and obviously the map $\eta_{V'} : A \to R \hat{\otimes} \mathrm{End}(V)$ is a homomorphism of k-algebras.

We would like to describe the kernel and the image of the map

$$\eta : A \to H \hat{\otimes} \mathrm{End}(V) \, .$$

To do this we need to consider the matric Massey products of the form

$$\langle \mathrm{Ext}^1_A(V), \dots, \mathrm{Ext}^1_A(V)\rangle, \quad \ell \text{ terms}, \ \ell \geq 2$$

by which we shall understand the partially defined matrix Massey products of the form

$$\mathrm{Ext}^1_A(V) \otimes \cdots \otimes \mathrm{Ext}^1_A(V) \cdots \to \mathrm{Ext}^2_A(V), \ \ell \text{ terms}, \ \ell \geq 2 \qquad (*)$$

the obvious generalizations of the matrix Massey products introduced in [La3]. Here we shall describe these products using Hochschild cohomology. We obtain in this way a more convenient way of describing the map η and maybe also an easier way of understanding the nature of the notion of Massey products.

For $\ell = 2$, the Massey product above is simply the cup product

$$\mathrm{Ext}^1_A(V) \otimes \mathrm{Ext}^1_A(V) \to \mathrm{Ext}^2_A(V)$$

defined by: Let $(\psi^1_{ij}), (\psi^2_{ij}) \in \mathrm{Ext}^1_A(V)$, and express ψ^k_{ij} as 1-cocycles in the Hochschild complex, i.e. $\bar{\psi}^1_{ij} \in \mathrm{Der}_k(A, \mathrm{Hom}_k(V_i, V_j))$, $\bar{\psi}^2_{ij} \in \mathrm{Der}_k(A, \mathrm{Hom}_k(V_i, V_j))$. The cup product $(\psi^1_{ij} \cup (\psi^2_{ij}) \in \mathrm{Ext}^2_A(V)$, now denoted

$$\langle (\psi^1_{ij}), (\psi^2_{ij})\rangle \in \mathrm{Ext}^2_A(V)$$

is defined by the 2-cocycle in the Hochschild complex

$$\langle (\psi^1_{ij}), (\psi^2_{ij})\rangle_{ik}(a,b) = \sum_j \bar{\psi}^1_{ij}(a) \circ \bar{\psi}^2_{jk}(b) \in \mathrm{Hom}_k(V_i, V_k)$$

Suppose $\langle (\psi^1_{ij}), (\psi^2_{ij})\rangle = 0$, this means that there exists, for each i, k a 1-cochain ϕ^{12}_{ik} in the Hochschild complex, i.e. a map

$$\phi^{12}_{ik} \in \mathrm{Hom}_k(A, \mathrm{Hom}_k(V_i, V_k))$$

such that $d\phi^{12}_{ik} = \langle (\psi^1_{ij}), (\psi^2_{ij})\rangle_{ik}$, i.e. such that for all $a, b \in A$,

$$a\phi^{12}_{ik}(b) - \phi^{12}_{ik}(ab) + \phi^{12}_{ik}(a)b = \sum_j \bar{\psi}^1_{ij}(a) \circ \bar{\psi}^2_{jk}(b)$$

Given classes $\psi^1 = (\psi^1_{ij}), \psi^2 = (\psi^2_{ij}), \psi^3 = (\psi^3_{ij}) \in \mathrm{Ext}^1_A(V)$ such that $\langle \psi^1, \psi^2\rangle = \langle \psi^2, \psi^3\rangle = 0$ there exists $\phi^{12} = (\phi^{12}_{ik}), \phi^{23} = (\phi^{23}_{ik}) \in \mathrm{Hom}_k(A, \mathrm{Hom}_k(V_i, V_k))$ such that

$$d\phi^{12} = \langle \psi^1, \psi^2\rangle, \quad d\phi^{23} = \langle \psi^2, \psi^3\rangle\,.$$

Then there exists a matrix Massey product

$$\langle \psi^1, \psi^2, \psi^3\rangle \in \mathrm{Ext}^2_A(V)$$

defined by the 2-cocycle

$$\langle \psi^1, \psi^2, \psi^3\rangle_{ik}(a,b) = \sum_j \phi^{12}_{ij}(a)\psi^3_{jk}(b) - \sum_j \psi^1_{ij}(a)\phi^{23}_{jk}(b)$$

in $\mathrm{Hom}_k(A \otimes_k A, \mathrm{Hom}_k(V_i, V_j))$.

As in [La3] there is a sequence of defining systems giving rise to the family of partially defined Massey products $(*)$.

Now if $a \in A$, then denote by $\tilde{a}_i \in \mathrm{Hom}_k(V_i, V_i)$ its action on V_i, $i = 1, \ldots, d$. Let $\mathrm{End}_0(V)$ be the diagonal matrix $(\mathrm{End}_k(V_i, V_i))$, contained in the matrix $\mathrm{End}_k(V) := (\mathrm{End}_k(V_i, V_j))$. Put,

$$\mathrm{End}(V)a = (\tilde{a}_1, \ldots, \tilde{a}_d) \in \mathrm{End}_0(V) \subseteq \mathrm{End}(V)$$

If $a \in A$ is such that $\mathrm{End}(V)a = 0$, this means that a acts trivially on each V_i. Let $\psi \in \mathrm{Ext}^1_A(V)$ be represented by 1-cocycles $\psi_{ij} \in \mathrm{Der}_k(A, \mathrm{End}_k(V_i, V_j))$. Then if $\mathrm{End}(V)a = \mathrm{End}(V)b = 0$, we have that

$$\psi_{ij}(ab) = a\psi_{ij}(b) + \psi_{ij}(a)b = 0\,.$$

This shows that $\psi \in \mathrm{Ext}^1_A(V)$ defines a unique k-linear map

$$\psi : \{a \in A \mid \mathrm{End}(V)a = 0\} \to \mathrm{End}_k(V)\,,$$

vanishing on all squares.

Let $a \in A$, $\mathrm{End}(V)a = 0$, and put

$$\mathrm{Ext}^1_A(V)a = 0$$

when $\psi(a) = 0$, $\forall \psi \in \mathrm{Ext}^1_A(V)$. Consider the sub k-vector space of A

$$K_2 = \{a \in A \mid \mathrm{End}(V)a = \mathrm{Ext}^1_A(V)a = 0\}\,.$$

Let $\sum \alpha_{ij}\psi^i \otimes \psi^j \in \mathrm{Ext}^1_A(V) \otimes \mathrm{Ext}^1_A(V)$ such that its Massey (cup-)product is zero, i.e. such that:

$$\sum \alpha_{ij}\langle \psi^i, \psi^j \rangle = 0\,.$$

Then there exists a 1-cochain $\phi \in \mathrm{Hom}_k(A, (\mathrm{Hom}_k(V_i, V_j)))$ such that

$$d\phi = \sum_{ij} \alpha_{ij}\langle \psi^i, \psi^j \rangle\,.$$

Since $d\phi = 0$ implies that ϕ represents an element of $\mathrm{Ext}^1_A(V)$ it is clear that ϕ defines a unique k-linear map

$$\phi : K_2 \to \mathrm{End}_k(V)\,.$$

Let us denote by

$$\ker\langle \mathrm{Ext}^1_A(V), \mathrm{Ext}^1_A(V)\rangle$$

the subset of $\mathrm{Ext}^1_A(V) \otimes \mathrm{Ext}^1_A(V)$ for which the Massery product (i.e. the cup product) is zero. Then we may put

$$\ker\langle \mathrm{Ext}^1_A(V), \mathrm{Ext}^1_A(V)\rangle a = 0$$

if for every $d\phi \in \ker\langle \mathrm{Ext}^1_A(V), \mathrm{Ext}^1_A(V)\rangle$, $\phi(a) = 0$.

Let

$$K_3 = \{a \in A \mid \mathrm{End}(V)a = \mathrm{Ext}^1_A(V)a = \ker\langle \mathrm{Ext}^1_A(V), \mathrm{Ext}^1_A(V)\rangle a = 0\}$$

Continuing in this way we find a sequence of ideals $\{K_n\}_{n \geq 1}$,
$K_1 = \ker\{A \to \mathrm{End}(V)\}$, and it is easy to prove the following

Theorem Let A be any k-algebra and let $V = \{V_i\}_{i=1}^r$ be a family of A-modules. Then the kernel of the canonical map

$$\eta : A \to H \bar{\otimes} \mathrm{End}_k(V)$$

is determined by the matrix Massey product structure of $\mathrm{Ext}_A^i(V)$, $i = 1, 2$. In fact

$$\ker \eta = \bigcap_{n \geq 1} K_n \,.$$

Proof: The A-module structure on $\tilde{V}$ defines a homomorphism of k-algebras

$$\eta : A \to \mathrm{End}_k(\tilde{V})$$

which, since the action of A and that of H commute, induces a homomorphism of k-algebras

$$\eta : A \to \mathrm{End}_H(\tilde{V}) =: H \bar{\otimes} \mathrm{End}_k(V)$$

Modulo $r(H)$ this is the map.

$$\eta : A \to \mathrm{End}_A(V) = \prod_{i=1}^{d} \mathrm{End}_A(V_i) \,,$$

and modulo $r(H)^2$ we may, using the above notations, write:

$$\eta_1 : A \to H_2 \otimes \mathrm{End}_k(V) = \prod_{i=1}^{d} \mathrm{End}_k(V_i) + (E_{ij}^1 \otimes \mathrm{Hom}_k(V_i, V_j))$$

as

$$\eta_1(a)_{ij} = \delta_{ij} \otimes \eta_0(a)_i + \sum_{\ell} t_{ij(\ell} \otimes \psi_{ij}^1(\ell)(a_i -) \,, \qquad \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Now, by construction H is the quotient of the formally free k-algebra T^1 generated by the independent variables $\{t_{ij}(\ell),\, \ell = 1, \ldots, \ell_{ij}\}$ as explained above. The relations of T^1 are generated by linear combinations of monomials in these variables of the form

$$y_{ik} = \sum_{r=1}^{\infty} \sum_{\underline{j}, \underline{l}} \alpha_{i, j_1, \ldots, j_{r-1}, k}^{\ell_1, \ldots, \ell_r} t_{ij_1}(\ell_1) t_{j_1 j_2}(\ell_2) \cdots t_{j_{r-1}, k}(\ell_r)$$

corresponding to elements

$$y_{ik} \in \mathrm{Ext}_A^2(V_i, V_k)^*$$

and the coefficients α are expressed in terms of partially, but inductively well defined, matric Massey products, such that

$$y_{ik}(\langle \psi_{ij_1}^1(\ell_1), \ldots, \psi_{j_{r-1},k}^1(\ell_r) \rangle) = \alpha_{ij_1, \ldots, j_{r-1}, k}^{\ell_1, \ldots, \ell_r} \,.$$

We therefore obtain a basis, as k-vector space, for H by considering the Massey products inductively defined on

$$\begin{aligned} D_{\underline{r}} \subseteq \operatorname{Ext}^1_A(V) \otimes \cdots \otimes \operatorname{Ext}^1_A(V) \\ \langle\,\rangle_{\underline{r}}: \quad D_{\underline{r}} \quad \longrightarrow \quad \operatorname{Ext}^2_A(V) \end{aligned}$$

and picking, in a coherent way, a basis for

$$\operatorname{coker}\{\operatorname{Ext}^2_A(V)^* \to D^*_{\underline{r}}\} = (\ker\langle\,\rangle_{\underline{r}})^* .$$

This is the conclusion of the Theorem. QED

Remark Let E_{ij} be an extension of V_i by V_j, then as a k-vector space $E_{ij} = V_j \oplus V_i$ and the right action by A is defined by: $(v_j, v_i) \in E_{ij}$, $a \in A$

$$(v_j, v_i) = (v_j a + \psi^1_{ij}(a, v_i), v_i a)$$

when the condition on $\psi^1_{ij} \in \operatorname{Hom}_k(A, \operatorname{Hom}_k(V_i, V_j))$ is simply that $\psi^1_{ij} \in \operatorname{Der}_k(A, \operatorname{Hom}_k(V_i, V_j))$ so that ψ^1_{ij} defines an element $\bar{\psi}^1_{ij}$ in $\operatorname{Ext}^1_A(V_i, V_j)$.

Suppose we consider an extension E_{ijk} of E_{ij} by V_k. Then as a k-vector space $E_{ijk} \simeq V_k \oplus E_{ij} = V_k \oplus V_j \oplus V_i$ and the action by A is defined by

$$(v_k, v_j, v_i)a = (v_k a + \phi(a, (v_j, v_i)), v_j a + \psi_{ij}(a, v_i), v_i a)$$

when by additivity

$$\phi(a, (v_j, v_i)) = \phi(a, (v_j, 0)) + \phi(a, (0, v_i)) .$$

Put

$$\psi^{0,1}_{jk}(a, v_j) = \phi(a, v_j, 0)), \quad \psi^{1,1}_{ik}(a, v_i) = \phi(a, (0, v_i)),$$

then the conditions on the action imply

$$\begin{aligned} \psi^{0,1}_{jk} &\in \operatorname{Der}_k(A, \operatorname{Hom}_k(V_j, V_k)) \\ \psi^{1,1}_{jk} &\in \operatorname{Hom}_k(A, \operatorname{Hom}_k(V_i, V_k)) \end{aligned}$$

and

$$d\psi^{1,1}_{ik} = \psi^{0,1}_{jk} \circ \psi^{1,0}_{ij} .$$

This means that $\bar{\psi}^{0,1}_{jk} \in \operatorname{Ext}^1_A(V_j, V_k)$ and that the cup product

$$\bar{\psi}^{0,1} \cup \bar{\psi}^{1,0} \in \operatorname{Ext}^2_A(V_i, V_k)$$

is zero.

Now, consider an extension $E_{ijk\ell}$ of E_{ijk} by V_ℓ. As before the action of A on $E_{ijk\ell}$ is given by

$$\begin{aligned} (v_\ell, v_k, v_j, v_i)a \;=\; & (v_\ell a + \phi(a, v_k, v_j, v_i), v_k \cdot a + \psi^2_{ik}(a, v_i) + \psi^{0,1}_{jk}(a, v_j), \\ & v_j a + \psi^1_{ij}(a, v_i), v_i \cdot a) . \end{aligned}$$

The conditions on ϕ are expressed by:

$$\begin{aligned}\psi_{k\ell}^{0,0,1}(a,v_k) &= \phi(a,v_k,0,0)\\ \psi_{j\ell}^{0,1,1}(a,v_j) &= \phi(a,0,v_j,0)\\ \psi_{ik}^{1,1,1}(a,v_i) &= \phi(a,0,0,v_i)\end{aligned}$$

$$\begin{aligned} d\psi_{j\ell}^{0,1,1} &= \psi_{k\ell}^{0,0,1}\circ\psi_{jk}^{0,1,0}\\ d\psi_{ik}^{1,1,1} &= \psi_{j\ell}^{0,1,1}\circ\psi_{ij}^{1,0,0}+\psi_{k\ell}^{0,0,1}\circ\psi_{ik}^{1,1,0}\end{aligned}$$

This means that $\bar{\psi}_{k\ell}^{0,0,1}\in \mathrm{Ext}_A^1(V_k,V_\ell)$, that the cup product $\bar{\psi}_{k\ell}^{0,0,1}\cup\bar{\psi}_{jk}^{0,1,0}\in \mathrm{Ext}_A^2(V_j,V_\ell)$ is zero, and that the Massey product

$$\langle\bar{\psi}_{k\ell}^{0,0,1},\bar{\psi}_{jk}^{0,1,0},\bar{\psi}_{ij}^{1,0,0}\rangle\in \mathrm{Ext}_A^2(V_i,V_\ell)$$

is zero.

It is clear how to continue. Notice, nevertheless, the fact that at each step the function

$$\phi(a,v_{i_n},v_{i_{n-1}},\dots,v_{i_1})\in V_{i_{n+1}}$$

is additive in the v_j's, and that we have not assumed, in any way, that the V_i's be different or of any special type.

Corollary Suppose the k-algebra A is of finite dimension, and let the family $V=\{V_i\}_{i=1}^r$ contain all simple representations, then

$$\eta : A\longrightarrow H\hat{\otimes}\mathrm{End}_k(V_i)$$

is injective.

Proof: Let $a\in A$, and suppose $\eta(a)=0$. Since A as a right A-module is an extension of the V_i's we may assume there are exact sequences of right A-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_1 & \longrightarrow & A & \longrightarrow & \bigoplus_{i\in I_1} V_i & \longrightarrow & 0\\ 0 & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & \bigoplus_{i\in I_2} V_i & \longrightarrow & 0\\ \vdots & & \vdots & & \vdots & & \vdots & & \\ 0 & \longrightarrow & Q_N & \longrightarrow & Q_{N-1} & \longrightarrow & \bigoplus_{i\in I_N} V_i & \longrightarrow & 0\end{array}$$

with $Q_N=\bigoplus_{i\in I_{N+1}} V_i$, $Q_{N+1}=0$.

Since $\mathrm{End}(V)a=0$ it follows from the firste exact sequence above that $1\cdot a=a\in Q_1$. Consider the exact sequence

$$0\longrightarrow\bigoplus_{i\in I_2}V_i\longrightarrow A/Q_2\longrightarrow\bigoplus_{i\in I_1}V_i\longrightarrow 0$$

Since $\mathrm{Ext}^1_A(V)a = 0$ it follows that $1 \cdot a = a \in Q_2$. In fact, multiplication by a is zero on V_i, $i = 1, \dots, r$ and on A/Q_2 it is therefore given by the elements in $\mathrm{Ext}^1_A(V)$. Continuing in this way, considering the extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{i \in I_3} V_i & \longrightarrow & A/Q_3 & \longrightarrow & A/Q_2 & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \\ 0 & \longrightarrow & \bigoplus_{i \in I_{N+1}} V_i & \longrightarrow & A & \longrightarrow & A/Q_N & \longrightarrow & 0 \end{array}$$

we deduce from the Lemma that multiplication by a on the right in the middle term is given by a cochain $\psi^n_{ij} \in \mathrm{Hom}_k(A, \mathrm{Hom}_k(V_i, V_j))$, $i \in I_1$, $j \in I_{n+1}$ such that

$$d\psi^n_{ik} = \sum_{\substack{p+q=n \\ j}} \psi^p_{ij} \circ \psi^q_{jk} \, .$$

But since, by the Theorem,

$$\ker \langle \mathrm{Ext}^1_A(V), \dots, \mathrm{Ext}^1_A(V) \rangle a = 0$$

this means that $\psi^n_{ij}(a) = 0$ for all i, j, and so by induction $a \in Q_n \Rightarrow a \in Q_{n+1}$, so $a = 0$. QED

Corollary Suppose A is an object of $\boldsymbol{a}_r$, and let $V = \{V_i\}_{i=1}^r$ be the family of simple representations, with $V_i \simeq k_i$. Then

$$A \simeq H$$

Proof: Obviously A is an $A \otimes_k A$-module, flat over A, therefore $A \in \mathrm{Def}_V(A)$. Since $\mathrm{End}(V) = \begin{pmatrix} k & \cdots & k \\ \vdots & & \vdots \\ k & \cdots & k \end{pmatrix}$ we see immediately that A as $A \otimes_k A$-module is versal. But then the unicity of the hull of Def_V gives us an isomorphism:

$$\phi : H \to A$$

QED

Corollary Let G be a finite reductive group, and put $A = k[G]$, k algebraically closed. Let $V = (V_i)_{i=1}^r$ be the set of irreducible representations. Then $H \simeq k^r$ and $A \simeq \bigoplus_{i=1}^{r} \mathrm{End}_k(V_i)$.

Proof: Since A is semi-simple all $\mathrm{Ext}^1_A(V_i, V_j) = 0$. See that the last isomorphism is exactly the η of the theorem. QED

2.4 Examples. Reconstructing an ordered set Λ and $k[\Lambda]$, from the category of simple modules

Let Λ be an ordered set, see (1.5), and let $A = k[\Lambda]$, $V = \{k_\lambda\}_{\lambda \in \Lambda}$. Then the Corollary above implies that $H \simeq k[\Lambda]$.

Example 1. By the general theory we know that $A = k[\Lambda]$ is the matrix algebra generated freely by the immediate relations $\lambda_1 > \lambda_2$, i.e. those for which $\{\lambda' \in \Lambda \mid \lambda_1 > \lambda' > \lambda_2\} = \emptyset$, modulo relations of the form

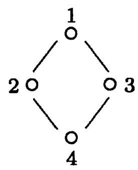
$$\begin{aligned} &(\lambda' > \lambda_2^1)(\lambda_2^1 > \lambda_3^1) \quad \cdots \quad (\lambda_{n_1}^1 > \lambda) \\ = &(\lambda' > \lambda_2^2)(\lambda_2^2 > \lambda_3^2) \quad \cdots \quad (\lambda_{n_2}^2 > \lambda) \end{aligned}$$

They correspond to the first obstructions, given by the n_i term well defined Massey products

$$\begin{aligned} \mathrm{Ext}_A^1(k_{\lambda'}, k_{\lambda_2^1}) \otimes \cdots \otimes \mathrm{Ext}_A^1(k_{\lambda_{n_1}^1}, k_\lambda) &\to \mathrm{Ext}_A^2(k_{\lambda'}, k_\lambda) \\ \mathrm{Ext}_A^1(k_{\lambda'}, k_{\lambda_2^2}) \otimes \cdots \otimes \mathrm{Ext}_A^1(k_{\lambda_{n_2}^2}, k_\lambda) &\to \mathrm{Ext}_A^2(k_{\lambda'}, k_\lambda) \end{aligned}$$

There are as many relations as there are base elements of $\mathrm{Ext}_A^2(k_{\lambda'}, k_\lambda)$.

2. Let's check this for the dimond, i.e. for Λ:



One easily computes the Ext's,

$$\mathrm{Ext}_A^1(k_{\lambda_i}, k_{\lambda_j}) = \begin{cases} 0 & i = j \\ k & \text{for } i = 1,\ j = 2,3 \\ k & \text{for } i = 2,3,\ j = 4 \end{cases}$$

$$\mathrm{Ext}_A^2(k_{\lambda_i}, k_{\lambda_j}) = \begin{cases} 0 & \text{for } (i,j) \neq (1,4) \\ k & \text{for } i = 1,\ j = 4 \end{cases}$$

The two cup-products

$$\mathrm{Ext}_A^1(k_{\lambda_1}, k_{\lambda_j}) \otimes \mathrm{Ext}_A^1(k_{\lambda_j}, k_{\lambda_4}) \to \mathrm{Ext}_A^2(k_{\lambda_1}, k_{\lambda_4}) \quad \text{for } j = 2,3,$$

are non-trivial. At the tangent level we have:

$$H_2 = \begin{pmatrix} k & k & k & 0 \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix}$$

Therefore H must be a quotient of the matrix ring,

$$T^1 = \begin{pmatrix} k & t_{12} \cdot k & t_{13} \cdot k & (t_{12}t_{24} \cdot k + t_{13}t_{34} \cdot k) \\ 0 & k & 0 & t_{24} \cdot k \\ 0 & 0 & k & t_{34} \cdot k \\ 0 & 0 & 0 & k \end{pmatrix}$$

The kernel of $T^1 \to H$ is given in terms of the cup products above. In fact, since we have $t_{13}^* \cup t_{34}^* = t_{12}^* \cup t_{24}^* = y^*$ where y^* is the generator of $\mathrm{Ext}_A^2(k_{\lambda_1}, k_{\lambda_4})$, the kernel of $T^1 \to H$ is simply $t_{13} \otimes t_{34} + t_{12} \otimes t_{24}$ such that

$$H = \begin{pmatrix} k & k & k & k \\ 0 & k & 0 & k \\ 0 & 0 & k & k \\ 0 & 0 & 0 & k \end{pmatrix} \simeq k[\Lambda]\,.$$

as it should.

In general, we may reconstruct Λ from the tangent space $\boldsymbol{t}_H$ and the Massey-products above.

The corresponding problem for finite groups, i.e. reconstructing G from $k[G]$ is called the isomorphism problem. Due to some nice examples of Dade, we know that this is hopeless. In fact there are two non isomorphic finite groups such that their group algebras are isomorphic for all fields.

3 Non commutative modular deformations

3.1 The modular (prorepresenting) substratum, its tangent space, and almost split sequences

Consider as above a family $V = \{V_i\}_{i=1}^r$ of A-modules, and consider the k-algebra

$$\mathrm{End}(V) = (\mathrm{Hom}_A(V_i, V_j)).$$

Suppose from now on that the modules V_i are different, i.e. non isomorphic, indecomposables, and that $\mathrm{End}_A(V_i)$ is a local ring with maximal ideal $\boldsymbol{m}_i$.

Lemma Under the above assumptions, the radical of $\mathrm{End}(V)$ has the form

$$r(V) = (r(V)_{ij}) = \begin{array}{c} \\ i- \end{array} \begin{array}{c} \begin{array}{ccccc} & & j & & \end{array} \\ \begin{pmatrix} \boldsymbol{m}_1\mathrm{End}(V_1) & & \vdots & & \\ \cdots\cdots & \boldsymbol{m}_i\mathrm{End}(V_i) & \vdots & \mathrm{Hom}_A(V_i, V_j) & \cdots\cdots \\ & & \vdots & \boldsymbol{m}_j\mathrm{End}(V_j) & \\ & & \vdots & \vdots & \boldsymbol{m}_r\mathrm{End}(V_r) \end{pmatrix} \end{array} \quad (i \neq j).$$

Proof: We need only check that $\boldsymbol{r}$ is an ideal, and this amounts to proving that if $\phi_{ij} \in \mathrm{Hom}(V_i, V_j)$ $i \neq j$ and $\phi_{ji} \in \mathrm{Hom}(V_i, V_j)$ then

$$\phi_{ji}\phi_{ij} \in \boldsymbol{m}_i \subseteq \mathrm{End}(V_i).$$

If not, $\phi_{ji}\phi_{ij}$ is an isomorphism, and we may as well assume that $\phi_{ji}\phi_{ij} = id_{V_i}$. But then $V_j \simeq V_i \oplus \ker \phi_{ji}$ which contradicts the indecomposability of V_j.

QED

In particular this lemma proves that

$$\bigcap_{n\geq 1} \boldsymbol{r}^n = 0$$

if A is Noetherian and all V_i are of finite type. We shall assume from now on that this is the case.

Obviously there is a left and a right action of $\mathrm{End}(V)$ on

$$\boldsymbol{t}_H = (\mathrm{Ext}^1_A(V_i, V_j)).$$

The difference between these actions defines the action of the Lie algebra $\mathrm{End}(V)$ on $\boldsymbol{t}_H$. The invariants of $\boldsymbol{t}_H$ under the Lie algebra $\boldsymbol{r}(V)$, defined by

$$\boldsymbol{t}_{H_0} := \{\xi \in \boldsymbol{t}_h | \forall \phi \in \boldsymbol{r}(V), \quad \phi\xi - \xi\phi = 0\},$$

is, in analogy with ordinary deformation theory, see [La-Pf], §2, and recall (1.4) above, the tangent space of the *modular*, or *prorepresentable* substratum H_0 of H.

Lemma Let $\xi \in \boldsymbol{t}_{H_0}$, with $\xi = (\xi_{ij})$, then for all $\phi = (\phi_{k\ell}) \in \boldsymbol{r}(V)$ we have for $i \neq j$

$$\begin{matrix} \phi_{\ell i}\xi_{ij} = 0 \\ \xi_{ij}\phi_{j\ell} = 0 \end{matrix} \quad \text{for all } \ell .$$

Moreover, for all i, j

$$\phi_{ij}\xi_{jj} = \xi_{ii}\phi_{ij} .$$

Proof: Just computation. QED

Definition In the above situation, an extension $\xi \in \mathrm{Ext}^1_A(V_i, V_j)$ is called a left almost split extension (resp. a right almost split extension), *lase* (resp. *rase*) for short, if for all $\phi_{ki} \in r(V)_{ki}$ (resp. $\phi_{jk} \in r(V)_{jk}$)

$$\phi_{ki}\xi = 0 \quad (\text{resp.} \quad \xi\phi_{jk} = 0).$$

An extension ξ which is both a lase and a rase is called an *ase*, an *almost split extension.*

This, of course is nothing but a trivial generalization of the notion of almost split sequence, due to Auslander.

Denote by $\mathrm{Ext}^1_\ell(V_i, V_j)$ (resp. $\mathrm{Ext}^1_r(V_i, V_j)$) the subspace of $\mathrm{Ext}^1_A(V_i, V_j)$ formed by the lase's (resp. rase's), and put

$$\begin{aligned} \boldsymbol{t}^\ell_H &= (\mathrm{Ext}^1_\ell(V_i, V_j) \subseteq \boldsymbol{t}_H \\ \boldsymbol{t}^r_H &= (\mathrm{Ext}^1_r(V_i, V_j) \subseteq \boldsymbol{t}_H \\ \boldsymbol{t}^a_H &= \boldsymbol{t}^\ell_H \cap \boldsymbol{t}^r_H =: (\mathrm{Ext}^1_a(V_i, V_j)) \subseteq \boldsymbol{t}_H. \end{aligned}$$

Observe that since the left and the right action of $\mathrm{End}(V)$ on $\boldsymbol{t}_H$ commute, $\mathrm{End}(V)$ acts at right on $\boldsymbol{t}_H^\ell$ and at left on $\boldsymbol{t}_H^r$. Moreover, by the lemma above

$$\boldsymbol{t}_H^a = \boldsymbol{t}_H^\ell \cap \boldsymbol{t}_H^r \subseteq \boldsymbol{t}_{H_0}.$$

Observe also that if $\mathrm{End}_A(V_i) = k + \boldsymbol{m}_i$ the diagonal part of $\boldsymbol{t}_{H_0}$ is exactly the tangent space of the deformation functor of the full subcategory of mod_A generated by V, see [La2].

3.2 The structure of the modular substratum, and the existence of almost split sequences

From now on, assume that $\boldsymbol{t}_H$ is a k-vectorspace of finite dimension. Then the radical $\boldsymbol{r}(V)$ of $\mathrm{End}(V)$ acts nilpotently on $\boldsymbol{t}_H$.

Corollary Given $i \in \{1, \dots, r\}$, assume there exists one $j \in \{1, \dots, r\}$ such that $\mathrm{Ext}_A^1(V_i, V_j) \neq 0$. Then there exists a $\tau(i) \in \{1, \dots, r\}$ such that

$$\mathrm{Ext}_r^1(V_i, V_{\tau(i)}) \neq 0.$$

Proof: This is simply Engels theorem for the right action of $\boldsymbol{r}(V)$ on $\boldsymbol{t}_H$.
QED

Theorem Suppose V is such that every extension $\xi \in \mathrm{Ext}_A^1(V_i, V_j)$ is of the form $0 \to V_j \to E \to V_i \to 0$ with E a direct sum of V_k's. Then, for every $i = 1, \dots, r$, such that there exists a $j = 1, \dots, r$ for which $\mathrm{Ext}_A^1(V_i, V_j) \neq 0$, there is a unique ase of the form

$$0 \to V_{\tau(i)} \to E_i \to V_i \to 0$$

Moreover, if we agree to put $\tau(i) = i$ for those i's for which $\mathrm{Ext}_A^1(V_i, V_k) = 0$ for all k, then

$$\tau : \{1, \dots, r\} \to \{1, \dots, r\}$$

is a permutation.

Proof: We already know that there exists a rase of the form

$$\xi_i : 0 \to V_{\tau(i)} \to E_i \to V_i = 0$$

Suppose ξ_i is not killed by the left action of $\boldsymbol{r}(V)$, then there exists a $\phi_{ki} \in \mathrm{Hom}_A(V_k, V_i)$ for $k \neq i$, or an element $\phi_{ii} \in \boldsymbol{m}_i \subseteq \mathrm{End}(V_i)$, and a commutative diagram,

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[d] & E_i \arrow[r] & V_i \arrow[r] & 0, \qquad \xi_i \\
 & E_k = V_k {}_{V_i}\!\times E_i \arrow[d] \arrow[ur, "\psi_{ki}"] & & & \\
 & V_k \arrow[d] \arrow[uurr, "\phi_{ki}"'] & & & \\
 & 0 & & &
\end{tikzcd}$$

Suppose $V_{\tau(i)} \to E_k$ is not split, then

$$0 \to E_k \to E_k \overset{V_{\tau(i)}}{\oplus} E_i \to V_i \to 0$$

is split, since ξ_i is a rase. Let pr.: $E_k \overset{V_{\tau(i)}}{\oplus} E_i \to E_k$ be the splitting. But then the two following diagrams commute:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[d] & E_i \arrow[r] \arrow[dl, shift left, "\psi_{ik}"] & V_i \arrow[r] \arrow[ddll, shift left, "\phi_{ik}"] & 0 \\
 & E_k \arrow[d] \arrow[ur, shift left, "\psi_{ki}"] & & & \\
 & V_k \arrow[d] \arrow[uurr, shift left, "\phi_{ki}"] & & & \\
 & 0 & & &
\end{tikzcd}$$

Here ψ_{ik} is the composition of $E_i \to E_k \overset{V_{\tau(i)}}{\oplus} E_i$ and the projection $E_k \overset{V_{\tau(i)}}{\oplus} E_i \to E_k$ and ϕ_{ik} the induced map.

This means that $(\phi_{ki}\phi_{ik})\xi_i = \xi_i$ which is impossible since $(\phi_{ki}\phi_{ik})$ acts nilpotently on $\mathrm{Ext}^1_A(V_i, V_{\tau(i)})$, and ξ is nonzero. Therefore $V_{\tau(i)} \to E_k$ splits and ξ_i is also a lase, therefore an ase.

The unicity and the permutation property follows immediately from the following: Assume there exist two ase's ξ_i and ξ_i' of the form:

$$
\xi_i : \quad
\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & V_{\tau(i)} \arrow[r] \arrow[drr, shift left, "\phi_i"] & E_i \arrow[r, "\rho_i"] \arrow[dr, shift left, "\psi_i"] & V_i \arrow[r] \arrow[u] & 0 \\
 & & & E'_i \arrow[u, "\rho'_i"'] \arrow[ul, shift left, "\psi'_i"] & \\
 & & & V_{\tau(i)'} \arrow[u] \arrow[uull, shift left, "\phi'_i"] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}
$$

Then, since ρ'_i is not split, there exist liftings ψ_i, ψ'_i inducing morphisms ϕ_i, ϕ'_i. But then $(\phi_i\phi'_i)\xi_i = \xi_i$ which means that ξ_i is zero. Therefore an ase is unique and in particular, $\tau(i) = \tau(i)'$. Dually we prove that $\tau(i) = \tau(i')$ implies $i = i'$, so that τ is a permutation.

We see that t_V^a looks like:

$$
\left(\mathrm{Ext}^1_a(V_i, V_j)\right)
$$

where $\mathrm{Ext}^1_a(V_i, V_j) = \begin{cases} 0 & \text{if } j \neq \tau(i) \\ k & \text{if } j = \tau(i) \end{cases}$ and some $\mathrm{Ext}^1_a(V_i, V_j) \neq 0$ QED

Corollary With the assumptions of the theorem above, we find that

$$
\begin{aligned}
t_{H_0} = \{(\alpha_{ij}) \text{ If } i \neq j \quad & \alpha_{ij} \in k, \ \ \alpha_{ij} = 0 \text{ if } j \neq \tau(i)\,. \text{ If } i = j, \\
& \alpha_{ii} \in \mathrm{Ext}^1_A(V_i, V_i) \text{ such that } \phi_{ij}\alpha_{ii} = \alpha_{jj}\phi_{ji}\,, \ \forall \phi_{ji} \in \mathrm{End}(V_j, V_i)\}\,.
\end{aligned}
$$

The story of almost split sequences in the sense of Auslander starts in [A1].

3.3 Examples

Let $A = k[x_0, x_1]/(f)$ where $f = x_0^3 + x_1^2$, i.e. A is the A_2 simple singularity. One finds two indecomposable maximal $C.M.$ A-modules, A and some M. $\mathrm{Ext}^1_A(M, M)$ is generated by two elements ξ_1 and ξ_2. Only one, ξ_1 say, is an ase. Moreover $\xi_1 \cup \xi_1 \neq 0$. Therefore

$$
H_0 = \begin{pmatrix} k & 0 \\ 0 & k[\epsilon] \end{pmatrix}
$$

and the modular versal family $\widetilde{V}_0$ looks like

$$
\widetilde{V}_0 = \begin{pmatrix} A & 0 \\ 0 & \widetilde{M} \end{pmatrix}
$$

where $k \otimes_{k[\epsilon]} \widetilde{M} = M$ and $\widetilde{M}$ as an A-module is isomorphic to $M \oplus A$, corresponding to the quiver

$$A \leftrightarrows M \circlearrowleft$$

Notice that the cycle in the quiver corresponds to the ϵ in H_0.

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