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THE STOCHASTIC WICK-TYPE BURGERS EQUATION

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Abstract

We study the multidimensional stochastic (Wick-type) Burgers equation

$$\begin{cases} \frac{\partial u_k}{\partial t} + \lambda \sum_{j=1}^{n} u_j \diamond \frac{\partial u_k}{\partial x_j} = \nu \Delta u_k + w_k(t,x) \ ; \ t > 0, x \in \mathbf{R}^n \\ u_k(0,x) = g_k(x) \ ; \ 1 \leq k \leq n \end{cases}$$

where $\diamond$ denotes the Wick product, λ and ν are constants ($\nu > 0,\ \lambda \neq 0$), Δ denotes the Laplacian and $\{w_k(t,x)\}_{k=1}^n$ are $(n+1)$-parameter stochastic processes (noise). We prove an existence and uniqueness result for the solution $u(t,x) = \{u_k(t,x)\}_{k=1}^n$, regarded as an $(n+1)$-parameter stochastic process with values in the Kondratiev space $(\mathcal{S})^{-1}$ of stochastic distributions.

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§0. INTRODUCTION

The purpose of this paper is to continue the work done in [HLØUZ 2] regarding the stochastic multidimensional Burgers equation in $\{u_k(t,x)\}_{k=1}^n$:

$$(0.1) \qquad \begin{cases} \frac{\partial u_k}{\partial t} + \lambda \sum\limits_{j=1}^{n} u_j \diamond \frac{\partial u_k}{\partial x_j} & = \nu \Delta u_k + w_k(t,x);\ t > 0,\ x \in \mathbf{R}^n \\ & u_k(0,x) = g_k(x);\ 1 \le k \le n \end{cases}$$

where $\diamond$ denotes the Wick product, λ and ν are constants ($\nu > 0,\ \lambda \neq 0$), Δ denotes the Laplacian and $w_k(t,x) = w_k(t,x)$ are $(n+1)$-parameter generalized stochastic processes; $1 \le k \le n$. (See details below). The use of the Wick product corresponds to an Ito/Skorohod interpretation of the equation (see e.g. [B1], [LØU 2] and also [HLØUZ 3]). We may regard $u = (u_1, \cdots, u_n)$ as the velocity field of a vorticity free fluid with viscosity ν, being exposed to the stochastic force $w = (w_1, \cdots, w_n)$.

In [HLØUZ 2] the following was proved: (For definition of functional processes etc. see §1). (Here - and in the following - all gradients are taken w.r.t. x).

A. Let $N = N(\phi, t, x, \omega)$ be a functional process and define $w = -\nabla N$. Assume that (0.1) has a solution of the form

$$u = -\nabla X$$

for some functional process $X = X(\phi, t, x, \omega)$. Moreover, assume that the functional processes

$$(0.2) \qquad Y := \operatorname{Exp}(\frac{\lambda}{2\nu} X)$$

as well as $Y \diamond X$ and $Y \diamond X^{\diamond 2}$ exist in $L^p(\mu)$ for some $p \ge 1$. Then Y solves the stochastic heat equation

$$(0.3) \qquad \begin{cases} \frac{\partial Y}{\partial t} = \nu \Delta Y + \frac{\lambda}{2\nu} Y \diamond (N + C); \quad t > 0,\ x \in \mathbf{R}^n \\ Y(0,x) = \operatorname{Exp}(\frac{\lambda}{2\nu} X(0,x)) \end{cases}$$

for some $C(t,\omega)$ not depending on x. (For simplicity we have written $Y(t,x)$ for $Y(\phi, x, x, \omega)$ etc.)

Furthermore, we proved the following:

B. Let either $N = W(\phi, t, x, \omega)$ ($(n+1)$-parameter white noise) or $N = \operatorname{Exp} W(\phi, t, x, \omega)$ ($(n+1)$-parameter positive noise), and assume that both $f(x) := Y(0,x)$ and $C(t,\omega) = C(t)$ are bounded and deterministic (do not depend on ω).

Then (0.3) has a unique $L^2(\mu)$ (respectively $L^1(\mu)$) functional process solution $Y(t,x)$ given by

$$(0.4) \qquad Y(t,x) = \hat{E}^x[f(b_{\alpha t}) \operatorname{Exp}(\int_0^t H(s, b_{\alpha s}) ds)],$$

where $\alpha = \sqrt{2\nu}$, $(b_t, \hat{P}^x)$ is standard Brownian motion in $\mathbf{R}^n$ ($\hat{E}^x$ denotes expectation w.r.t. $\hat{P}^x$) and

$$H(t,x) = H(t,x,\omega) = \frac{\lambda}{2\nu}(N(\phi,t,x,\omega) + C(t)). \tag{0.5}$$

Thus we see that if u is a gradient solution of (0.1) then we can use it to construct an explicit solution of the stochastic heat equation (0.3). Moreover, by other methods the unique solution of (0.3) can be found explicitly. In particular, this proves that under the given assumptions there exists at most one functional process solution (of gradient form) of the stochastic Burgers equation (0.1).

The purpose of the present paper is to complete the analysis by proving that one can reverse the Wick-Cole-Hopf transformation (0.2) to construct a solution of the stochastic Burgers equation from the stochastic heat equation and hence obtain a uniqueness *and* existence result for equation (0.1). In order to accomplish this we consider processes and operations in the *Kondratiev space* $(\mathcal{S})^{-1}$ *of stochastic distributions.* This space has already found several applications in stochastic partial (and ordinary) differential equations. See e.g. [B 2], [HLØUZ 4] and [Ø]. For the stochastic Burgers equation this approach has the following advantages:

a) The transformation from the stochastic Burgers equation to the stochastic heat equation can be performed with fewer assumptions than given above if done in the space $(\mathcal{S})^{-1}$. (See Theorem 2.1).

b) The stochastic heat equation can be solved explicitly in $(\mathcal{S})^{-1}$ for a general $(\mathcal{S})^{-1}$ potential (Theorem 3.1).

c) Most importantly, in $(\mathcal{S})^{-1}$ one can also construct the *converse* transformation from the stochastic heat equation to the stochastic Burgers equation. (See Theorem 4.1).

By combining a), b) and c) we obtain a uniqueness and existence result (Theorem 5.1) for the stochastic Burgers equation (0.1). Moreover, we obtain this under weaker assumptions than what was needed for the uniqueness result in [HLØUZ 2].

One-dimensional Burgers equations with ordinary product instead of Wick product have been studied in [BCJ-L], [DDT] and [DG].

§1. WHITE NOISE, WICK PRODUCTS AND STOCHASTIC DISTRIBUTIONS

Here we briefly recall some of the basic definitions and results that we need from white noise calculus. For more information the reader is referred to [HKPS] and [KLS].

In the following we fix the parameter dimension d and let $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$ denote the Schwartz space of rapidly decreasing smooth (C^∞) functions on $\mathbf{R}^d$. The dual $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^d)$ is the space of tempered distributions. By the Bochner-Minlos theorem [GV] there exists a probability measure μ on the Borel subsets $\mathcal{B}$ of $\mathcal{S}'$ with the property that

$$\int_{\mathcal{S}'} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2}; \forall\phi \in \mathcal{S} \tag{1.1}$$

where $\langle\omega, \phi\rangle$ denotes the action of $\omega \in \mathcal{S}'$ on $\phi \in \mathcal{S}$ and $\|\phi\|^2 = \int_{\mathbf{R}^d} |\phi(x)|^2 dx$. The triple $(\mathcal{S}', \mathcal{B}, \mu)$ is called *the white noise probability space.*

The white noise process is the map $W : \mathcal{S} \times \mathcal{S}' \to \mathbf{R}$ defined by

$$W(\phi, \omega) = W_\phi(\omega) = \langle\omega, \phi\rangle; \quad \omega \in \mathcal{S}', \phi \in \mathcal{S} \tag{1.2}$$

Expressed in terms of Ito integrals with respect to d-parameter Brownian motion B we have

$$W_\phi(\omega) = \int_{\mathbf{R}^d} \phi(x) dB_x(\omega) \quad ; \phi \in \mathcal{S}. \tag{1.3}$$

The *Hermite polynomials* are defined by

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}); \quad n = 0, 1, 2, \cdots \tag{1.4}$$

and the *Hermite functions* are defined by

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x) \quad ; \quad n \geq 1 \tag{1.5}$$

In the following we let $\{e_1, e_2, \cdots\} \subset \mathcal{S}$ denote a fixed orthonormal basis for $L^2(\mathbf{R}^d)$. For many purposes the basis can be arbitrary, but for us it is convenient to assume that the e_n's are obtained by taking tensor products of $\xi_k(x)$. Define

$$\theta_j(\omega) := W_{e_j}(\omega) = \int_{\mathbf{R}^d} e_j(x) dB_x(\omega) \quad ; \quad j = 1, 2, \cdots \tag{1.6}$$

If $\alpha = (\alpha_1, \cdots, \alpha_m)$ is a multi-index of non-negative integers we put

$$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) \tag{1.7}$$

The *Wiener-Ito chaos expansion theorem* says that any $X \in L^2(\mu)$ can be (uniquely) written

$$X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \tag{1.8}$$

Moreover,

$$\|X\|^2_{L^2(\mu)} = \sum_{\alpha} \alpha! c_\alpha^2 \quad \text{where} \quad \alpha! = \alpha_1!\alpha_2!\cdots\alpha_m! \tag{1.9}$$

The Hida test function space $(\mathcal{S})$ and the Hida distribution space $(\mathcal{S})^*$ can be given the following characterization, due to T.-S. Zhang [Z]:

THEOREM 1.1 ([Z])

Part a): A function $f = \sum_\alpha c_\alpha H_\alpha \in L^2(\mu)$ belongs to $(\mathcal{S})$ if and only if

$$\sup_\alpha c_\alpha^2 \alpha! (2\mathrm{N})^{\alpha k} < \infty \quad \forall k < \infty \tag{1.10}$$

where

$$(2\mathrm{N})^\alpha := \prod_{j=1}^m (2^d \beta_1^{(j)} \cdots \beta_d^{(j)})^{\alpha_j} \quad \text{if } \alpha = (\alpha_1, \cdots, \alpha_m) \tag{1.11}$$

Here $\beta^{(j)} = (\beta_1^{(j)}, \cdots, \beta_d^{(j)})$ is multi-index nr. j in the fixed ordering of all d-dimensional multi-indices $\beta = (\beta_1, \cdots, \beta_d)$, related to the basis $\{e_j\}$ by

$$e_j = \xi_{\beta_1^{(j)}} \otimes \cdots \otimes \xi_{\beta_d^{(j)}}. \tag{1.12}$$

Part b): A formal series $F = \sum_\alpha b_\alpha H_\alpha$ belongs to $(\mathcal{S})^*$ if and only if

$$\sup_\alpha b_\alpha^2 \alpha! (2\mathrm{N})^{-\alpha q} < \infty \quad \text{for some} \quad q < \infty \tag{1.13}$$

The action of $F = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^*$ on $f = \sum_\alpha c_\alpha H_\alpha \in (\mathcal{S})$ is given by

$$\langle F, f \rangle = \sum_\alpha \alpha! b_\alpha c_\alpha \tag{1.14}$$

EXAMPLE The pointwise (or singular) white noise W_x is defined by

$$W_x(\omega) = \sum_{k=1}^\infty e_k(x) H_{\epsilon_k}(\omega) = \sum_{k=1}^\infty e_k(x) h_1(\theta_k) \tag{1.15}$$

where $\epsilon_k = (0, 0, \cdots, 0, 1)$ with 1 on k'th place.

In this case

$$b_\alpha = b_{\epsilon_k} = e_k(x) \text{ if } \alpha = \epsilon_k \text{ for some } k$$
$$b_\alpha = 0 \text{ if } \alpha \neq \epsilon_k \text{ for all } k$$

Moreover, if $\alpha = \epsilon_k$ we have

$$(2\mathbf{N})^\alpha = 2^d \beta_1^{(k)} \cdots \beta_d^{(k)}$$

So in this case we get

$$\sup_\alpha b_\alpha^2 \alpha! (2\mathbf{N})^{-\alpha q} = \sup_k e_k^2(x) (2^d \beta_1^{(k)} \cdots \beta_d^{(k)})^{-q} < \infty$$

for all $q > 0$, since

$$\sup_{t \in \mathbf{R}} |\xi_k(t)| = 0(k^{-\frac{1}{12}}) \quad ([\text{HiP}])$$

We conclude that $W_x(\omega) \in (\mathcal{S})^*$.

Note that if $1 < p < \infty$ we have

$$(\mathcal{S}) \subset L^p(\mu) \subset (\mathcal{S})^* \tag{1.16}$$

However,

$$L^1(\mu) \not\subset (\mathcal{S})^* \quad \text{(see e.g. [HLØUZ 1])} \tag{1.17}$$

For our purposes it turns out to be convenient to work with the *Kondratiev* spaces $(\mathcal{S})^1$ and $(\mathcal{S})^{-1}$ which are related to $(\mathcal{S})$ and $(\mathcal{S})^*$ as follows:

$$(\mathcal{S})^1 \subset (\mathcal{S}) \subset (\mathcal{S})^* \subset (\mathcal{S})^{-1} \tag{1.18}$$

The spaces $(\mathcal{S})$ and $(\mathcal{S})^{-1}$ were originally constructed on spaces of sequences by Kondratiev [K] and later extended by him and several other authors. See [KLS] and the references there. We recall here their basic properties, stated in forms which are convenient for our purposes. For details and proofs we refer to [KLS].

DEFINITION 1.2 [KLS]

Part a): For $0 \le \rho \le 1$ let $(\mathcal{S})^\rho$ (the *Kondratiev space of stochastic test functions*) consist of all

$$f = \sum_\alpha c_\alpha H_\alpha \in L^2(\mu) \quad \text{such that}$$

$$\|f\|_{\rho,k}^2 := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbf{N})^{\alpha k} < \infty \quad \text{for all } k < \infty \tag{1.19}$$

Part b): The *Kondratiev space of stochastic distributions*, $(\mathcal{S})^{-\rho}$, consists of all formal expansions

$$F = \sum_\alpha b_\alpha H_\alpha$$

such that

$$\sum_{\alpha} b_\alpha^2 (\alpha!)^{1-\rho}(2\mathbf{N})^{-\alpha q} < \infty \quad \text{for some } q < \infty \tag{1.20}$$

The family of seminorms $\|f\|^2_{\rho,k}$; $k = 1, 2, \cdots$ gives rise to a topology on $(\mathcal{S})^\rho$ and we can then regard $(\mathcal{S})^{-\rho}$ as the dual of $(\mathcal{S})^\rho$ by the action

$$\langle F, f\rangle = \sum_{\alpha} b_\alpha c_\alpha \alpha! \tag{1.21}$$

if $F = \sum b_\alpha H_\alpha \in (\mathcal{S})^{-\rho}$ and $f = \sum c_\alpha H_\alpha \in (\mathcal{S})^\rho$.

REMARKS.

1) Regarding (1.21), note that

$$\begin{aligned}
&\sum_{\alpha} |b_\alpha c_\alpha|\alpha! = \sum_{\alpha} |b_\alpha c_\alpha|(\alpha!)^{\frac{1-\rho}{2}}(\alpha!)^{\frac{1+\rho}{2}} \cdot (2\mathbf{N})^{\frac{\alpha k}{2}}(2\mathbf{N})^{-\frac{\alpha k}{2}} \\
&\leq [\sum_{\alpha} b_\alpha^2 (\alpha!)^{1-\rho}(2\mathbf{N})^{-\alpha k}]^{\frac{1}{2}} \cdot [\sum_{\alpha} c_\alpha^2 (\alpha!)^{1+\rho}(2\mathbf{N})^{\alpha k}]^{\frac{1}{2}} \\
&< \infty \quad \text{for } k \text{ large enough.}
\end{aligned}$$

2) Putting $\rho = 0$ we see by comparing (1.19), (1.20) with (1.10), (1.13) that $(\mathcal{S}) = (\mathcal{S})^0$ and $(\mathcal{S})^* = (\mathcal{S})^{-0}$. So for general $\rho \in [0, 1]$ we have

$$(\mathcal{S})^1 \subset (\mathcal{S})^\rho \subset (\mathcal{S})^0 = (\mathcal{S}) \subset (\mathcal{S})^* = (\mathcal{S})^{-0} \subset (\mathcal{S})^{-\rho} \subset (\mathcal{S})^{-1} \tag{1.22}$$

(Observe that with this notation $(\mathcal{S})^0$ and $(\mathcal{S})^{-0}$ are different spaces).

DEFINITION 1.3

The Wick product $F \diamond G$ of two elements

$$\begin{aligned}
&F = \sum_{\alpha} a_\alpha H_\alpha, G = \sum_{\beta} b_\beta H_\beta \text{ in } (\mathcal{S})^{-1} \text{ is defined by} \\
&F \diamond G = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta}
\end{aligned} \tag{1.23}$$

From Def. 1.2 we get

LEMMA 1.4

(i) $F, G \in (\mathcal{S})^{-1} \Rightarrow F \diamond G \in (\mathcal{S})^{-1}$
(ii) $f, g \in (\mathcal{S})^1 \Rightarrow f \diamond g \in (\mathcal{S})^1$

The *Hermite transform* [LØU 1-3] has a natural extension to $(\mathcal{S})^{-1}$:

DEFINITION 1.5 If $F = \sum_\alpha b_\alpha H_\alpha \in (\mathcal{S})^{-1}$ then the Hermite transform of $F, \mathcal{H}F = \tilde{F}$, is defined by

(1.24) $$\tilde{F}(z) = \mathcal{H}F(z) = \sum_\alpha b_\alpha z^\alpha \quad \text{(whenever convergent)}$$

where $z = (z_1, z_2, \cdots) \in \mathbf{C}^{\mathbf{N}}$ (the space of all sequences of complex numbers) and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m} \text{ if } \quad \alpha = (\alpha_1, \cdots, \alpha_m).$$

If $F \in (\mathcal{S})^{-\rho}$ for $\rho < 1$ then it is easy to see that $(\mathcal{H}F)(z_1, z_2, \cdots)$ converges for all finite sequences $(z_1, \cdots, z_m)$ of complex numbers.

If $F \in (\mathcal{S})^{-1}$, however, we can only obtain convergence of $\mathcal{H}F(z_1, z_2, \cdots)$ in a neighbourhood of the origin: We have

$$\sum_\alpha |b_\alpha||z^\alpha| \leq [\sum_\alpha b_\alpha^2 (2\mathbf{N})^{-\alpha q}]^{\frac{1}{2}} \cdot [\sum_\alpha |z^\alpha|^2 (2\mathbf{N})^{\alpha q}]^{\frac{1}{2}},$$

where the first factor on the right hand side converges for q large enough. For such a value of q we have convergence of the second factor if

(1.25) $$z \in \mathbf{B}_q(\delta) := \{\zeta = (\zeta_1, \zeta_2, \cdots) \in \mathbf{C}^{\mathbf{N}} \, ; \, \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2\mathbf{N})^{\alpha q} < \delta^2\}$$

for some $\delta < \infty$.

The next result is an immediate consequence of Def. 1.3 and Def. 1.5:

LEMMA 1.6 If $F, G \in (\mathcal{S})^{-1}$ then

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)$$

for all z such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exist.

The topology on $(\mathcal{S})^{-1}$ can conveniently be expressed in terms of Hermite transforms as follows:

LEMMA 1.7 [KLS]

The following are equivalent
(i) $X_n \to X$ in $(\mathcal{S})^{-1}$
(ii) $\exists \delta > 0, q < \infty, M < \infty$ such that

$$\mathcal{H}X_n(z) \to \mathcal{H}X(z) \quad \text{as } n \to \infty \quad \text{for } z \in \mathbf{B}_q(\delta)$$

and

$$|\mathcal{H}X_n(z)| \leq M \quad \text{for all } n = 1, 2, \cdots; z \in \mathbf{B}_q(\delta).$$

THEOREM 1.8 [KLS] (Characterization theorem for $(\mathcal{S})^{-1}$)

Suppose $g(z_1, z_2, \cdots)$ is a bounded analytic function on $\mathbf{B}_q(\delta)$ for some $\delta > 0, q < \infty$. Then there exists $X \in (\mathcal{S})^{-1}$ such that

$$\mathcal{H}X = g$$

From this we deduce the following useful result:

COROLLARY 1.9

Suppose $g = \mathcal{H}X$ for some $X \in (\mathcal{S})^{-1}$. Let f be an analytic function in a neighbourhood of $\zeta_0 = g(0)$ in $\mathbf{C}$. Then there exists $Y \in (\mathcal{S})^{-1}$ such that

$$\mathcal{H}Y = f \circ g$$

Proof. Let $r > 0$ be such that f is bounded analytic on $\{\zeta \in \mathbf{C}; |\zeta - \zeta_0| < r\}$. Then choose $\delta > 0$ and $q < \infty$ such that the function $z \to g(z)$ is bounded analytic on $\mathbf{B}_q(\delta)$ and such that $|g(z) - \zeta_0| < r$ for $z \in \mathbf{B}_q(\delta)$. Then $f \circ g$ is bounded analytic in $\mathbf{B}_q(\delta)$, so the result follows from Theorem 1.8.

EXAMPLE 1.10

a) Let $X \in (\mathcal{S})^{-1}$. Then $X \diamond X = X^{\diamond 2} \in (\mathcal{S})^{-1}$ and more generally $X^{\diamond n} \in (\mathcal{S})^{-1}$ for all natural numbers n. Define *the Wick exponential of* X, Exp X, by

$$\text{Exp } X = \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}$$

Then by Corollary 1.9 applied to $f(z) = e^z$ we see that Exp $X \in (\mathcal{S})^{-1}$ also.

b) In particular, if we choose $X = W_x$ (the singular white noise) then $K_0 := \text{Exp}W_x$ is in fact in $(\mathcal{S})^*$. As suggested in [LØU 1], [LØU 3] the process $K_0(x, \omega)$ is a natural model for stochastic permeability in connection with fluid flow in porous media.

c) Other useful applications of Lemma 1.9 include the *Wick logarithm* $Y =$ Log X, which is defined (in $(\mathcal{S})^{-1}$) for all $X \in (\mathcal{S})^{-1}$ with $\tilde{X}(0) \neq 0$. For such X we have

$$\text{Exp}(\text{Log } X) = X$$

and for all $Z \in (\mathcal{S})^{-1}$ we have

$$\text{Log}(\text{Exp } Z) = Z$$

d) Similarly we note that the *Wick-inverse* $X^{\diamond(-1)}$ and more generally the *Wick powers* $X^{\diamond\gamma}(\gamma \in \mathbf{R})$ exist in $(\mathcal{S})^{-1}$ for all $X \in (\mathcal{S})^{-1}$ with $\tilde{X}(0) \neq 0$.

REMARK.

The connection between the $\mathcal{H}$-transform and the $\mathcal{S}$-*transform* [HKPS] is

$$\mathcal{H}F(z_1, z_2, \cdots, z_m) = (\mathcal{S}F)(z_1 e_1 + \cdots + z_m e_m) \tag{1.26}$$

(see e.g. [LØU 1])

REMARK. We can define what we could call the *generalized expectation* of an arbitrary $F \in (\mathcal{S})^{-1}$, in spite of the fact that such an F need not even be in $L^1(\mu)$: If $F_0 \in L^p(\mu)$ for $p > 1$ then the action of F_0 on an element $\psi \in (\mathcal{S})^1$ is given by

$$\langle F_0, \psi \rangle = E[F_0 \psi] = \int_{\mathcal{S}'} F_0(\omega)\psi(\omega)d\mu(\omega), \tag{1.27}$$

so if $\psi \equiv 1$ then $\langle F_0, \psi \rangle = \langle F_0, 1 \rangle$ gives us the expectation of F_0. On the other hand, if a general $F \in (\mathcal{S})^{-1}$ has the chaos expansion

$$F = \sum_\alpha b_\alpha H_\alpha$$

then by (1.21) we have $\langle F, 1 \rangle = b_0 = \tilde{F}(0)$. Hence we define

$$\tilde{F}(0) = b_0 = \langle F, 1 \rangle \tag{1.28}$$

to be the *generalized expectation* of $F \in (\mathcal{S})^{-1}$.

From now on we will use the notation

$$E[F] = E_\mu[F] := \tilde{F}(0) \tag{1.29}$$

for the generalized expectation of $F \in (\mathcal{S})^{-1}$. Note that with this definition we have

$$E[F \diamond G] = E[F] \cdot E[G] \quad \text{for all} \quad F, G \in (\mathcal{S})^{-1}. \tag{1.30}$$

and

$$E[\operatorname{Exp}(X)] = \exp E[X] \quad \text{for all } X \in (\mathcal{S})^{-1}. \tag{1.31}$$

More generally, from the chaos expansion of F we see that the Hermite transform gives us all the *actions*

$$\langle F, \langle \cdot, \phi \rangle^{\diamond n} \rangle$$

of $F \in (\mathcal{S})^{-1}$ on $\langle \cdot, \phi \rangle^{\diamond n} \in (\mathcal{S})^{1}$. Therefore, although F need not exist as a random variable, it exists as a stochastic distribution: Given a stochastic test function we can compute its associated average.

$(\mathcal{S})^{-1}$ processes

The Kondratiev space $(\mathcal{S})^{-1}$ of stochastic distributions turns out to be the right space to work in, not just for the Burgers equation discussed in this paper, but for several stochastic (partial or ordinary) differential equations. See e.g. [B 3], [HLØUZ 4]. Therefore, the solution we seek will be a function

$$u = u(t,x) : \mathbf{R}^{1+n} \to (\mathcal{S})^{-1},$$

which may be regarded as a (stochastic) distribution valued stochastic process. We call such functions $(\mathcal{S})^{-1}$ *processes.*

The derivative of an $(\mathcal{S})^{-1}$ process $f(t)$ w.r.t. t at $t = t_0 \in \mathbf{R}$ will then (if it exists) be the element $\eta = \eta(t_0) \in (\mathcal{S})^{-1}$ with the property that

$$\frac{f(t_0+h) - f(t_0)}{h} \to \eta(t_0) \quad \text{in} \quad (\mathcal{S})^{-1} \quad \text{as} \quad h \to 0. \tag{1.31}$$

If this holds we write $\eta(t_0) = \frac{df}{dt}(t_0)$ (or $\frac{\partial f}{\partial t}(t_0)$). By the characterization of the topology of $(\mathcal{S})^{-1}$ in terms of the Hermite transform (Lemma 1.7) this is equivalent to

$$\frac{\tilde{f}(t_0+h;z) - \tilde{f}(t_0;z)}{h} \to \tilde{\eta}(t_0;z) \tag{1.32}$$

pointwise boundedly for $z \in \mathbf{B}_q(\delta)$; for some $q < \infty, \delta > 0$.

For this it suffices that

$$\frac{d}{dt}\tilde{f}(t;z) = \tilde{\eta}(t;z) \quad \text{for } t = t_0 \tag{1.33}$$

pointwise for each $z \in \mathbf{B}_q(\delta)$, if we also have that

$$t \to \frac{d}{dt}\tilde{f}(t;z)$$

is continuous and uniformly bounded, for $z \in \mathbf{B}_q(\delta)$ and t in a neighbourhood of t_0. For if this holds, we can write

$$\frac{\tilde{f}(t_0+h;z) - \tilde{f}(t_0;z)}{h} = \frac{1}{h}\int_{t_0}^{t_0+h} \frac{d}{ds}\tilde{f}(s;z)ds \quad \text{for small } h$$

and therefore this expression is uniformly bounded for $z \in \mathbf{B}_q(\delta)$ as $h \to 0$. If $\frac{df}{dt}$ exists and is t-continuous, we say that the $(\mathcal{S})^{-1}$ process $f(t)$ is C^1. A similar notation, C^k, is used for higher order derivatives, $k = 2, 3, \cdots$ (and for continuity if $k = 0$) and for several variables, $C^{k_1, k_2, \cdots, k_m}$.

In [HLØUZ 1-2] a different kind of solution concept is studied: An L^p *functional process* ($p \geq 1$) is a map

$$X : \mathcal{S} \times \mathbf{R}^d \times \mathcal{S}' \to \mathbf{R}$$

such that

(i) $x \to X(\phi, x, \omega)$ is (Borel) measurable for all $\phi \in \mathcal{S}$, $\omega \in \mathcal{S}'$
and
(ii) $\omega \to X(\phi, x, \omega)$ belongs to $L^p(\mu)$ for all $\phi \in \mathcal{S}$, $x \in \mathbf{R}^d$.

Intuitively, $X(\phi, x, \omega)$ is the result of measuring the quantity X using the test function ("window") ϕ shifted to the point x and in the "experiment" ω.

EXAMPLE. White noise W may be regarded as an L^p functional process (for any $p < \infty$) by putting

$$W(p, x, \omega) = W_{\phi_x}(\omega),$$

where

$$\phi_x(y) = \phi(y - x).$$

REMARK For L^1 functional processes the definition of Wick product must be extended, since $L^1(\mu)$ is not contained in $(\mathcal{S})^{-1}$. See [HLØUZ 1].

§2. FROM THE STOCHASTIC BURGERS EQUATION TO THE STOCHASTIC HEAT EQUATION.

This transformation was performed in [HLØUZ 2], but only in the context of functional processes and hence with more assumptions than will be needed within $(\mathcal{S})^{-1}$:

THEOREM 2.1 (The Wick-Cole-Hopf transformation (I))

Let $N = N(t, x)$ be an $(\mathcal{S})^{-1}$-valued $C^{0,1}$-process and define

$$w = -\nabla N. \tag{2.1}$$

Assume that there exists an $(\mathcal{S})^{-1}$-valued $C^{1,3}$-process $X(t, x)$ such that

$$u = -\nabla X \tag{2.2}$$

solves the multidimensional Burgers equation

$$\begin{cases} \frac{\partial u_k}{\partial t} + \lambda \sum\limits_{j=1}^{n} u_j \diamond \frac{\partial u_k}{\partial x_j} & = \nu \Delta u_k + w_k(t, x);\ t > 0;\ x \in \mathbf{R}^n \\ & u_k(0, x) = g_k(x);\ 1 \leq k \leq n \end{cases} \tag{2.3}$$

Then the *Wick-Cole-Hopf transform* Y of u defined by

$$Y := \operatorname{Exp}(\frac{\lambda}{2\nu}X) \tag{2.4}$$

solves the stochastic heat equation

$$\begin{cases} \frac{\partial Y}{\partial t} = \nu\Delta Y + \frac{\lambda}{2\nu}Y\diamond[N+C];\ t>0;\ x\in\mathbf{R}^n \\ Y(0,x) = \operatorname{Exp}(\frac{\lambda}{2\nu}X(0,x)) \end{cases} \tag{2.4}$$

for some t-continuous $(\mathcal{S})^{-1}$-valued process $C(t)$ (independent of x).

Proof. The proof in the present $(\mathcal{S})^{-1}$-case follows the proof in [HLØUZ 2] with only minor modifications. For completeness we give the argument:

Substituting (2.1) and (2.3) in (2.2) we get

$$-\frac{\partial}{\partial x_k}(\frac{\partial X}{\partial t}) + \lambda\sum_j \frac{\partial X}{\partial x_j}\diamond\frac{\partial}{\partial x_j}(\frac{\partial X}{\partial x_k}) = -\nu\sum_j\frac{\partial^2}{\partial x_j^2}(\frac{\partial X}{\partial x_k}) - \frac{\partial N}{\partial x_k} \tag{2.5}$$

or

$$\frac{\partial X}{\partial t} = \frac{\lambda}{2}\sum_j(\frac{\partial X}{\partial x_j})^{\diamond 2} + \nu\Delta X + N + C, \tag{2.6}$$

where $C = C(t)$ is a t-continuous, x-independent $(\mathcal{S})^{-1}$-process.

Basic Wick calculus rules give that

$$\frac{\partial Y}{\partial t} = \frac{\lambda}{2\nu}Y\diamond\frac{\partial X}{\partial t}, \tag{2.7}$$

$$\frac{\partial Y}{\partial x_j} = \frac{\lambda}{2\nu}Y\diamond\frac{\partial X}{\partial x_j} \tag{2.8}$$

Hence

$$\begin{aligned} \Delta Y &= \sum_j\frac{\partial}{\partial x_j}(\frac{\partial Y}{\partial x_j}) = \sum_j\frac{\partial}{\partial x_j}(\frac{\lambda}{2\nu}Y\diamond\frac{\partial X}{\partial x_j}) \\ &= \sum_j(\frac{\lambda}{2\nu})^2 Y\diamond(\frac{\partial X}{\partial x_j})^{\diamond 2} + \sum_j\frac{\lambda}{2\nu}Y\diamond\frac{\partial^2 X}{\partial x_j^2} \\ &= \frac{\lambda}{2\nu}Y\diamond[\frac{\lambda}{2\nu}\sum_j(\frac{\partial X}{\partial x_j})^{\diamond 2} + \Delta X] \end{aligned} \tag{2.9}$$

Now apply (2.7), (2.6) and (2.9) to get

$$\begin{aligned}\frac{\partial Y}{\partial t} &= \frac{\lambda}{2\nu} Y \diamond [\frac{\lambda}{2} \sum_j (\frac{\partial X}{\partial x_j})^{\diamond 2} + \nu \Delta X + N + C] \\ &= \frac{\lambda}{2\nu} Y \diamond [\frac{\lambda}{2\nu} \sum_j (\frac{\partial X}{\partial x_j})^{\diamond 2} + \Delta X]\nu + \frac{\lambda}{2\nu} Y \diamond [N + C] \\ &= \nu \Delta Y + \frac{\lambda}{2\nu} Y \diamond [N + C], \quad \text{as claimed.}\end{aligned}$$

§3. $(\mathcal{S})^{-1}$ - SOLUTION OF THE STOCHASTIC HEAT EQUATION

We now consider the stochastic heat equation (2.4) obtained by performing the Wick-Cole-Hopf transformation:

THEOREM 3.1
Suppose that $H(t,x)$ and $f(x)$ are continuous $(\mathcal{S})^{-1}$-processes. Then the stochastic heat equation

$$\begin{cases} \frac{\partial Y}{\partial t} = \nu \Delta Y + H \diamond Y \ ; \ t > 0, \ x \in \mathbf{R}^n \\ Y(0,x) = f(x) \ ; \ x \in \mathbf{R}^n \end{cases} \tag{3.1}$$

has the unique $(\mathcal{S})^{-1}$-solution

$$Y(t,x) = \hat{E}^x[f(b_{\alpha t}) \diamond \operatorname{Exp}(\int_0^t H(s, b_{\alpha s}) ds)] \tag{3.2}$$

where $\alpha = \sqrt{2\nu}$, $(b_t, \hat{P}^x)$ is standard Brownian motion in $\mathbf{R}^n$ and $\hat{E}^x$ denotes expectation with respect to $\hat{P}^x$.

Proof. Taking Hermite transforms of (3.1) we get the equation

$$\begin{cases} \frac{\partial \tilde{Y}}{\partial t} = \nu \Delta \tilde{Y} + \tilde{H} \cdot \tilde{Y} \quad ; \ t > 0, \ x \in \mathbf{R}^n, \ z \in \mathbf{B}_q(\delta) \\ \tilde{Y}(0,x) = \tilde{f}(x) \quad ; \ x \in \mathbf{R}^n, \ z \in \mathbf{B}_q(\delta) \end{cases} \tag{3.3}$$

where

$$\tilde{Y} = \tilde{Y}(t,x) = \tilde{Y}(t,x;z_1,z_2,\cdots) \quad ; \quad z \in \mathbf{B}_q(\delta)$$

denotes the Hermite transform of Y etc. and $\mathbf{B}_q(\delta)$ is some neighbourhood of 0 in $\mathbf{C}^{\mathbf{N}}$, as defined in §1. Fix $z \in \mathbf{B}_q(\delta)$. Then by the complex version of the Feynman-Kac formula the solution $\tilde{Y}(t,x,z)$ of (3.3) can be written

$$\tilde{Y}(t,x;z) = \hat{E}^x[\tilde{f}(b_{\alpha t};z) \exp[\int_0^t \tilde{H}(s, b_{\alpha s}; z) ds]] \tag{3.4}$$

with $(b_t, \hat{P}^x)$ as described above.

This is the Hermite transform of the $(\mathcal{S})^{-1}$ process

$$Y(t,x) = \hat{E}^x[f(b_{\alpha t}) \diamond \operatorname{Exp}(\int_0^t H(s, b_{\alpha s})ds]],$$

and the proof is complete.

REMARK. The equation

$$\frac{\partial u}{\partial t} = \Delta u + W \diamond u, \text{ where } W \text{ is white noise,}$$

has been studied in [NZ]. They prove the existence of a solution of a type they call *generalized Wiener functionals.*

§4. FROM THE STOCHASTIC HEAT EQUATION TO THE STOCHASTIC BURGERS EQUATION

Using the fact that an analytic function composed with a Hermite transform (of an $(\mathcal{S})^{-1}$-element) is again the Hermite transform (Corollary 1.9), we can now construct the inverse of the transformation in §2:

THEOREM 4.1 (The Wick-Cole-Hopf transformation (II))

Suppose that $Y(t,x)$ is an $(\mathcal{S})^{-1}$-process which solves the stochastic heat equation

(4.1) $$\begin{cases} \frac{\partial Y}{\partial t} = \nu\Delta Y + H \diamond Y & ; \quad t > 0, x \in \mathbf{R}^n \\ Y(0,x) = f(x) & ; \quad x \in \mathbf{R}^n \end{cases}$$

where $f(x), H(t,x)$ are given $(\mathcal{S})^{-1}$-processes, continuously differentiable w.r.t x, $H(t,x)$ continuous w.r.t. t and (see (1.29))

(4.2) $$E_\mu[f(x)] > 0 \quad \text{for all} \quad x \in \mathbf{R}^n$$

Then

(4.3) $$u(t,x) := -\frac{2\nu}{\lambda}\nabla(\operatorname{Log} Y(t,x)) \in (\mathcal{S})^{-1} \quad \text{for all} \quad t \geq 0, x \in \mathbf{R}^n,$$

where Log denotes "Wick-log" (Example 1.10d)) and the gradient is taken with respect to x (in $(\mathcal{S})^{-1}$).

Moreover, $u = (u_1, \cdots, u_n)$ solves the stochastic Burgers equation

$$\begin{cases} \frac{\partial u_k}{\partial t} + \lambda \sum\limits_{j=1}^{n} u_j \diamond \frac{\partial u_k}{\partial x_j} = \nu \Delta u_k + w_k \; ; \; t > 0, \; x \in \mathbf{R}^n \\ u_k(0,x) = g_k(x) \; ; \; x \in \mathbf{R}^n, \end{cases} \tag{4.4}$$

where

$$w_k(t,x) = -\frac{2\nu}{\lambda} \cdot \frac{\partial H}{\partial x_k}(t,x), \tag{4.5}$$

and

$$g_k(x) = -\frac{2\nu}{\lambda} f(x)^{\diamond(-1)} \diamond \frac{\partial f}{\partial x_k}(x); \quad 1 \leq k \leq n \tag{4.6}$$

Proof. By Theorem 3.1 the solution Y of (4.1) is given by

$$Y(t,x) = \hat{E}^x[f(b_{\alpha t}) \diamond \operatorname{Exp}(\int_0^t H(s, b_{\alpha s}) ds)] \quad ; \quad \alpha = \sqrt{2\nu}. \tag{4.7}$$

Therefore, by (1.30)

$$E_\mu[Y(t,x)] = \hat{E}^x[E_\mu[f(b_{\alpha t})] \cdot \exp(\int_0^t E_\mu[H(s, b_{\alpha s})] ds)] > 0$$

for all t, x.

We conclude that the Wick-log of Y, $X = \operatorname{Log} Y$, exists in $(\mathcal{S})^{-1}$, by Example 1.10d. We can therefore reverse the argument in the proof of Theorem 2.1:

Put

$$X(t,x) = \frac{2\nu}{\lambda} \operatorname{Log} Y(t,x) \tag{4.8}$$

and

$$u(t,x) = -\nabla X(t,x) \quad \text{(gradient w.r.t. } x) \tag{4.9}$$

Then

$$Y(t,x) = \operatorname{Exp}(\frac{\lambda}{2\nu} X(t,x)), \tag{4.10}$$

namely

$$u(t,x) = -\frac{2\nu}{\lambda}\nabla \mathrm{Log}\, Y(t,x), \tag{5.3}$$

where Y is given by (4.7), i.e.

$$Y(t,x) = \hat{E}^x[h(b_{\alpha t})^{\diamond \frac{\lambda}{2\nu}} \diamond \mathrm{Exp}(\frac{\lambda}{2\nu}\int_0^t N(s,b_{\alpha s})ds)], \quad \alpha = \sqrt{2\nu}. \tag{5.4}$$

b) (Uniqueness): Moreover, this process u in (5.3)-(5.4) is the only solution of (5.2) of *gradient form*, i.e. which is the gradient w.r.t. x of some continuously x-differentiable $(\mathcal{S})^{-1}$ process.

Proof.

a) Apply Theorem 4.1 to the case when

$$H(t,x) = \frac{\lambda}{2\nu}N(t,x) \quad , \quad f(x) = h(x)^{\diamond \frac{\lambda}{2\nu}} \tag{5.5}$$

b) If u solves (5.2) and

$$u(t,x) = -\nabla X(t,x)$$

then by Theorem 2.1 the process $Y = \mathrm{Exp}(\frac{\lambda}{2\nu}X)$ solves (3.1) with $H = \frac{\lambda}{2\nu}[N + C]$ for some x-independent $C(t) \in (\mathcal{S})^{-1}$, and with

$$f(x) = \mathrm{Exp}(\frac{\lambda}{2\nu}X(0,x)).$$

By Theorem 3.1 Y is unique. Then $X = \frac{2\nu}{\lambda}\mathrm{Log}\, Y$ is unique up to a constant and therefore u is unique. That completes the proof.

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