Hereditarily Effective Typestreams*

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Abstract

We prove that the hierarchy of hereditarily effective typestreams, that are effective models of inductivly defined types, has the length of the first recursivly inaccessible ordinal.

1 Introduction

In a series of papers [2, 3, 6] interpretations of types defined by dependent sums and products, strictly positive and generally positive inductive definitions are introduced.

In this paper we will consider the effective version of these hierarchies and characterise the complexity of the hierarchy of typestreams.

We will have to assume familiarity with the basic definitions in [3, 6].

The fundament of all our constructions are the domains S and D defined as the ideals of the partial preorderings $(|S|, \preceq)$ and $(|D|, \preceq)$.
S is a domain of syntactic forms with an interpretation map $I(s)$ as a subdomain of D for $s \in S$.
S contains atomic representatives B, N, C and O for the base types **Boole**, $\mathbb{N}$, Constant (singleton) and $\emptyset$, and is closed under the operations of taking the terms for dependent sums and products. More precisely, if $s_1 \in S$ and $F_s : I(s_1) \to S$ is continuous, then (Π, s_1, F_s) and (Σ, s_1, F_s) are in S.

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In the previous papers we have defined the class S_{wf} of well founded types, the class TS of typestreams and the class T_{IND} of inductivly defined types. In this paper we will consider the hereditarily effective typestreams and characterise the complexity of this class.

The concept of a typestream first appeared in the unpublished [4]. The results from [4] were presented at the EC-Twinning meeting in München in 1992. When Stan Wainer later that summer visited Oslo, we discussed the possibility of making an effective version of some of the results from [4]. Together we stated the main theorem of this paper in the setting of [4] as a very plausible conjecture. Later, the author worked out the details. The proof given in this paper is essentially a rewriting of the proof obtained in '92.

Our result is a semantical analogue to the proof-theoretical characterisations of Martin-Löf type theory with induction and one universe that were independently found by Griffor and Rathjen in [1] and by Setzer in [7]. These results show that the formal theory of the next admissible has the same logical strength as M-L-theory with induction and one universe. Our result shows that the canonical model of the theory based on effective domains will be as complex as the minimal structure that is both admissible and closed under 'the next admissible' operator.

2 Hereditarily effective typestreams

As mentioned, we let the domain S be the set of ideals in a preordering, likewise with D.

Definition 1 a) Let S^R be the set of ideals in S that are recursivly enumerable (r.e.).

b) Let D^R be the set of r.e. subsets of D.

c) For $s \in S^R$ let $I^R(s)$ be the r.e. elements of $I(s)$.

Typestreams are defined as generalisations of types defined by strictly positive induction. The idea is that we can define the total elements of nonwellfounded type-expressions as long as we know the total elements of all types used negativly, and that this is exactly what we do in a strictly positive induction. The set of typestreams is defined in stages, and we may define the hereditarily effective typestreams in the same way by restricting all quantifiers to r.e. sets.

Definition 2 By induction on the ordinal number α, we define the hereditarily effective typestreams of level α, TS^R_α as follows:
Assume that TS^R_β are defined for all $\beta < \alpha$.
Assume further that $I^R(s)_{\mathrm{TOT}}$ is defined for all $s \in TS^R_\beta$ and all $\beta < \alpha$.
Let TS^R_α be the largest subset of S^R such that if $s \in TS^R_\alpha$, then s is of one of the forms

$$[O],\ [B],\ [C],\ [N],\ (\Sigma, s_1, F_s) \text{ or } (\Pi, s_1, F_s),$$

and if s is of one of the two latter forms, then $s_1 \in TS^R_\beta$ for some $\beta < \alpha$, and for all $x \in I^R(s)_{\mathrm{TOT}}$ we have that $F_s(x) \in TS^R_\alpha$.

We define $I^R(s)_{\mathrm{TOT}}$ simultanously for all

$$s \in TS^R_\alpha \setminus \bigcup_{\beta<\alpha} TS^R_\beta$$

as the least family of sets satisfying:

$I^R([X])_{\mathrm{TOT}}$ is the canonical set when $X = O$, B, C or N.

$I^R(\Pi, s_1, F_s)_{\mathrm{TOT}}$ is the set of elements z of $I^R(\Pi, s_1, F_s)$ such that $z(x) \in I^R(F_s(x))_{\mathrm{TOT}}$ for all $x \in I^R(s_1)_{\mathrm{TOT}}$.

$I^R(\Sigma, s_1, F_s)_{\mathrm{TOT}}$ is the set of elements z of $I^R(\Sigma, s_1, F_s)$ such that $\pi_0(z) \in I^R(s_1)_{\mathrm{TOT}}$ and $\pi_1(z) \in I^R(F_s(\pi_0(z)))_{\mathrm{TOT}}$.

In [3, 5] the complexity of the noneffective version of the hierarchy of well founded types is proved to be the same as semirecursion in the functional ${}^3E$, and when we go from well founded types to typestreams, the complexity does not increase. In the effective case, the complexity of the hierarchy of well founded types, and the complexity of the hierarchy of typestreams will differ. The closure ordinal of the first hierarchy will be the first nonrecursive ordinal, while the closure ordinal of the second one will be the first recursivly inaccessible ordinal.

3 The complexity

Let ω_α be the α'th admissible ordinal, with $\omega_0 = \omega$.

Lemma 1 *If α is an ordinal, then $TS^R_\alpha \in L_{\omega_\alpha}$ and $\{I^R(s)_{\mathrm{TOT}}\}_{s \in TS^R_\alpha} \in L_{\omega_\alpha}$.*

Proof:
The proof is trivial observing that TS^R_α is simply and uniformly definable in

$$\{TS^R_\beta\}_{\beta<\alpha}, \{I^R(s)_{\mathrm{TOT}}\}_{s \in TS^R_\beta, \beta<\alpha}$$

and that the inductive definition of the total elements will close off at the next admissible.

Lemma 2 *If α is an admissible ordinal and TS^R_β and $\{I^R(s)_{\mathrm{TOT}}\}_{s \in TS^R_\beta}$ are in L_α for all $\beta < \alpha$, then*

$$TS^R_\alpha \subseteq \bigcup_{\beta<\alpha} TS^R_\beta.$$

Proof:
Let $s \in TS^R_\alpha$. The definition of s will give a Δ_1-definable set of types used negativly, and all these will be in some TS^R_β for some $\beta < \alpha$. Since α is admissible, the set of such β's will have a bound $\alpha_0 < \alpha$. Then $s \in TS^R_{\alpha_0+1}$.

This shows that the first recursivly inaccessible ordinal is an upper bound for the hierarchy.

Our main result is the converse. We will use the fact that the first recursivly inaccessible ordinal is the closure ordinal of the functional E_1 defined by:

Definition 3 **a)** If σ is a finite sequence of natural numbers, we let $\lceil \sigma \rceil$ denote its sequence number, and we let $\sigma * n$ denote the sequence obtained by adding n at the end. We let $<>$ denote the empty sequence. (Below we will represent the sequences as total elements in a certain typestream. Then we will keep this notation).

b) If $f : \mathbb{N} \to \mathbb{N}$, we let $<> \in T_f$ if $f(\lceil <> \rceil) = 0$, and recursivly we let $\sigma * n \in T_f$ if $\sigma \in T_f$ and $f(\lceil \sigma * n \rceil) = 0$

c) $E_1(f) = 0$ if T_f is well founded, 1 otherwise.

Proposition
The closure ordinal for recursion in E_1 is the first recursivly inaccessible ordinal.

This is standard generalised recursion theory.

We will use the standard notation $\{e\}(E_1, \mathbf{n}) \downarrow$ and $\{e\}(E_1, \mathbf{n}) = m$ to mean that the Kleene algorithm with index e and input E_1 and the finite number sequence $\mathbf{n}$ takes a value or takes the value m resp. We assume for the sake of simplicity that the algorithms are organised so that the functional E_1 always commes first in the list of entries. This is not in complete accordance with Kleene's definition.

Theorem 1 *Let $e, \mathbf{n}$ be given.*
Uniformly recursive in $e, \mathbf{n}$ we can find a

$$T(e, \mathbf{n}) \in S^R$$

and a continuous map

$$v(e, \mathbf{n}) : D^R \to \mathbb{N}$$

such that

$$\{e\}(E_1, \mathbf{n}) \downarrow \Leftrightarrow T(e, \mathbf{n}) \in TS^R$$

and in this case

- $I^R(T(e, \mathbf{n}))_{\mathrm{TOT}} \neq \emptyset$
- *$v(e, \mathbf{n})$ is constant $\{s\}(E_1, \mathbf{n})$ on $I^R(T(e, \mathbf{n}))_{\mathrm{TOT}}$.*

Proof:
The proof is by cases following Kleene's S1-S9. We may use the proof of Theorem 6 in [6] exept for case 8, application of E_1.
The proof is a combination of a definition that is valid thanks to the recursion theorem, and an argument by induction on the length of the computation in E_1. We combine these two steps in discussing the one open case

$$\{e\}(E_1, \mathbf{n}) = E_1(\lambda i \{e_1\}(E_1, i, \mathbf{n})).$$

For simplicity, we let $f(i) = \{e_1\}(E_1, i, \mathbf{n})$, and without loss of generality we may assume that if $f(\lceil\sigma\rceil) = 0$, and if $\tau \preceq \sigma$, then $f(\lceil\tau\rceil) = 0$.
We also let $T(\sigma) = T(e_1, \lceil\sigma\rceil, \mathbf{n})$ and $v(\sigma) = v(e_1, \lceil\sigma\rceil, \mathbf{n})$.

First we let SEQ be the typestream representation of the finite sequences of natural numbers, i.e. the solution to the equation

$$X = I([C]) \oplus (X \times \mathbb{N}).$$

We let SEQ^+ be the canonical typestream of nonempty sequences, i.e. the solution to the equation

$$X = \mathbb{N} \oplus (X \times \mathbb{N}).$$

For $\tau \in \mathrm{SEQ}^+$ we get $\tau^- \in \mathrm{SEQ}$ by removing the last entry of the sequence.

Let W be a typestream that solves the equation

$$W = I([C]) \oplus (I(\Sigma, \mathrm{SEQ}^+, \lambda\tau.T(\tau^-)) \to W)$$

with total elements W_{TOT}.
The total elements of W will be well founded trees with all leaves of the form $[c]$, and with branchings over the disjoint union of the $I^R(T(\tau^-))_{\mathrm{TOT}}$ as τ varies over SEQ^+.
If the tree T_f can be bounded by some total $w \in W$, we know that T_f is well founded.

Let $\sigma \in \mathrm{SEQ}$ and let $w \in W_{\mathrm{TOT}}$.
We define $X_{\sigma,w}$ as the typestream solution to the following set of equations:

$$X_{\sigma,w} = \Pi(x \in T(\sigma))\Pi(n \in \mathbb{N})Z_{\sigma,w,x,n}$$

$$w = left([c]) \Rightarrow Z_{\sigma,w,x,n} = I([O])$$

$$w = right(u) \wedge v(\sigma)(x) \neq 0 \Rightarrow z_{\sigma,w,x,n} = I([C])$$

$$w = right(u) \wedge v(\sigma)(x) = 0 \Rightarrow Z_{\sigma,w,x,n} = X_{\sigma * n, u(\sigma * n, x)}$$

By induction on the rank of w we see that $X_{\sigma,w}$ is a well founded type that uniformly in σ and w can be represented by an element in S^R. So we may form the typestream

$$Y = (\Sigma, W, \lambda w \in W.X_{<>,w}).$$

Claim
Y contains a total element if and only if the tree T_f is well founded.

Proof:
Let $w \in W_{\mathrm{TOT}}$ and let t be total in $X_{\sigma,w}$.
By induction on the rank of w we will show that T_f is well founded below σ.
If w is a leaf we have no total t, so there is nothing to prove.
If w is not a leaf, and if $\sigma \in T_f$ we have a total element $x \in I^R(T(\sigma))$, and for each $n \in \mathbb{N}$ we have that $t(x,n)$ is a total element in $X_{\sigma * n, u(\sigma * n, x)}$.
By the induction hypothesis for y this shows that T_f is well founded below $\sigma * n$ for all n, so T_f is well founded below σ.
If $\sigma \neq T_f$, then T_f is well founded below σ, and the proof by induction is complete.
This shows that if (w,t) is a total element of Y, then t demonstrates that T_f is well founded.

Conversly, if T_f is well founded, we let
$$w(\sigma * n, x) = \begin{cases} left([c]) & \text{if} \quad v(\sigma)(x) \neq 0 \\ right(w_{\sigma * n}) & \text{if} \quad v(\sigma)(x) = 0 \end{cases}$$
where
$$w_\sigma(\tau * m, y) = \begin{cases} w(\tau * m) & \text{if} \quad \sigma \preceq \tau \\ left([c]) & \text{otherwise} \end{cases}$$
By induction on the rank of σ in T_f we see that

$$w_\sigma \in I(\Sigma, \mathrm{SEQ}^+, \lambda\tau.T(\tau^-)) \to W$$

and that $w(\sigma * n, x) \in W_{\mathrm{TOT}}$ when $x \in I^R(T(\sigma))_{\mathrm{TOT}}$.
Thus $right(w) \in W_{\mathrm{TOT}}$.
It remains to show that $X_{<>,right(w)}$ is nonempty. Let
$$t_\sigma(x,n) = \begin{cases} [c] & \text{if} \quad v(\sigma) \neq 0 \\ t_{\sigma * n} & \text{if} \quad v(\sigma) = 0 \end{cases}$$
t_σ is recursive uniformly in σ. By induction on the rank of σ in T_f, we show that $t_\sigma \in X_{\sigma,right(w_\sigma)})_{\mathrm{TOT}}$:

i) $f(\lceil \sigma \rceil) \neq 0$.
Then $X_{\sigma,right(w_\sigma)} = \Pi(x \in T(\sigma))\Pi(n \in \mathbb{N})I([C])$ and $t_\sigma(x,n) = 0$ for $x \in I^R(T(\sigma))_{\mathrm{TOT}}$, so $t_\sigma \in X_{\sigma,right(w_\sigma)}$.

ii) $f(\lceil \sigma \rceil) = 0$.
Then $X_{\sigma,right(w_\sigma)} = \Pi(x \in T(\sigma))\Pi(n \in \mathbb{N})X_{\sigma * n, w_\sigma(\sigma * n, x)}$.

But $w_\sigma(\sigma * n, x) = w(\sigma * n, x) = right(w_{\sigma * n})$ so
$X_{\sigma,right(w_\sigma)} = \Pi(x \in T(\sigma))\Pi(n \in \mathbb{N})X_{\sigma * n,right(w_{\sigma * n})}$.
Moreover $t_\sigma(x, n) = t_{\sigma * n} \in)X_{\sigma * n,right(w_{\sigma * n})})_{\mathrm{TOT}}$ by the induction hypothesis, and it follows that $t_\sigma \in (X_{\sigma,right(w_\sigma)})_{\mathrm{TOT}}$.

Since $w = w_{<>}$, this shows that $t_{<>} \in (X_{<>,right(w)})_{\mathrm{TOT}}$ and the claim is proved.

Now let $T(f)$ be a representative in TS^R of $Y \oplus (Y \to \emptyset)$.
Let $v(f)(left(x)) = 0$ and let $v(f)(right(x)) = 1$.
Then $I^R(T(f))_{\mathrm{TOT}} \neq \emptyset$ and $v(f)$ is constant $E_1(f)$ on $I^R(T(f))_{\mathrm{TOT}}$.
This completes the proof.

References

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