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HOMOGENEOUS SUPERMANIFOLDS ASSOCIATED WITH THE COMPLEX PROJECTIVE LINE

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Abstract. One studies homogeneous complex supermanifolds whose reduction is the complex projective line $\mathbb{CP}^1$. For odd dimensions 1 and 2 a complete classification is given.

0. Introduction

In [5] S.Lie published his classification of actions of local complex Lie groups on the complex line, proving that any finite dimensional transitive Lie algebra of holomorphic vector fields defined on an open set of $\mathbb{C}$ possess a basis $\mathbf{e}, \mathbf{h}, \mathbf{f}$ or $\mathbf{e}, \mathbf{h}$, or $\mathbf{e}$, where $\mathbf{e}, \mathbf{h}, \mathbf{f}$ can be expressed, after an appropriate choice of the local coordinate z, by

$$\mathbf{e} = -\frac{d}{dz}, \mathbf{h} = 2z\frac{d}{dz}, \mathbf{f} = z^2\frac{d}{dz}.$$

It follows, in particular, that any Lie algebra of holomorphic vector fields of the type considered can actually be globalized to the Lie algebra of fundamental vector fields corresponding to the standard action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{CP}^1$, to the standard action of the affine group on $\mathbb{C}$ or to the standard action of $\mathbb{C}$ on itself.

It is quite natural to consider the similar problem for holomorphic Lie superalgebras of vector fields on the superspace $\mathbb{C}^{1|m}$, where $m > 0$. The goal of this paper is to make a first step in this direction. We study here homogeneous supermanifolds $(M, \mathcal{O})$ of dimension $1|m$ such that $M = \mathbb{CP}^1$. In the cases $m = 1, 2$, the explicit description of the Lie superalgebra of holomorphic vector fields is given and the 1-cohomology of the tangent sheaf is calculated. It turns out that in the dimension $1|2$ there exists only one non-split homogeneous supermanifold of the form $(\mathbb{CP}^1, \mathcal{O})$. This supermanifold was constructed by V.P.Palamodov as one of the first examples of non-split complex supermanifolds (see [1]). For $m \geq 3$, the classification is not yet completed. We remark that in [3] a 1-parameter family of non-split homogeneous supermanifolds of dimension $1|4$, having $\mathbb{CP}^1$ as their reduction, was constructed.

Key words and phrases. Homogeneous supermanifold, Lie superalgebra, holomorphic vector bundle.

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1. Generalities about supermanifolds

Under *vector superspace* and *superalgebras* we mean $\mathbb{Z}_2$-graded vector spaces and algebras, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \bar{0}, \bar{1}$. The word 'supermanifold' is used in the sense of Berezin-Leites, having in mind the complex-analytic version of the theory (see [6]). Thus, a *supermanifold* of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space $(M, \mathcal{O})$, where $\mathcal{O}$ is a sheaf of commutative superalgebras on a topological space M, supposing that $(M, \mathcal{O})$ is locally isomorphic to a *superdomain* in $\mathbb{C}^{n|m}$, i.e. to a ringed space of the form $(U, \mathcal{F}_{n|m})$, where U is an open subset of $\mathbb{C}^n$, $\mathcal{F}_{n|m} = \bigwedge_{\mathcal{F}_n}(\xi_1, ..., \xi_m)$ and $\mathcal{F}_n$ is the sheaf of germs of holomorphic functions in $\mathbb{C}^n$. Here we assume that the functions from $\mathcal{F}_n$ are even elements of the structure sheaf, while ξ_j are odd ones. Let $x_1, ..., x_n$ denote the standard coordinates in $\mathbb{C}^n$. If we identify an open subspace $(\tilde{U}, \mathcal{O})$ of $(M, \mathcal{O})$ with the superdomain $(U, \mathcal{F}_{n|m})$ in $\mathbb{C}^{m|n}$, then we get the elements $x_i\,(i = 1, ..., n)$, $\xi_j\,(j = 1, ..., m)$ of $\Gamma(\tilde{U}, \mathcal{O})$ called the *local coordinates* on $\tilde{U}$. In the intersection of two coordinate domains we can express the local coordinates in one domain by the local coordinates in another one; one gets the so-called *transition functions.*

Let $(M, \mathcal{O})$ be a supermanifold and $\mathcal{J} \subset \mathcal{O}$ the subsheaf of ideals generated by the subsheaf $\mathcal{O}_{\bar{1}}$ of odd elements or, which is the same, $\mathcal{J} = \mathcal{O}_{\bar{1}} + (\mathcal{O}_{\bar{1}})^2$. One denotes $\mathcal{O}_{\mathrm{rd}} = \mathcal{O}/\mathcal{J}$. Then $M_{\mathrm{rd}} = (M, \mathcal{O}_{\mathrm{rd}})$ is a usual complex analytic manifold of dimension n called the *reduction* of $(M, \mathcal{O})$, and we have a morphism $\mathrm{red}\colon M_{\mathrm{rd}} \to (M, \mathcal{O})$, taking the odd local coordinates ξ_j to 0 and the even ones x_i to certain local coordinates $X_1, ..., X_n$ on M_{rd}.

The simplest class of supermanifolds are the so-called split ones. Let $(M, \mathcal{F})$ be a complex manifold and $\mathcal{E}$ a locally free analytic sheaf on it. Defining $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{E}$, we get a supermanifold $(M, \mathcal{O})$. A supermanifold is called *split* if it is isomorphic to a supermanifold of this form. The structure sheaf $\mathcal{O}$ of a split supermanifold admits the $\mathbb{Z}$-grading $\mathcal{O} = \bigoplus_{p\geq 0} \mathcal{O}_p$, where $\mathcal{O}_p \simeq \bigwedge^p_{\mathcal{F}} \mathcal{E}$; the $\mathbb{Z}_2$-grading on it is defined from the $\mathbb{Z}$-grading by reducing mod 2. In what follows, we often omit the index $\mathcal{F}$ while denoting the exterior powers, the tensor products etc. of the sheaves of $\mathcal{F}$-modules. We recall a construction that associates with any supermanifold $(M, \mathcal{O})$ a split one. Consider the filtration

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots \tag{1}$$

by the powers of the sheaf $\mathcal{J}$ introduced above. The associated graded sheaf

$$\mathrm{gr}\,\mathcal{O} = \bigoplus_{p\geq 0} \mathrm{gr}_p\,\mathcal{O},$$

where $\mathrm{gr}_p\,\mathcal{O} = \mathcal{J}^p/\mathcal{J}^{p+1}$, gives rise to the split supermanifold $(M, \mathrm{gr}\,\mathcal{O})$. In fact, $\mathrm{gr}\,\mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{F} = \mathrm{gr}_0\,\mathcal{O} = \mathcal{O}_{\mathrm{rd}}$ and $\mathcal{E} = \mathrm{gr}_1\,\mathcal{O}$. Clearly, $(M, \mathcal{O})$ and $(M, \mathrm{gr}\,\mathcal{O})$ have the same dimension.

Let $(M, \mathcal{O})$ be a supermanifold of dimension $m|n$. In what follows, we usually denote by $\mathcal{F}$ the structure sheaf $\mathcal{O}_{\mathrm{rd}}$ of its reduction. The complex manifold $M_{\mathrm{rd}} = (M, \mathcal{F})$ will be often denoted by M.

Using local coordinates x_i, ξ_j, we see that $\mathcal{O}_x \simeq \mathbb{C}\{x_1, \dots, x_n\} \otimes \bigwedge_{\mathbb{C}}(\xi_1, \dots, \xi_m)$ for any $x \in M$ is a local superalgebra, and $\mathfrak{m}_x = (x_1, \dots, x_n, \xi_1, \dots, \xi_m)$ is its maximal ideal. The vector superspace $T_x(M,\mathcal{O}) = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called the *tangent space* to $(M,\mathcal{O})$ at the point x. The vector space $T_x(M,\mathcal{O})_{\bar 0}$ is identified with the usual tangent space $T_x(M)$ to the complex manifold $M = M_{\mathrm{rd}}$.

Denote by $\mathcal{T} = \mathcal{D}er\,\mathcal{O}$ the sheaf of derivations of the structure sheaf $\mathcal{O}$. Its stalk at $x \in M$ is the Lie superalgebra $\mathfrak{der}_{\mathbb{C}}\mathcal{O}_x = \mathfrak{der}_{\bar 0}\mathcal{O}_x \oplus \mathfrak{der}_{\bar 1}\mathcal{O}_x$ of derivations of the superalgebra $\mathcal{O}_x$ (the summands with indices $\bar 0$ and $\bar 1$ consist of even and odd derivations respectively). The sheaf $\mathcal{T}$ is called the *tangent sheaf* and its sections *holomorphic vector fields* on $(M,\mathcal{O})$. The set $\mathfrak{v}(M,\mathcal{O}) = \Gamma(M,\mathcal{O})$ of all holomorphic vector fields is finite-dimensional if M is compact. We regard it as a complex Lie superalgebra with the bracket

$$[X,Y] = YX + (-1)^{p(X)p(Y)+1}XY.$$

Fix a point $x \in M$. Any $\delta \in \mathfrak{der}\,\mathcal{O}_x$ satisfies $\delta(\mathfrak{m}_x^2) \subset \mathfrak{m}_x$ and hence defines a linear mapping $\tilde\delta : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathcal{O}_x/\mathfrak{m}_x = \mathbb{C}$ which is an element of $T_x(M,\mathcal{O})$. This permits us to define an even linear mapping

$$\mathrm{ev}_x : \mathfrak{v}(M,\mathcal{O}) \to T_x(M,\mathcal{O}) \text{ by}$$
$$\mathrm{ev}_x(v) = \tilde v_x.$$

We note that, in contrast with the classical case, a vector field v is not, in general, uniquely determined by its values $\tilde v_x$ at all $x \in M$.

We also make some remarks concerning vector fields on the split supermanifolds. If $(M,\mathcal{O})$ is split then $\mathcal{T}$ is a $\mathbb{Z}$-graded sheaf of Lie superalgebras, the grading being given by

$$\mathcal{T} = \bigoplus_{p\ge -1} \mathcal{T}_p,$$

where

$$\mathcal{T}_p = \mathcal{D}er_p\mathcal{O} = \{\delta \in \mathcal{T} \mid \delta(\mathcal{O}_q) \subset \mathcal{O}_{q+p} \text{ for all } q \in \mathbb{Z}\}. \tag{2}$$

Hence $\mathfrak{v}(M,\mathcal{O}) = \bigoplus_{p\ge -1}\mathfrak{v}_p(M,\mathcal{O})$ is a $\mathbb{Z}$-graded Lie superalgebra. We denote by ε the *grading vector field*, i.e. the element of $\mathfrak{v}(M,\mathcal{O})_0$ given by

$$\varepsilon(f) = pf \text{ for } f \in \mathcal{O}_p.$$

We have $\mathcal{O} = \bigwedge \mathcal{E}$, where $\mathcal{E}$ is a locally free analytic sheaf on M. In the odd local coordinates $\xi_1,\dots,\xi_m$, forming a basis of local sections of $\mathcal{E}$, the grading vector field is expressed by

$$\varepsilon = \sum_{j=1}^m \xi_j \frac{\partial}{\partial \xi_j}.$$

The corresponding adjoint operator acts in $\mathfrak{v}(M,\mathcal{O})$ as follows:

$$\mathrm{ad}\,\varepsilon(v) = [\varepsilon, v] = pv \text{ for } v \in \mathfrak{v}(M,\mathcal{O})_p.$$

Since $(M,\mathcal{O})$ is split, $\mathcal{T}$ can be regarded as an analytic sheaf on the complex manifold $M = (M,\mathcal{F})$. It were useful to interprete $\mathcal{T}$ directly in terms of the sheaf

$\mathcal{E}$. A partial description of $\mathcal{T}_p$, $p \geq -1$ is given by the following exact sequence of locally free analytic sheaves on M (see [7,8]):

$$0 \to \mathcal{E}^* \otimes \bigwedge^{p+1} \mathcal{E} \xrightarrow{\alpha} \mathcal{T}_p \xrightarrow{\beta} \Theta \otimes \bigwedge^{p} \mathcal{E} \to 0, \tag{3}$$

where $\Theta = \mathcal{D}er\mathcal{F}$ is the tangent sheaf of the manifold M. In particular, in the case $p = -1$ we have an isomorphism

$$\mathcal{T}_{-1} \simeq \mathcal{H}om_{\mathcal{F}}(\mathcal{E}, \mathcal{F}) = \mathcal{E}^*.$$

The mapping β is the restriction of a derivation of degree p onto the subsheaf $\mathcal{F}$, and α identifies any sheaf homomorphism $\mathcal{E} \to \bigwedge^{p+1} \mathcal{E}$ with a derivation of degree p that is zero on $\mathcal{F}$.

One sees easily that (3) splits locally (but not globally!), and hence $\mathcal{T}$ is a $\mathbb{Z}$-graded locally free analytic sheaf on M, too. Over a coordinate neighborhood with local coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_m$ we may take the vector fields

$$\xi_{j_1} \dots \xi_{j_{p+1})} \tfrac{\partial}{\partial \xi_i}, j_1 < \dots < j_{p+1}; \ \xi_{j_1} \dots \xi_{j_p} \tfrac{\partial}{\partial x_i}, j_1 < \dots < j_p$$

as basic sections of $\mathcal{T}_p$. We denote by $\mathbf{ST}(M, \mathcal{O})$ the corresponding $\mathbb{Z}$-graded holomorphic vector bundle over M (the *supertangent bundle*). Its fibre at $x \in M$ is isomorphic to $T_x(M, \mathcal{O}) \otimes \bigwedge_{\mathbb{C}}(\xi_1, \dots, \xi_m)$. The tangent bundle over M will be denoted by $\mathbf{T}(M)$.

The general case can be essentially reduced to the split one in the following way. Endow the tangent sheaf $\mathcal{T}$ with the following filtration:

$$\mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \dots \supset \mathcal{T}_{(m)} \supset \mathcal{T}_{(m+1)} = 0, \tag{4}$$

where

$$\mathcal{T}_{(p)} = \{\delta \in \mathcal{T} \mid \delta(\mathcal{O}) \subset \mathcal{J}^p, \delta(\mathcal{J}) \subset \mathcal{J}^{p+1}\} \text{ for } p \geq 0.$$

Thus we obtain a filtered sheaf of Lie superalgebras. One sees that the associated graded sheaf of Lie superalgebras $\operatorname{gr}\mathcal{T}$ is naturally isomorphic to $\tilde{\mathcal{T}} = \mathcal{D}er \operatorname{gr}\mathcal{O}$ (see [7]). Hence for any $p \geq -1$ we have the exact sequence

$$0 \to \mathcal{T}_{(p+1)} \to \mathcal{T}_{(p)} \to \tilde{\mathcal{T}}_p \to 0. \tag{5}$$

2. Actions on supermanifolds

In order to avoid using the rather complicated machinery of Lie supergroups, we shall deal with Lie superalgebras. Let $(M, \mathcal{O})$ be a supermanifold and $\mathfrak{g}$ a complex (finite dimensional) Lie superalgebra. An *action* of $\mathfrak{g}$ on $(M, \mathcal{O})$ is an arbitrary Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{v}(M, \mathcal{O})$. The action φ is called *effective* if $\operatorname{Ker}\varphi = 0$. If an action φ is given, then with any $x \in M$ the linear mapping $\varphi^x = \operatorname{ev}_x \circ \varphi : \mathfrak{g} \to T_x(M, \mathcal{O})$ is associated. The set $\mathfrak{g}_x = \operatorname{Ker}\varphi^x$ is a subalgebra of $\mathfrak{g}$, called the *stabilizer* of x. The action φ is called *transitive* if φ^x is surjective for any $x \in M$. In this case one also says that $(M, \mathcal{O})$ is a *homogeneous space* of the Lie superalgebra $\mathfrak{g}$.

With any action $\varphi : \mathfrak{g} \to \mathfrak{v}(M, \mathcal{O})$ one associates the natural homomorphism $\varphi_0 : \mathfrak{g}_{\bar{0}} \to \mathfrak{v}(M, \mathcal{O})_{\bar{0}} \to \mathfrak{v}(M)$, where $\mathfrak{v}(M)$ is the Lie algebra of holomorphic vector fields on M. If M is compact, then this homomorphism is the differential of a holomorphic action Φ_0 of the simply connected complex Lie group G_0 with tangent algebra $\mathfrak{g}$ on M. Clearly, Φ_0 is transitive in the usual sense if φ is transitive. It follows that the stabilizers of a transitive action on a compact supermanifold are conjugate by inner automorphisms of $\mathfrak{g}$ (i.e. the automorphisms from the group generated by $\exp(\operatorname{ad} u)$, for $u \in \mathfrak{g}_{\bar{0}}$).

If M is compact, then there is the natural action $\varphi = \mathrm{id}$ of the finite dimensional Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ on $(M, \mathcal{O})$. The supermanifold $(M, \mathcal{O})$ is called *homogeneous* if this action is transitive, i.e. if the mapping $\mathrm{ev}_x : \mathfrak{v}(M, \mathcal{O}) \to T_x(M, \mathcal{O})$ is surjective for any $x \in M$.

Returning to the general case, we now suppose that an action φ of a Lie superalgebra $\mathfrak{g}$ on a supermanifold $(M, \mathcal{O})$ is given. We are going to define an action on the split supermanifold $(M, \operatorname{gr} \mathcal{O})$. To do this, we note that the filtration (4) gives rise to the filtration

$$\mathfrak{g} = \mathfrak{g}_{(-1)} \supset \mathfrak{g}_{(0)} \supset \dots \supset \mathfrak{g}_{(m)} \supset \mathfrak{g}_{(m+1)} = 0, \tag{6}$$

defined by

$$\mathfrak{g}_{(p)} = \mathfrak{g} \cap \varphi^{-1}(\Gamma(M, \mathcal{T}_{(p)})) = \{u \in \mathfrak{g} \mid \varphi(u)(\mathcal{O}) \subset \mathcal{J}^p,\ \varphi(u)(\mathcal{J}) \subset \mathcal{J}^{p+1}\}.$$

Clearly, $\mathfrak{g}$ becomes a filtered Lie superalgebra, and φ determines a homomorphism $\tilde{\varphi}$ of the correspondent graded Lie superalgebra $\tilde{\mathfrak{g}}$ into the graded Lie superalgebra $\mathfrak{v}(M, \operatorname{gr} \mathcal{O})$, i.e. an action of $\tilde{\mathfrak{g}}$ on $(M, \operatorname{gr} \mathcal{O})$.

In particular, if we consider the natural action $\varphi = \mathrm{id}$ of $\mathfrak{g} = \mathfrak{v}(M, \mathcal{O})$ on a compact supermanifold $(M, \mathcal{O})$, then $\mathfrak{g}_{(p)} = \Gamma(M, \mathcal{T}_{(p)})$, and $\tilde{\varphi}$ will be the injective homomorphism $\tilde{\mathfrak{g}} \to \mathfrak{v}(M, \operatorname{gr} \mathcal{O})$ that is implied by the exact sequence (5).

Proposition 1. *For any transitive action of a Lie superalgebra $\mathfrak{g}$ on $(M, \mathcal{O})$, the corresponding action of the subalgebra $\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{g}}$ on $(M, \operatorname{gr} \mathcal{O})$ is transitive. If a compact supermanifold $(M, \mathcal{O})$ is homogeneous, then $(M, \operatorname{gr} \mathcal{O})$ is homogeneous, too.*

Proof. Fix a point $x \in M$ and denote $\tilde{\mathcal{O}} = \operatorname{gr} \mathcal{O}$. Since $(M, \mathcal{O})$ is always locally splittable, we may identify $\tilde{\mathcal{O}}|U$ with $\mathcal{O}|U$ over a neighborhood U of x and, hence, $\tilde{\mathcal{O}}_x$ with $\mathcal{O}_x$. It is sufficient to choose local coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_m$ of $(M, \mathcal{O})$ in U and to identify x_i with $x_i + \mathcal{J}$ and ξ_j with $\xi_j + \mathcal{J}^2$. As a result, we have an identification of $T_x(M, \tilde{\mathcal{O}})$ with $T_x(M, \mathcal{O})$. Let $\tilde{\mathrm{ev}}_x : \mathfrak{v}(M, \tilde{\mathcal{O}}) \to T_x(M, \tilde{\mathcal{O}})$ be the evaluating mapping for $(M, \tilde{\mathcal{O}})$.

Let φ be an action of $\mathfrak{g}$ on $(M, \mathcal{O})$. Consider the filtration (6) of $\mathfrak{g}$ determined by φ. Clearly, $\mathfrak{g}_{\bar{0}} \subset \mathfrak{g}_{(0)}$ and hence $\mathfrak{g}_{(0)} = \mathfrak{g}_{\bar{0}} \oplus (\mathfrak{g}_{(0)} \cap \mathfrak{g}_{\bar{1}})$. We have $\varphi(u)(\mathcal{O}_x) \subset \mathfrak{m}_x$ for all $u \in \mathfrak{g}_{(0)} \cap \mathfrak{g}_{\bar{1}}$, and so $\mathrm{ev}_x(\varphi(u)) = 0$ for these u. Therefore, we get a mapping $\tilde{\mathfrak{g}}_0 \to T_x(M, \mathcal{O})_{\bar{0}}$ which, as is easy to check, coincides with $\tilde{\mathrm{ev}}_x \circ \tilde{\varphi}$. It follows that $\tilde{\mathrm{ev}}_x(\tilde{\varphi}(\tilde{\mathfrak{g}}_0)) = \mathrm{ev}_x(\varphi(\mathfrak{g}_{\bar{0}}))$.

We also see from the above that $\mathrm{ev}_x \circ \varphi : \mathfrak{g}_{\bar{1}} \to T_x(M, \mathcal{O})_{\bar{1}}$ induces a mapping $\mathfrak{g}_{\bar{1}} / (\mathfrak{g}_{\bar{1}} \cap \mathfrak{g}_{(0)}) \to T_x(M, \mathcal{O})_{\bar{1}}$. Since $\mathfrak{g} = \mathfrak{g}_{\bar{1}} + \mathfrak{g}_{(0)})$, we have an isomorphism $\tilde{\mathfrak{g}}_{-1} =$

$\mathfrak{g}/\mathfrak{g}_{(0)} \to \mathfrak{g}_{\bar{1}}/(\mathfrak{g}_{\bar{1}} \cap \mathfrak{g}_{(0)})$. Hence we get a mapping $\tilde{\mathfrak{g}}_{-1} \to T_x(M, \mathcal{O})_{\bar{1}}$ which, as is easy to check, coincides with $\tilde{\text{ev}}_x \circ \tilde{\varphi}$. It follows that $\tilde{\text{ev}}_x(\tilde{\varphi}(\tilde{\mathfrak{g}}_{-1})) = \text{ev}_x(\varphi(\mathfrak{g}_{\bar{1}}))$.

Now we make some remarks concerning the action of the group G_0 on the sheaves involved. Suppose that M is compact and that an action φ of $\mathfrak{g}$ on $(M, \mathcal{O})$ is given. Let us denote by $\text{Aut}(M, \mathcal{O})$ and $\text{Bih}\, M$ the groups of automorphisms of $(M, \mathcal{O})$ and of biholomorphic transformations of M, respectively. Then, as in the classical Lie theory, it is possible to integrate the homomorphism $\varphi : \mathfrak{g}_{\bar{0}} \to \mathfrak{v}(M, \mathcal{O})_{\bar{0}}$, getting a homomorphism $\Phi : G_0 \to \text{Aut}(M, \mathcal{O})$ that induces the action $\Phi_0 : G_0 \to \text{Bih}\, M$. The corresponding action of G_0 on $\mathcal{O}$ preserves the parities and hence leaves invariant the filtration (1). As a result, we get actions of G_0 on the sheaf $\mathcal{E}$, on the supermanifold $(M, \text{gr}\, \mathcal{O})$ and on the holomorphic vector bundle $\mathbf{E}$ corresponding to $\mathcal{E}$. If Φ_0 is transitive then $\mathbf{E}$ becomes a homogeneous vector bundle.

Now, the action Φ induces an action of G_0 on $\mathcal{T}$ preserving the parities. It follows that G_0 preserves the filtration (4) and induces an action on $\text{gr}\, \mathcal{T}$ which, clearly, coincides with the action induced on the tangent sheaf $\tilde{\mathcal{T}}$ of $(M, \text{gr}\, \mathcal{O})$ if we identify this sheaf with $\tilde{\mathcal{T}}$ by the above isomorphism. One sees easily that the homomorphisms in the exact sequence (3) (corresponding to $\tilde{\mathcal{T}}$) and in (5) are G_0-equivariant.

3. The case when $M = \mathbb{CP}^1$

As usually, we cover $\mathbb{CP}^1$ by two affine charts U_0, U_1 with local coordinates X and $Y = \frac{1}{X}$ respectively. On any supermanifold of the form $(\mathbb{CP}^1, \mathcal{O})$ of dimension $1|m$ we may choose local coordinate systems $x, \xi_1, \ldots, \xi_m$ in U_0 and $y, \eta_1, \ldots, \eta_m$ in U_1 in such a way that $\text{red}(x) = X$, $\text{red}(y) = Y$.

First we consider the simplest case when the supermanifold is split. Then $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{E}$ is the sheaf of sections of a holomorphic vector bundle $\mathbf{E}$ of rank m over $M = \mathbb{CP}^1$. As is well known, we always have

$$\mathbf{E} = \bigoplus_{j=1}^{m} \mathbf{E}_j,$$

where $\mathbf{E}_j$ are holomorphic line bundle. Now, any line bundle over $\mathbb{CP}^1$ is determined by its degree $k \in \mathbb{Z}$. We denote by $\mathbf{L}(k)$ the line bundle of degree k and by $\mathcal{F}(k)$ its sheaf of holomorphic sections (usually denoted by $\mathcal{O}(k)$ in algebraic geometry). Then the transition function of $\mathbf{L}(k)$ in $U_0 \cap U_1$ has the form

$$z_0 = X^k z_1,$$

where z_i, $i = 0, 1$, is the coordinate in the fibre over U_i. In what follows we denote the degree of $\mathbf{E}_j$ by $-k_j$ and assume that the sequence $k_1, \ldots, k_m$ is not decreasing. Thus, we have

$$\begin{aligned} \mathbf{E} &= \bigoplus_{j=1}^{m} \mathbf{L}(-k_j), \\ \mathcal{E} &= \bigoplus_{j=1}^{m} \mathcal{F}(-k_j), \\ k_1 &\geq \ldots \geq k_m. \end{aligned} \tag{7}$$

For the corresponding supermanifold $(\mathbb{CP}^1, \bigwedge_{\mathcal{F}} \mathcal{E})$, we may choose as even local coordinates $x = X$, $y = Y$ on U_0, U_1 respectively and as odd ones the basic sections ξ_j over U_0 and η_j over U_1 of the locally free analytic sheaf $\mathcal{F}(-k_j)$. Then the transition functions in $U_0 \cap U_1$ will have the form

$$\begin{aligned} y &= x^{-1}, \\ \eta_j &= x^{-k_j}\xi_j,\ j = 1, \dots, m. \end{aligned} \tag{8}$$

We would like to know when this supermanifold is a homogeneous one, and to describe the Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ of all holomorphic vector fields in this case. To answer these questions, we shall first construct an action of $\mathfrak{sl}_2(\mathbb{C})$ on $(M, \mathcal{O})$.

Consider the standard action of the Lie group $\mathrm{SL}_2(\mathbb{C})$ on $M = \mathbb{CP}^1$. It is well known that its image is the whole group $\mathrm{Bih}\,\mathbb{CP}^1$. Therefore the Lie algebra $\mathfrak{v}(M)$ of holomorphic vector fields on M is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. It is also known that the following vector fields form a basis of this Lie algebra:

$$E = -\frac{\partial}{\partial X},\ H = 2X\frac{\partial}{\partial X},\ F = X^2\frac{\partial}{\partial X}. \tag{9}$$

We have

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = H.$$

It is easy to lift these vector fields to our supermanifold, i.e. to construct vector fields $\mathbf{e}$, $\mathbf{h}$, $\mathbf{f} \in \mathfrak{v}(M, \mathcal{O})$ inducing the vector fields E, H, F on M. (This means that any holomorphic vector bundle on $\mathbb{CP}^1$ is homogeneous.) We put

$$\mathbf{e} = -\frac{\partial}{\partial x},\ \mathbf{h} = 2x\frac{\partial}{\partial x} + \nabla_k,\ \mathbf{f} = x^2\frac{\partial}{\partial x} + x\nabla_k, \tag{10}$$

where

$$\nabla_k = \sum_{i=1}^m k_i\xi_i\frac{\partial}{\partial \xi_i},\ k = (k_1, \dots, k_m).$$

One checks easily that these vector fields are holomorphic on the whole $\mathbb{CP}^1$ and that

$$[\mathbf{h}, \mathbf{e}] = 2\mathbf{e},\ [\mathbf{h}, \mathbf{f}] = -2\mathbf{f},\ [\mathbf{e}, \mathbf{f}] = \mathbf{h}. \tag{11}$$

Thus, the span $\mathfrak{s} = <\mathbf{e}, \mathbf{h}, \mathbf{f}>$ is a subalgebra of $\mathfrak{v}(M, \mathcal{O})$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Under the adjoint representation of $\mathfrak{s}$, the Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ splits into the direct sum of irreducible $\mathfrak{s}$-submodules. Hence, it is generated, as the $\mathfrak{s}$-module, by the highest vectors of this representation. We are going to describe them.

First we regard $\mathbf{h}$ as a derivation of the superalgebra $\mathcal{O}(U_0)$ and prove that it is "diagonalizable" with integer eigenvalues.

Proposition 2. *The function*

$$\varphi = x^r\xi_{j_1}\dots\xi_{j_s},\ r, s \geq 0,\ j_1 < \dots < j_s, \tag{12}$$

is an eigenvector of $\mathbf{h}$ *in* $\mathcal{O}(U_0)$ *with the eigenvalue*

$$\lambda = 2r + k_{j_1} + \ldots + k_{j_s}. \tag{13}$$

Any eigenvalue λ *of* $\mathbf{h}$ *in* $\mathcal{O}(U_0)$ *has the form (13), and the correspondent eigenspace* $\mathcal{O}(U_0)_\lambda$ *is spanned by all functions (12) satisfying (13). In particular,* $\mathcal{O}(U_0)_\lambda$ *is of finite dimension.*

Proof. The first assertion is checked by a direct calculation. To prove the second one, we use the power series expansion. We can write any $\varphi \in \mathcal{O}(U_0)$ in the form

$$\varphi = \sum_{r=0}^{\infty} a_r x^r,$$

where $a_r \in A = \bigwedge(\xi_1, \ldots, \xi_m)$. Then $\mathbf{h}(\varphi) = \lambda\varphi$, $\lambda \in \mathbb{C}$, implies

$$\nabla_k a_r = (\lambda - 2r)a_r, r = 0, 1, \ldots$$

Clearly, ∇_k is diagonalizable in A with basic eigenvectors $\xi_{j_1} \ldots \xi_{j_s}$ $(j_1 < \ldots < j_s)$ and corresponding eigenvalues $k_{j_1} + \ldots + k_{j_s}$. Therefore, if $a_r \neq 0$, then $\lambda - 2r = k_{j_1} + \ldots + k_{j_s}$, and $a_r =< \xi_{j_1} \ldots \xi_{j_s} \mid 2r + k_{j_1} + \ldots + k_{j_s} = \lambda >$.

We prove now a similar proposition concerning the adjoint operator $\operatorname{ad}\mathbf{h}$ in the Lie superalgebra $\mathcal{T}(U_0)$.

Proposition 3. *The vector field*

$$v = x^r \xi_{j_1} \ldots \xi_{j_s} \frac{\partial}{\partial x}, \; r, s \geq 0, \; j_1 < \ldots < j_s, \tag{14}$$

is an eigenvector of $\operatorname{ad}\mathbf{h}$ *in* $\mathcal{T}(U_0)$ *with the eigenvalue*

$$\lambda = 2 - (2r + k_{j_1} + \ldots + k_{j_s}). \tag{15}$$

The vector field

$$w = x^r \xi_{j_1} \ldots \xi_{j_s} \frac{\partial}{\partial \xi_i}, \; r, s \geq 0, \; j_1 < \ldots < j_s, \tag{16}$$

is an eigenvector of $\operatorname{ad}\mathbf{h}$ *in* $\mathcal{T}(U_0)$ *with the eigenvalue*

$$\lambda = k_i - (2r + k_{j_1} + \ldots + k_{j_s}). \tag{17}$$

Any eigenvalue λ *of* $\operatorname{ad}\mathbf{h}$ *in* $\mathcal{T}(U_0)$ *has the form (15) or (17), and the correspondent eigenspace* $\mathcal{T}(U_0)_\lambda$ *is spanned by all vector fields (14) and (16) satisfying (15) or (17), respectively. In particular,* $\dim \mathcal{T}(U_0)_\lambda < \infty$.

Proof. For any $v = \varphi\frac{\partial}{\partial x} \in \mathcal{T}(U_0)$ we have

$$[\mathbf{h}, v](x) = 2\varphi - \mathbf{h}(\varphi); \; [\mathbf{h}, v](\xi_j) = 0, \; j = 1, \ldots, m.$$

Thus, $[\mathbf{h}, v] = \lambda v$ if and only if $\mathbf{h}(\varphi) = (2 - \lambda)\varphi$. If φ has the form (12) then, by Proposition 2, λ satisfies (15). Quite similarly one investigates the vector fields w of the form (16). The last assertion follows, as in Proposition 2.

It is important to know, which vector fields over U_0 actually lie in $\mathfrak{v}(M, \mathcal{O})$. For the eigenvectors the folllowing is true.

Proposition 4. *Let v be a vector field of the form (14) such that $k_j = 0$ for all $j \neq j_1, \dots, j_s$. Then v is holomorphic on the whole M if and only if*

$$r + k_{j_1} + \dots + k_{j_s} \leq 2. \tag{19}$$

Otherwise, the condition is that

$$r + k_{j_1} + \dots + k_{j_s} \leq 1. \tag{20}$$

A vector field w of the form (16) is holomorphic on M if and only if

$$r + k_{j_1} + \dots + k_{j_s} \leq k_i. \tag{21}$$

Proof. For a vector field v of the form (14) we see from (8) that

$$v(y) = -y^{2-(r+k_{j_1}+\dots+k_{j_s})}\eta_{j_1}\dots\eta_{j_s},$$
$$v(\eta_j) = -k_j y^{1-(r+k_{j_1}+\dots+k_{j_s})}\eta_{j_1}\dots\eta_{j_s}\eta_j.$$

This implies our assertion. One argues quite similarly for the vector fields (16).

An eigenvector v from $\mathcal{T}(U_0)$ or $\mathfrak{v}(M, \mathcal{O})$ is called a *highest vector* if it satisfies

$$[\mathbf{e}, v] = 0.$$

This condition is, clearly, expressed by

$$\frac{\partial}{\partial x} v(x) = \frac{\partial}{\partial x} v(\xi_j) = 0, \ j = 1, \dots, m.$$

The corresponding eigenvalue λ is called a *highest weight.* By the classical theory of representations, for any highest vector $v \in \mathfrak{v}(M, \mathcal{O})$ we have $\lambda \geq 0$, and the irreducible submodule generated by v has dimension $\lambda + 1$.

These remarks and Proposition 4 imply the following

Proposition 5. *A vector field of the form (14) is a highest vector of $\mathcal{T}(U_0)$ if and only if it is expressed by*

$$v = \xi_{j_1}\dots\xi_{j_s}\frac{\partial}{\partial x}, \ j_1 < \dots < j_s, \tag{22}$$

where the corresponding highest weight

$$\lambda = 2 - (k_{j_1} + \dots + k_{j_s}) \geq 0.$$

If $k_j = 0$ for all $j \neq j_1, \dots, j_s$ then $b \in \mathfrak{v}(M, \mathcal{O})$. Otherwise, v is holomorphic on M if and only if

$$k_{j_1} + \dots + k_{j_s} \leq 1.$$

Under this condition, we have $\lambda \geq 1$.

Any highest vector from $w \in \mathcal{T}(U_0)$ of the form (16) actually lies in $\mathfrak{v}(M, \mathcal{O})$ and is expressed by

$$w = \xi_{j_1}\dots\xi_{j_s}\frac{\partial}{\partial \xi_i}, \ j_1 < \dots < j_s, \tag{23}$$

where the corresponding highest weight

$$\lambda = k_i - (k_{j_1} + \dots + k_{j_s}) \geq 0.$$

Now we prove the following homogeneity criterion.

Proposition 6. *The supermanifold* $(\mathbb{CP}^1, \bigwedge_{\mathcal{F}} \mathcal{E})$*, where* $\mathbf{E}$ *is given by (7), is homogeneous if and only if* $k_i \geq 0$ *for all* $i = 1, \ldots, m$.

Proof. Since $\mathrm{SL}_2(\mathbb{C})$ acts on M transitively and this action can be lifted to $\mathcal{O}$, the mapping $\mathrm{ev}_x : \mathfrak{v}(M, \mathcal{O})_{\bar{0}} \to T_x(M, \mathcal{O})_{\bar{0}}$ is surjective for all $x \in M$.

Let us denote by o the point $X = 0$ of U_0. Consider the mapping $\mathrm{ev}_o : \mathfrak{v}(M, \mathcal{O})_{\bar{1}} \to T_o(M, \mathcal{O})_{\bar{1}}$. If it is surjective, then for any i there exists a vector field $v \in \mathfrak{v}(M, \mathcal{O})$ such that $\mathrm{ev}_o(v) = \frac{\partial}{\partial \xi_i}$. Then we may suppose that

$$v = \frac{\partial}{\partial \xi_i} \tag{24}$$

on the whole U_0. In fact, since $(M, \mathcal{O})$ is split, $\mathfrak{v}(M, \mathcal{O})$ possesses a natural $\mathbb{Z}$-grading. One sees easily that $\mathrm{ev}_o(w) = 0$ for all vector fields w of positive odd degrees. It follows that we may choose v as a vector field of degree -1, but then it is clear that (24) is true. Then, by (8), $v(\eta_i) = y_i^{k_i}$ is holomorphic on U_1 which implies that $k_i \geq 0$.

Conversely, suppose that $k_i \geq 0$ for all $i = 1, \ldots, m$. Then the vector fields $\frac{\partial}{\partial \xi_i}$ and $\frac{\partial}{\partial \eta_i}$ lie in $\mathfrak{v}(M, \mathcal{O})$ for all i. The vectors $\mathrm{ev}_x(\frac{\partial}{\partial \xi_i})$ span $T_x(M, \mathcal{O})_{\bar{1}}$ for all $x \in M$ except of the point $Y = 0$ in U_1, and the vectors $\mathrm{ev}_x(\frac{\partial}{\partial \eta_i})$ span the odd tangent vector space at this remaining point. Thus, the supermanifold is homogeneous.

4. Homogeneous supermanifolds of dimension 1|1

Here we study the homogeneous supermanifolds $(M, \mathcal{O})$, where $M = \mathbb{CP}^1$, of odd dimension 1. It is well known that all they are split. By Proposition 6, we have

$$\mathcal{O} = \bigwedge \mathcal{F}(-k), \ k \geq 0.$$

Denoting $\xi = \xi_1$, we write (10) in the form

$$\mathbf{e} = -\frac{\partial}{\partial x}, \ \mathbf{h} = 2x\frac{\partial}{\partial x} + k\xi\frac{\partial}{\partial \xi}, \ \mathbf{f} = x^2\frac{\partial}{\partial x} + kx\xi\frac{\partial}{\partial \xi}.$$

By Propositions 3 and 5, any highest vector in $\mathfrak{v}(M, \mathcal{O})$ is a linear combination of the following vector fields:

In $\mathfrak{v}(M, \mathcal{O})_{\bar{0}}$: $\mathbf{e}$ $(\lambda = 2)$; $\varepsilon = \xi\frac{\partial}{\partial \xi}$ $(\lambda = 0)$.
In $\mathfrak{v}(M, \mathcal{O})_{\bar{1}}$: $\frac{\partial}{\partial \xi}$ $(\lambda = k)$; $\xi\frac{\partial}{\partial x}$, $k = 0, 1, 2$ $(\lambda = 2 - k)$.

The Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ is $\mathbb{Z}$-graded by the degrees $-1, 0, 1$, and so

$$\begin{aligned} \mathfrak{v}(M, \mathcal{O})_{\bar{0}} &= \mathfrak{v}(M, \mathcal{O})_0, \\ \mathfrak{v}(M, \mathcal{O})_{\bar{1}} &= \mathfrak{v}(M, \mathcal{O})_{-1} \oplus \mathfrak{v}(M, \mathcal{O})_1. \end{aligned}$$

Clearly, we have for all $k \geq 0$

$$\mathfrak{v}(M, \mathcal{O})_0 = \mathfrak{s} \oplus < \varepsilon > \simeq \mathfrak{gl}_2(\mathbb{C}).$$

It is also easy to see that the odd part is as follows:

For $k = 0$:

$$\mathfrak{v}(M,\mathcal{O})_{-1} =< \frac{\partial}{\partial \xi} >,\ \mathfrak{v}(M,\mathcal{O})_1 =< \xi\frac{\partial}{\partial x}, x\xi\frac{\partial}{\partial x}, x^2\xi\frac{\partial}{\partial x} >,\ \dim \mathfrak{v}(M,\mathcal{O}) = 4|4.$$

For $k = 1$:

$$\mathfrak{v}(M,\mathcal{O})_{-1} =< \frac{\partial}{\partial \xi}, x\frac{\partial}{\partial \xi} >,\ \mathfrak{v}(M,\mathcal{O})_1 =< \xi\frac{\partial}{\partial x}, x\xi\frac{\partial}{\partial x} >,\ \dim \mathfrak{v}(M,\mathcal{O}) = 4|4.$$

For $k = 2$:

$$\mathfrak{v}(M,\mathcal{O})_{-1} =< \frac{\partial}{\partial \xi}, x\frac{\partial}{\partial \xi}, x^2\frac{\partial}{\partial \xi} >,\ \mathfrak{v}(M,\mathcal{O})_1 =< \xi\frac{\partial}{\partial x} >,\ \dim \mathfrak{v}(M,\mathcal{O}) = 4|4.$$

For $k \geq 3$:

$$\mathfrak{v}(M,\mathcal{O})_{-1} =< \frac{\partial}{\partial \xi}, x\frac{\partial}{\partial \xi}, \dots, x^k\frac{\partial}{\partial \xi} >,\ \dim \mathfrak{v}(M,\mathcal{O}) = 4|k+1.$$

Let us add some comments to the cases $k = 0,1,2$.

For $k = 0$ we have $\mathcal{O} = \mathcal{F} \otimes A$, where $A = \bigwedge_{\mathbb{C}}$, and hence

$$\mathfrak{v}(M,\mathcal{O}) = A\mathfrak{s} + \mathfrak{der}\, A.$$

For $k = 2$ we have $\mathcal{O} = \Omega$ – the sheaf of holomorphic forms on M (ξ is identified with the 1-form dX). Then $\xi\frac{\partial}{\partial x}$ is the exterior derivative, and $x^q\frac{\partial}{\partial \xi}$ is the interior multiplication by the vector field $X^q\frac{\partial}{\partial X}$, $q = 0,1,2$. The Lie superalgebra $\mathfrak{v}(M,\mathcal{O})$ is isomorphic to the previous one, after changing the $\mathbb{Z}$-grading.

For $k = 1$ we have $\mathfrak{v}(M,\mathcal{O}) \simeq \mathfrak{pgl}_{2|1}(\mathbb{C}) = \mathfrak{gl}_{2|1}(\mathbb{C})/ < 1_3 >$, and $(M,\mathcal{O})$ is the projective superline $\mathbb{CP}^{1|1}$ of $(1|0)$-subspaces in $\mathbb{C}^{2|1}$ defined in [6].

5. Split homogeneous supermanifolds of dimension $1|2$

Here we study the split homogeneous supermanifolds $(M,\mathcal{O})$, where $M = \mathbb{CP}^1$, of odd dimension 2. By Proposition 6, we have

$$\mathcal{O} = \bigwedge(\mathcal{F}(-k_1) \oplus \mathcal{F}(-k_2)),\ k_1 \geq k_2 \geq 0.$$

The Lie superalgebra $\mathfrak{v}(M,\mathcal{O})$ is $\mathbb{Z}$-graded by the degrees $-1,0,1,2$, and so

$$\mathfrak{v}(M,\mathcal{O})_{\bar 0} = \mathfrak{v}(M,\mathcal{O})_0 \oplus \mathfrak{v}(M,\mathcal{O})_2,$$
$$\mathfrak{v}(M,\mathcal{O})_{\bar 1} = \mathfrak{v}(M,\mathcal{O})_{-1} \oplus \mathfrak{v}(M,\mathcal{O})_1.$$

We shall distinct the cases $k_1 = k_2$ and $k_1 > k_2$.

Suppose that $k_1 = k_2 = k \geq 0$. If $k = 0$ then $\mathcal{O} = \mathcal{F} \otimes A$, where $A = \bigwedge_{\mathbb{C}}(\xi_1,\xi_2)$, whence

$$\mathfrak{v}(M,\mathcal{O}) = A\mathfrak{s} + \mathfrak{der}\, A.$$

It is easy to write down a basis of this Lie superalgebra.

Let now $k \geq 1$. By Propositions 3 and 5, any highest vector in $\mathfrak{v}(M, \mathcal{O})$ is a linear combination of the following vector fields:

In $\mathfrak{v}(M, \mathcal{O})_{\bar{0}}$: e $(\lambda = 2)$; $\xi_i \frac{\partial}{\partial \xi_j}$ $(\lambda = 0)$; $\xi_1\xi_2 \frac{\partial}{\partial x}$, $k = 1$, $(\lambda = 0)$.
In $\mathfrak{v}(M, \mathcal{O})_{\bar{1}}$: $\frac{\partial}{\partial \xi_j}$ $(\lambda = k)$; $\xi_j \frac{\partial}{\partial x}$, $k = 1$ $(\lambda = 1)$.

For $k = 1$, we see easily that $(M, \mathcal{O}) = \mathbb{CP}^{1|2}$ (the projective superline of $(1|0)$-subspaces in $\mathbb{C}^{2|2}$, see [6]). In this case all the highest vectors listed above are holomorphic on M and hence generate the $\mathfrak{s}$-module $\mathfrak{v}(M, \mathcal{O})$. As a result we get

$$\begin{aligned}
&\mathfrak{v}(M, \mathcal{O})_0 = \mathfrak{s} \oplus < \xi_1 \frac{\partial}{\partial \xi_2}, \xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2}, \xi_2 \frac{\partial}{\partial \xi_1} > \oplus < \varepsilon > \\
&\simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}, \\
&\mathfrak{v}(M, \mathcal{O})_2 = < \xi_1\xi_2 \frac{\partial}{\partial x} >, \\
&\mathfrak{v}(M, \mathcal{O})_{-1} = < \frac{\partial}{\partial \xi_1}, x\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, x\frac{\partial}{\partial \xi_2} >, \\
&\mathfrak{v}(M, \mathcal{O})_1 = < \xi_1 \frac{\partial}{\partial x}, x\xi_1 \frac{\partial}{\partial x}, \xi_2 \frac{\partial}{\partial x}, x\xi_2 \frac{\partial}{\partial x} >, \\
&\dim \mathfrak{v}(M, \mathcal{O}) = 8|8.
\end{aligned}$$

It follows that

$$\mathfrak{v}(M, \mathcal{O}) \simeq \mathrm{H}_4 \oplus < \varepsilon >,$$

where H_4 is a Cartan type Lie superalgebra denoted in [4] as H(4). (Notice that in [8,9] it was claimed erroneously that $\mathfrak{v}(\mathbb{CP}^{1|2}) \simeq \mathfrak{pgl}_{2|2}$.)

For $k \geq 2$ we get

$$\begin{aligned}
&\mathfrak{v}(M, \mathcal{O})_0 = \mathfrak{s} \oplus < \xi_1 \frac{\partial}{\partial \xi_2}, \xi_1 \frac{\partial}{\partial \xi_1} - \xi_2 \frac{\partial}{\partial \xi_2}, \xi_2 \frac{\partial}{\partial \xi_1} > \oplus < \varepsilon > \\
&\simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}, \\
&\mathfrak{v}(M, \mathcal{O})_2 = 0, \\
&\mathfrak{v}(M, \mathcal{O})_{-1} = < \frac{\partial}{\partial \xi_1}, x\frac{\partial}{\partial \xi_1}, \ldots, x^k \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, x\frac{\partial}{\partial \xi_2}, \ldots, x^k \frac{\partial}{\partial \xi_2} >, \\
&\mathfrak{v}(M, \mathcal{O})_1 = 0, \\
&\dim \mathfrak{v}(M, \mathcal{O}) = 7|2(k+1).
\end{aligned}$$

Suppose that $k_1 > k_2 \geq 0$. Then the highest vectors are linear combinations of the following vector fields:

In $\mathfrak{v}(M, \mathcal{O})_{\bar{0}}$: e $(\lambda = 2)$; $\xi_2 \frac{\partial}{\partial \xi_1}$ $(\lambda = k_1 - k_2)$; $\xi_j \frac{\partial}{\partial \xi_j}$ $(\lambda = 0)$.
In $\mathfrak{v}(M, \mathcal{O})_{\bar{1}}$: $\frac{\partial}{\partial \xi_j}$, $j = 1, 2$ $(\lambda = k_j)$; $\xi_j \frac{\partial}{\partial x}$, $k_j \leq 2$ $(\lambda = 2 - k_j)$; $\xi_1\xi_2 \frac{\partial}{\partial \xi_1}$, $k_2 = 0$ $(\lambda = 0)$.

Here we get

$$\begin{aligned}
\mathfrak{v}(M,\mathcal{O})_0 &= (\mathfrak{s}\oplus < \xi_1\frac{\partial}{\partial\xi_1}, \xi_2\frac{\partial}{\partial\xi_2} >) + < \xi_2\frac{\partial}{\partial\xi_1}, x\xi_2\frac{\partial}{\partial\xi_1}, \dots, x^{k_1-k_2}\xi_2\frac{\partial}{\partial\xi_1} >,\\
\mathfrak{v}(M,\mathcal{O})_2 &= 0,\\
\mathfrak{v}(M,\mathcal{O})_{-1} &= < \frac{\partial}{\partial\xi_1}, x\frac{\partial}{\partial\xi_1}, \dots, x^{k_1}\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}, x\frac{\partial}{\partial\xi_2}, \dots, x^{k_2}\frac{\partial}{\partial\xi_2} > .
\end{aligned}$$

The component $\mathfrak{v}(M,\mathcal{O})_1$ is non-zero only in the following cases:

$k_1 \geq 3,\ k_2 = 0:$

$$\begin{aligned}
\mathfrak{v}(M,\mathcal{O})_1 &= < \xi_2\frac{\partial}{\partial x}, 2x\xi_2\frac{\partial}{\partial x} - k_1\xi_1\xi_2\frac{\partial}{\partial\xi_1}, x^2\xi_2\frac{\partial}{\partial x} - k_1x\xi_1\xi_2\frac{\partial}{\partial\xi_1} >\\
&\oplus < \xi_1\xi_2\frac{\partial}{\partial\xi_1} >,
\end{aligned}$$

$k_1 = 2, k_2 = 0:$

$$\begin{aligned}
\mathfrak{v}(M,\mathcal{O})_1 &= < \xi_1\frac{\partial}{\partial x} >\\
&\oplus < \xi_2\frac{\partial}{\partial x}, x\xi_2\frac{\partial}{\partial x} - \xi_1\xi_2\frac{\partial}{\partial\xi_1}, x^2\xi_2\frac{\partial}{\partial x} - 2x\xi_1\xi_2\frac{\partial}{\partial\xi_1} >\\
&\oplus < \xi_1\xi_2\frac{\partial}{\partial\xi_1} >,
\end{aligned}$$

$k_1 = 1,\ k_2 = 0:$

$$\begin{aligned}
\mathfrak{v}(M,\mathcal{O})_1 &= < \xi_1\frac{\partial}{\partial x}, x\xi_1\frac{\partial}{\partial x} >\\
&\oplus < \xi_2\frac{\partial}{\partial x}, 2x\xi_2\frac{\partial}{\partial x} - \xi_1\xi_2\frac{\partial}{\partial\xi_1}, x^2\xi_2\frac{\partial}{\partial x} - x\xi_1\xi_2\frac{\partial}{\partial\xi_1} >\\
&\oplus < \xi_1\xi_2\frac{\partial}{\partial\xi_1} >,
\end{aligned}$$

$k_1 > k_2 = 1:$

$$\mathfrak{v}(M,\mathcal{O})_1 = < \xi_2\frac{\partial}{\partial x}, 2x\xi_2\frac{\partial}{\partial x} - k_1\xi_1\xi_2\frac{\partial}{\partial\xi_1} > .$$

We have

$$\dim\mathfrak{v}(M,\mathcal{O}) = \begin{cases} k_1 - k_2 + 5|k_1 + k_2 + 6\\ k_1 - k_2 + 5|k_1 + k_2 + 7\\ k_1 - k_2 + 5|k_1 + k_2 + 8\\ k_1 - k_2 + 5|k_1 + k_2 + 4, \end{cases}$$

respectively.

6. Non-split homogeneous supermanifolds of dimension $1|2$

Here we determine all homogeneous non-split versions of the supermanifolds studied in the previous section. It turns out that there exists only one homogeneous supermanifold of this kind.

First we will make some general remarks. Let $(M, \bigwedge \mathcal{E})$ be a split supermanifold of dimension $n|2$. In [6], Ch.4,Sec.2,Prop.9, a family of supermanifolds was constructed that contains all the supermanifolds having $(M, \bigwedge \mathcal{E})$ as the associated split supermanifold. This family is parametrized by elements $\zeta \in H^1(M, \Theta \otimes \bigwedge^2 \mathcal{E})$. The class ζ, corresponding to a supermanifold $(M, \mathcal{O})$ of the family, is the characteristic class of the extension

$$0 \to \mathcal{J}_{\bar{0}} \to \mathcal{O}_{\bar{0}} \stackrel{\text{red}}{\to} \mathcal{F} \to 0, \tag{25}$$

where $\mathcal{J}_{\bar{0}} \simeq \bigwedge^2 \mathcal{E}$ as sheaves of $\mathcal{F}$-modules.

Suppose that there is an action of a group G on $(M, \mathcal{O})$ such that the induced action on M is transitive. Then (cf. Sect. 2) we have a G-action on $(M, \operatorname{gr} \mathcal{O})$ and on the corresponding cohomology groups.

Lemma 1. *In the situation described above we have*

$$\zeta \in H^1(M, \Theta \otimes \bigwedge^2 \mathcal{E})^G.$$

Proof. To calculate the characteristic class of (25), we may proceed as follows. Choose an open set $U \subset M$ such that the sequence (25) admits a splitting $q : \mathcal{F}|U \to \mathcal{O}_{\bar{0}}|U$, $\text{red} \circ q = \text{id}$. Then $(gU)_{g \in G}$ is an open covering of M, and over any gU we have the splitting q_g of (25) given by

$$q_g(\gamma) = g(q(g^{-1}(\gamma))),\ \gamma \in \mathcal{F}|gU.$$

Then ζ is determined by the cocycle $z = (z_{g,h}),\ g, h \in G$ of our covering with values in $\Theta \oplus \bigwedge^2 \mathcal{E}$ defined by

$$z_{g,h}(\gamma) = q_h(\gamma) - q_g(\gamma),\ \gamma \in \mathcal{F}|(gU \cap hU).$$

It is sufficient to prove that z is G-invariant. But, for any $a \in G$, we have

$$\begin{aligned}(az)_{g,h}(\gamma) &= a(z_{a^{-1}g,a^{-1}h}(a^{-1}\gamma) = a(q_{a^{-1}h}(a^{-1}\gamma) - q_{a^{-1}g}(a^{-1}\gamma))\\ &= q_h(\gamma) - q_g(\gamma) = z_{g,h}(\gamma).\end{aligned}$$

We deduce from this the following

Proposition 7. *Let $(M, \mathcal{O})$, $M = \mathbb{CP}^1$, be a non-split supermanifold such that $\operatorname{gr} \mathcal{O} \simeq \bigwedge(\mathcal{O}(k_1) \oplus \mathcal{O}(k_2))$. Then $(M, \mathcal{O})$ admits an action of $\mathfrak{sl}_2(\mathbb{C})$, inducing on $(M, \operatorname{gr} \mathcal{O})$ the action defined in Sec.3, if and only if $k_1 + k_2 = 4$. In this case the supermanifold $(M, \mathcal{O})$, together with the action, is determined uniquely, up to isomorphism. Its transition functions in the covering (U_0, U_1) are given by*

$$\begin{aligned}y &= x^{-1} + x^{-3}\xi_1\xi_2,\\ \eta_j &= x^{-k_j}\xi_j,\ j = 1, 2.\end{aligned} \tag{26}$$

Proof. Since $\Theta \simeq \mathcal{O}(2)$, we have $\Theta \otimes \bigwedge^2 \mathcal{E} \simeq \mathcal{O}(2-(k_1+k_2))$. It follows that $H^1(M, \Theta \otimes \bigwedge^2 \mathcal{E}) \neq 0$ if and only if $2-(k_1+k_2) \le -2$, i.e., if $k_1+k_2 \ge 4$ (see [6], Ch.4,Sec.10). To determine the action of $G_0 = \mathrm{SL}_2(\mathbb{C})$ on this cohomology group, one may use the theorem of Bott [2]. By (9), the stabilizer $(\mathfrak{g}_0)_o$ of the point $o \in U_0$ given by $X = 0$ coincides with $\mathfrak{b} =< \mathbf{h}, \mathbf{f} >$ which is the Borel subalgebra of $\mathfrak{g}_0$ corresponding to the negative root $-\alpha$, where $\alpha(\mathbf{h}) = 2$. Let π be the fundamental weight of $\mathfrak{g}_0$, $\pi(\mathbf{h}) = 1$. Then the homogeneous line bundle $\mathbf{T}(M) \otimes \bigwedge^2 \mathbf{E}$ is determined by the representation of $\mathfrak{b}$ with highest weight $2 - (k_1+k_2)$. By the theorem of Bott, $H^1(M, \Theta \otimes \bigwedge^2 \mathcal{E})$ is an irreducible G_0-module with highest weight k_1+k_2-4. Non-zero invariants exist if and only if $k_1+k_2 = 4$, and in this case $H^1(M, \Theta \otimes \bigwedge^2 \mathcal{E}) \simeq \mathbb{C}$. Applying Lemma 1, we see that the desired action on $(M, \mathcal{O})$ exists if and only if $k_1+k_2 = 4$. Besides, $(M, \mathcal{O})$ is determined uniquely, since the supermanifolds, corresponding to the classes ζ and $c\zeta$, $c \in \mathbb{C}^\times$, are isomorphic.

Suppose that $k_1+k_2 = 4$. We want to express the cocycle z, defining $(M, \mathcal{O})$, and the transition functions of this supermanifold. The cocycle $z = (z_{01})$ can be given in the covering (U_0, U_1) by

$$z_{01} = \theta(X)\frac{\partial}{\partial X} \otimes \xi_1\xi_2 = Y^{-2}\theta(Y^{-1})\frac{\partial}{\partial Y} \otimes \eta_1\eta_2,$$

where $\theta \in \mathcal{F}(U_0 \cap U_1)$. If θ is holomorphic in U_0 or $Y^{-2}\theta(Y^{-1})$ is holomorphic in U_1, then z is a cobord. If it is not the case, then, expanding θ into the Laurent series, we see that $\theta(X) = cX^{-1}$, where $c \in \mathbb{C}^\times$. Thus we may write

$$z_{01} = X^{-1}\frac{\partial}{\partial X} \otimes \xi_1\xi_2 = Y^{-1}\frac{\partial}{\partial Y} \otimes \eta_1\eta_2.$$

Consider now the sheaf $\mathcal{O}$, corresponding to this cocycle. Let $q_i : \mathcal{F}|U_i \to \mathcal{O}_{\bar{0}}|U_i$, $i = 0, 1$, be the local splittings of (25). Then we may take $x = q_0(X)$, $y = q_1(Y)$ as even coordinates in U_0, U_1, respectively. Then

$$z_{01}(X) = q_0(X) - q_1(X) = x - y^{-1}.$$

Identifying $\mathcal{J}_{\bar{0}}$ with $\bigwedge^2 \mathcal{E}$, we get

$$x - y^{-1} = x^{-1}\xi_1\xi_2,$$

whence

$$y^{-1} = x - x^{-1}\xi_1\xi_2$$

or

$$y = x^{-1} + x^{-3}\xi_1\xi_2.$$

It follows that the transition functions have the form (26).

Now we have to construct an action of $\mathfrak{sl}_2(\mathbb{C})$ on $(M, \mathcal{O})$, inducing on $(M, \operatorname{gr}\mathcal{O})$ the action defined in Sec.3. This can be done in a unique way. In fact, if $\mathbf{e}, \mathbf{h}, \mathbf{f}$

denote the fundamental vector fields of the action to be constructed, then, clearly, $\mathbf{e}(\xi_j), \mathbf{h}(\xi_j), \mathbf{f}(\xi_j)$ must have the same form as in (10). In addition,

$$\mathbf{e}(x) = -1 + \alpha\xi_1\xi_2,\ \mathbf{h}(x) = \beta\xi_1\xi_2,\ \mathbf{f}(x) = \gamma\xi_1\xi_2,$$

where α, β, γ depend on x only. Using (26), we get

$$\mathbf{e}(y) = x^{-2} + 3x^{-4}\xi_1\xi_2 - x^{-2}\alpha\xi_1\xi_2.$$

One easily sees that

$$x^{-1} = y - y^{-1}\eta_1\eta_2,$$

whence

$$x^{-2} = y^2 - 2\eta_1\eta_2.$$

Therefore

$$\mathbf{e}(y) = y^2 + \eta_1\eta_2 - \alpha y^{-2}\eta_1\eta_2.$$

This shows that $\mathbf{e}(y) \in \mathcal{O}(U_1)$ if and only if $\alpha = 0$. One verifies quite similarly that $\mathbf{h}(y),\ \mathbf{f}(y) \in \mathcal{O}(U_1)$ if and only if $\beta = 0, \gamma = 1$, respectively. Thus, $\mathbf{e}, \mathbf{h}, \mathbf{f}$ necessarily have the following expression:

$$\mathbf{e} = -\frac{\partial}{\partial x},\ \mathbf{h} = 2x\frac{\partial}{\partial x} + \nabla_k,\ \mathbf{f} = (x^2 + \xi_1\xi_2)\frac{\partial}{\partial x} + x\nabla_k. \tag{27}$$

One easily checks that (11) is satisfied.

Now we are able to characterize the homogeneous supermanifolds of the class in consideration.

Theorem 1. *The only non-split homogeneous supermanifold $(\mathbb{CP}^1, \mathcal{O})$ of dimension $1|2$ is the superquadric of isotropic $(1|0)$-subspaces in the vector space $\mathbb{C}^{3|2}$ endowed with a non-degenerate even symmetric bilinear form. The Lie superalgebra $\mathfrak{v}(M, \mathcal{O})$ has dimension $6|6$ and is isomorphic to $\mathfrak{osp}_{3|2}(\mathbb{C})$; its even and odd parts are spanned by the following vector fields:*

$$\begin{aligned}
\mathfrak{v}(M,\mathcal{O})_{\bar 0} &= < \mathbf{e}, \mathbf{h}, \mathbf{f} > \oplus < \xi_1\frac{\partial}{\partial\xi_2}, \xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}, \xi_2\frac{\partial}{\partial\xi_1} > \\
&\simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}),\\
\mathfrak{v}(M,\mathcal{O})_{\bar 1} &= < \frac{\partial}{\partial\xi_1}, \xi_2\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial\xi_1}, x\xi_2\frac{\partial}{\partial x} + (x^2 - \xi_1\xi_2)\frac{\partial}{\partial\xi_1},\\
&\frac{\partial}{\partial\xi_2}, -\xi_1\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial\xi_2}, -x\xi_1\frac{\partial}{\partial x} + (x^2 - \xi_1\xi_2)\frac{\partial}{\partial\xi_2} > .
\end{aligned}$$

Proof. We have the following three possibilities:

$$(k_1, k_2) = (4,0),\ (3,1),\ (2,2).$$

It is easy to check that the mapping $\mathrm{ev}_x : \mathfrak{v}(M,\mathcal{O})_{\bar 0} \to T_x(M,\mathcal{O})_{\bar 0}$ is always surjective for any $x \in M$. Consider now the basic highest vectors in $\mathcal{T}(U_0)_{\bar 1}$. By Proposition 5, these are

$$\frac{\partial}{\partial \xi_1}\,(\lambda = k_1);\ \frac{\partial}{\partial \xi_2}\,(\lambda = k_2);\ x\frac{\partial}{\partial \xi_1}\,(\lambda = k_1 - 2);\ x\frac{\partial}{\partial \xi_2},\ k_1 = k_2 = 2\,(\lambda = 0).$$

Using (26), we get

$$\begin{aligned} &\frac{\partial}{\partial \xi_1}(y) = x^{-3}\xi_2,\ \frac{\partial}{\partial \xi_1}(\eta_1) = x^{-k_1},\ \frac{\partial}{\partial \xi_1}(\eta_2) = 0,\\ &\frac{\partial}{\partial \xi_2}(y) = -x^{-3}\xi_1,\ \frac{\partial}{\partial \xi_2}(\eta_1) = 0,\ \frac{\partial}{\partial \xi_2}(\eta_2) = x^{-k_2}. \end{aligned} \tag{28}$$

If $k_1 = 4$, then $\frac{\partial}{\partial \xi_2}(y) = -y^{-1}\eta_1$, and so $\frac{\partial}{\partial \xi_2}$ is not holomorphic. Since there are no other odd highest vectors with $\lambda = 4$, this implies that $\mathfrak{v}(M,\mathcal{O})$ acts non-transitively.

If $k_1 = 3$, then $\frac{\partial}{\partial \xi_2}(\eta_2) = y - y^{-1}\eta_1\eta_2$, and so $\mathfrak{v}(M,\mathcal{O})$ act non-transitively again.

In the case $k_1 = k_2 = 2$, we have

$$\frac{\partial}{\partial \xi_1} = y\eta_2\frac{\partial}{\partial y} + \frac{\partial}{\partial \eta_1},\ \frac{\partial}{\partial \xi_2} = -y\eta_1\frac{\partial}{\partial y} + \frac{\partial}{\partial \eta_2}.$$

This shows that $\mathrm{ev}_x : \mathfrak{v}(M,\mathcal{O})_{\bar 1} \to T_x(M,\mathcal{O})_{\bar 1}$ is surjective for any $x \in U_0$. To show this for $x \in U_1$, one should consider the vector fields $\frac{\partial}{\partial \eta_j}$, $j = 1,2$, that are holomorphic, too. Thus $(M,\mathcal{O})$ is homogeneous in this case.

Suppose now that $k_1 = k_2 = 2$. As the theory of representations shows, the highest vectors $\frac{\partial}{\partial \xi_i}$, $i = 1,2$, generate two irreducible submodules of odd vector fields $< \frac{\partial}{\partial \xi_i}, [\mathbf{f}, \frac{\partial}{\partial \xi_i}], [\mathbf{f}, [\mathbf{f}, \frac{\partial}{\partial \xi_i}]] >$, where $\mathbf{f}$ is given by (27). On the other hand, we see from (28) that $x\frac{\partial}{\partial \xi_1}(\eta_1) = x\frac{\partial}{\partial \xi_2}(\eta_2)]x^{-1} = y - y^{-1}\eta_1\eta_2$ is not holomorphic on U_1. This implies that $\mathfrak{v}(M,\mathcal{O})_{\bar 1}$ has the form indicated in the statement of the theorem. The Lie algebra $\mathfrak{v}(M,\mathcal{O})_{\bar 0}$ is calculated as in Sec. 4, taking into account the fact that $\sum_{j=1}^{2} \xi_j \frac{\partial}{\partial \xi_j}$ is not holomorphic on U_1.

It is not difficult to prove that the non-split homogeneous supermanifold $(M,\mathcal{O})$ constructed above can be described as the super-Grassmannian of isotropic $(1|0)$-subspaces in the vector space $\mathbb{C}^{3|2}$ endowed with a non-degenerate even symmetric bilinear form (see [6]) or, equivalently, as the submanifold of $\mathbb{CP}^{2|2}$ given in homogeneous coordinates $z_0, z_1, z_2, \zeta_1, \zeta_2$ by the equation

$$z_0^2 + z_1 z_2 + \zeta_1\zeta_2 = 0.$$

7. The supertangent bundle and the 1-cohomology

In previous sections, we studied vector fields on split supermanifolds, using the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. Another way is the direct study of the supertangent bundle $\mathbf{ST}(M,\mathcal{O})$. In this section, we determine this vector bundle explicitly for $m = 1,2$ and, as an application, we calculate the group $H^1(M,\mathcal{T})$ for homogeneous supermanifolds studied in Sections 4,5,6.

Proposition 8. . *Suppose that $\mathcal{O} = \mathcal{F}(-k)$, $k \in \mathbb{Z}$. Then*

$$\begin{aligned}
\mathbf{ST}(M,\mathcal{O})_{-1} &\simeq \mathbf{L}_k, \\
\mathbf{ST}(M,\mathcal{O})_0 &\simeq \begin{cases} \mathbf{L}_0 \oplus \mathbf{L}_2 \textit{if } k = 0 \\ 2\mathbf{L}_1 \textit{if } k \neq 0, \end{cases} \\
\mathbf{ST}(M,\mathcal{O})_1 &\simeq \mathbf{L}_{2-k}.
\end{aligned}$$

Proof. Clearly, in $U_0 \cap U_1$ we have

$$\begin{aligned}
&\frac{\partial}{\partial \eta} = x^k \frac{\partial}{\partial \xi}, \\
&\eta \frac{\partial}{\partial \eta} = \xi \frac{\partial}{\partial \xi}, \ \frac{\partial}{\partial y} = -x^2 \frac{\partial}{\partial x} - kx\xi \frac{\partial}{\partial \xi}, \\
&\eta \frac{\partial}{\partial y} = -x^{2-k} \xi \frac{\partial}{\partial \xi}.
\end{aligned}$$

This obviously implies our assertion in degrees -1 and 1. The transition functions of $\mathbf{ST}(M,\mathcal{O})_0$ are given by the matrix

$$\begin{pmatrix} 1 & -kx \\ 0 & -x^2 \end{pmatrix}.$$

If $k = 0$, then $\mathbf{ST}(M,\mathcal{O})_0 \simeq \mathbf{L}_0 \oplus \mathbf{L}_2$. If $k \neq 0$, then we can write

$$\begin{pmatrix} \frac{x}{k} & -1 \\ -\frac{1}{k} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -kx \\ 0 & -x^2 \end{pmatrix} \cdot \begin{pmatrix} k & 0 \\ \frac{1}{x} & 1 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

whence $\mathbf{ST}(M,\mathcal{O})_0 \simeq 2\mathbf{L}_1$.

Corollary. *Suppose that $k \geq 0$. Then*

$$\begin{aligned}
H^1(M,\mathcal{T}_p) &= 0 \textit{for } p = -1, 0, \\
H^1(M,\mathcal{T}_1) &= \mathbb{C}^{k-3}.
\end{aligned}$$

We mean here that a vector space of negative dimension is 0.

Proposition 9. *Suppose that $\mathcal{O} = \mathcal{F}(-k_1) \oplus \mathcal{F}(-k_2)$, where $k_1, k_2 \in \mathbb{Z}$. Then*

$$\begin{aligned}
\mathbf{ST}(M,\mathcal{O})_{-1} &\simeq \mathbf{L}_{k_1} \oplus \mathbf{L}_{k_2}, \\
\mathbf{ST}(M,\mathcal{O})_0 &\simeq \begin{cases} \mathbf{L}_0 \oplus 2\mathbf{L}_1 \oplus \mathbf{L}_{k_1-k_2} \oplus \mathbf{L}_{k_2-k_1} \textit{if } k_1 \neq 0 \textit{ or } k_2 \neq 0 \\ 3\mathbf{L}_0 \oplus \mathbf{L}_2 \textit{if } k_1 = k_2 = 0, \end{cases} \\
\mathbf{ST}(M,\mathcal{O})_1 &= \simeq \begin{cases} 2\mathbf{L}_{1-k_1} \oplus 2\mathbf{L}_{1-k_2} \textit{if } k_1, k_2 \neq 0 \\ \mathbf{L}_{-k_1} \oplus \mathbf{L}_{2-k_1} \oplus 2\mathbf{L}_1 \textit{if } k_1 \neq 0, \ k_2 = 0 \\ \mathbf{L}_{-k_2} \oplus \mathbf{L}_{2-k_2} \oplus 2\mathbf{L}_1 \textit{if } k_1 = 0, \ k_2 \neq 0 \\ 2\mathbf{L}_0 \oplus 2\mathbf{L}_2 \textit{if } k_1 = k_2 = 0. \end{cases} \\
\mathbf{ST}(M,\mathcal{O})_2 &\simeq \mathbf{L}_{2-(k_1+k_2)}.
\end{aligned}$$

Proof. Clearly, we have

$$\frac{\partial}{\partial \eta_i} = x^{k_i}\frac{\partial}{\partial \xi_i},\ i=1,2,$$

$$\eta_i\frac{\partial}{\partial \eta_i} = \xi_i\frac{\partial}{\partial \xi_i},\ i=1,2,\ \frac{\partial}{\partial y} = -x^2\frac{\partial}{\partial x} - x\sum_{j=1}^{2} k_j\xi_j\frac{\partial}{\partial \xi_j},$$

$$\eta_1\eta_2\frac{\partial}{\partial \eta_1} = x^{-k_2}\xi_1\xi_2\frac{\partial}{\partial \xi_1},\ \eta_1\eta_2\frac{\partial}{\partial \eta_2} = x^{-k_1}\xi_1\xi_2\frac{\partial}{\partial \xi_2},$$

$$\eta_1\frac{\partial}{\partial y} = -x^{2-k_1}\xi_1\frac{\partial}{\partial x} - k_2x^{1-k_1}\xi_1\xi_2\frac{\partial}{\partial \xi_2},$$

$$\eta_2\frac{\partial}{\partial y} = -x^{2-k_2}\xi_2\frac{\partial}{\partial x} + k_1x^{1-k_2}\xi_1\xi_2\frac{\partial}{\partial \xi_1},$$

$$\eta_1\eta_2\frac{\partial}{\partial y} = -x^{2-(k_1+k_2)}\xi_1\xi_2\frac{\partial}{\partial x}.$$

In degrees -1,2 this proves our assertion.

The transition functions of $\mathbf{ST}(M,\mathcal{O})_0$ are given by the matrix

$$\begin{pmatrix} 1 & 0 & -k_1x \\ 0 & 1 & -k_2x \\ 0 & 0 & -x^2 \end{pmatrix} \oplus \begin{pmatrix} x^{k_1-k_2} & 0 \\ 0 & x^{k_2-k_1} \end{pmatrix}.$$

If, e.g., $k_2 \neq 0$, then, as in the proof of Proposition 8, the first summand can be replaced by

$$\begin{pmatrix} 1 & 0 & -k_1x \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 & -k_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix}.$$

This proves our assertion for $\mathbf{ST}(M,\mathcal{O})_0$.

The transition functions of $\mathbf{ST}(M,\mathcal{O})_1$ are given by the matrix

$$\begin{pmatrix} x^{-k_1} & -k_2x^{1-k_1} \\ 0 & -x^{2-k_1} \end{pmatrix} \oplus \begin{pmatrix} x^{-k_2} & k_1x^{1-k_2} \\ 0 & -x^{2-k_2} \end{pmatrix}.$$

If $k_2 \neq 0$, then, by the proof of Proposition 8, we may replace the first summand by

$$x^{-k_1}\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} x^{1-k_1} & 0 \\ 0 & x^{1-k_1} \end{pmatrix}.$$

If $k_1 \neq 0$, then the same operation can be made with the second summand. This implies our assertion.

Corollary. *Suppose that* $k_1 \geq k_2 \geq 0$. *Then*

$$H^1(M,\mathcal{T}_{-1}) = 0,$$

$$H^1(M,\mathcal{T}_0) \simeq \mathbb{C}^{k_1-k_2-1},$$

$$H^1(M,\mathcal{T}_1) \simeq \begin{cases} \mathbb{C}^{k_1-1}\oplus\mathbb{C}^{k_1-3} \textit{if } k_2=0 \\ 2\mathbb{C}^{k_1-2}\oplus 2\mathbb{C}^{k_1-3} \textit{if } k_2=0, \end{cases}$$

$$H^1(M,\mathcal{T}_2) \simeq \mathbb{C}^{k_1+k_2-3}.$$

Now we determine the tangent sheaf 1-cohomology of the non-split homogeneous supermanifold described in the previous section.

Theorem 2. *Let* $(\mathbb{CP}^1, \mathcal{O})$ *be the non-split homogeneous supermanifold of dimension* $1|2$. *Then* $H^1(\mathbb{CP}^1, \mathcal{T}) = 0$, *and hence the supermanifold is rigid.*

Proof. We use the cohomology exact sequences associated with (5) for $p = 2, 1, 0, -1$, the cohomology of $\tilde{\mathbf{T}}_p$ being determined above. By Corollary of Proposition 9, we have $H^1(M, \mathcal{T}_{(2)}) = H^1(M, \mathcal{T}_2) \simeq \mathbb{C}^{1|0}$. Since $H^0(M, \tilde{\mathcal{T}}_1) = 0$ by Sect.5 and $H^1(M, \tilde{\mathcal{T}}_1) = 0$ by Corollary of Proposition 9, we obtain, using (5) for $p = 1$, that $H^1(M, \mathcal{T}_{(1)}) \simeq \mathbb{C}^{1|0}$. Consider now the exact sequence, corresponding to $p = 0$:

$$H^0(M, \mathcal{T}_{(0)}) \xrightarrow{p} H^0(M, \tilde{\mathcal{T}}_0) \xrightarrow{\delta} H^1(M, \mathcal{T}_{(1)}) \to H^1(M, \mathcal{T}_{(0)}) \to H^1(M, \tilde{\mathcal{T}}_0) \to 0.$$

We see from the proof of Theorem 1 that $< \varepsilon >$ does not lie in $\operatorname{Im} p$. Therefore δ maps $< \varepsilon >$ onto $H^1(M, \mathcal{T}_{(1)})$. Since $H^1(M, \tilde{\mathcal{T}}_0) = 0$ by Corollary of Proposition 9, we see that $H^1(M, \mathcal{T}_{(0)}) = 0$. Further, $H^1(M, \tilde{\mathcal{T}}_{-1}) = 0$ by Corollary of Proposition 9, and the exact sequence, corresponding to $p = -1$, shows that $H^1(M, \mathcal{T}) = 0$.

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