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Surfaces of Degree 10 in the Projective Fourspace via Linear systems and Linkage

by

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Contents

0 Introduction

Ellingsrud and Peskine showed in [ElP] that there are finitely many components of the Hilbert scheme of $\mathbf{P}^4$ containing smooth surfaces not of general type. The upper bound for the degree of such surfaces has recently been reduced to 105 (cf. [BF]). Quite a bit of work has been put into trying to construct such surfaces of high degree. So far the record is 15, which coincides with the conjectural upper bound. This paper concerns the classification of surfaces of degree 10 and sectional genus 9 and 10. The surfaces of degree at most 9 are described through classical work dating from the last century up to recent years: [Ba], [Ro], [Io], [Ok], [Al], [AR]. Surfaces of degree at least 11 have been considered systematically recently in [Po]. In degree 10 there are the abelian surfaces discovered by Comessatti [Co], rediscovered by Horrocks and Mumford [HM] in the seventies as zero-sections of an indecomposable rank 2 vector bundle on $\mathbf{P}^4$. Beside surfaces of sectional genus at least 11, which can be linked to smooth surfaces of smaller degree, other surfaces of degree 10 were considered only recently. Serrano gave examples of bielliptic surfaces of degree 10 in [S], which have sectional genus 6 like the abelian ones. The Hilbert schemes of abelian and bielliptic surfaces are now well understood (cf. [BHM], [BaM], [H], [HKW], [HL], [HV], [L], [R], [ADHPR]). The second author determined numerical invariants and gave some examples of surfaces with sectional genus 8,9 and 10. There are two families of surfaces of genus 8, one of rational surfaces, and one of non-minimal Enriques surfaces. A construction using syzygies of both of them is given in [DES]. The linear system of surfaces of the first family is described in [Ra], cf. also [Al], while for the second it is described in [Br]. The purpose of this paper is to describe the remaining components of the Hilbert scheme, i.e., to describe the smooth surfaces in $\mathbf{P}^4$ of degree 10 and sectional genus 9 and 10. The results in themselves are to be considered interesting from the perspective of the diversity of techniques with which we present them. Thus we use relations between multisecants, linear systems, syzygies and linkage to describe the geometry of each surface. We want in fact to stress the importance of multisecants and syzygies for the study of these surfaces. Adjunction which provided efficient arguments for the classification of surfaces of smaller degrees, here appears to be less effective and will play almost no role in the proofs.

We show that there are 8 different families of smooth surfaces of degree 10 and sectional genus 9 and 10. The families are determined by numerical data such as the sectional genus π, the Euler characteristic $\chi = \chi(\mathcal{O}_S)$, the number N_6 of 6-secants to the surface S and the number N_5 of 5-secants to the surface which meet a general plane. For each type we describe the linear system of hyperplane sections on S, the resolution of the ideal, the geometry of the surface in terms of curves on the surface and hypersurfaces containing the surface, and the liaison class; in particular minimal elements in the even liaison class. Each type corresponds to an irreducible component of the Hilbert scheme, and the dimension is computed. The following table collects the numerical data and give references to the points in this paper where the corresponding additional information can be found.

S	π	χ	N_6	N_5	Cohomology	Birational type	Linear system	construc– tion	linkage	Hilbert scheme
A	9	1	1	6	(3.4)	rational	(2.1)	(4.5)	(4.3)	42
B	9	1	0	6	(3.3)	rational	(2.1)	(2.10)	(4.2)	42
C	9	2	3	12	(3.7)	K3	(2.12)	(2.14)	(4.7)	45
D	9	2	1	12	(3.6)	K3	(2.19)	(4.10)	(4.9)	44
E	9	2	0	12	(3.5)	elliptic	(2.23)	(2.31)	(4.12)	44
F	9	3	3	18	(3.9)	gen.type	(2.32)	(2.33)	(4.13)	47
G	10	3	0	2	(3.11)	elliptic	(2.34)	(4.15)	(4.14)	51
H	10	4	1	6	(3.12)	gen.type	(2.34)	(4.20)	(4.18)	53

The organization of the paper follows the different approaches to a description of the surfaces rather than each surface one by one. Thus we focus on the different methods used. In the first section we use 6-secants and plane curves to get information on the Hartshorne-Rao module of the surface. In the second section we use a geometric approach to describe linear systems and special curves on S and special hypersurfaces containing S. In the third section the constructions via the Eagon-Northcott complex performed in [DES] is recovered together with the resolution of the ideal of S. In the fourth section we use both the resolution of the ideal and the geometry of the surface to describe the minimal element in the even liaison class of S.

Notation and basic results:

Adjunction formula 0.1. $2p_a(C) - 2 = C^2 + C \cdot K$.

Proof. See [Ha, Prop. 1.5].□

For curves C, D and $C \cup D$ on a smooth surface S the adjunction immediately gives the following addition formula for the arithmetic genus:

$$p_a(C \cup D) = p_a(C) + p_a(D) + C \cdot D - 1. \tag{0.2.}$$

This formula quickly yields the following

Lemma 0.3. *Let C be a non-planar curve of degree d and arithmetic genus p on a smooth surface. If $d = 5$, then $p \leq 3$ with equality only if C decomposes into a plane quartic and a line meeting in a point. When $d = 6$, then $5 \leq p \leq 6$ only if C decomposes into a plane quintic curve and a line, while $3 \leq p \leq 4$ implies that C spans a $\mathbf{P}^3$ unless $p = 3$ and C decomposes into a plane quartic curve and a conic, or two lines, meeting the quartic in a point.*

Proof. Straightforward from (0.2) and the genus bound for irreducible curves.□

Theorem (Riemann-Roch) 0.4.

$$\chi(\mathcal{O}_S(C)) = h^0(\mathcal{O}_S(C)) - h^1(\mathcal{O}_S(C)) + h^0(\mathcal{O}_S(K - C)) = \frac{1}{2}(C^2 - C \cdot K) + \chi(S).$$

Proof. See [Ha, Th.1.6].□

Hodge index theorem 0.5. *If H is an ample divisor and D is a divisor on S such that $H \cdot D = 0$, then either $D^2 < 0$, or D is numerically equivalent to 0.*

Proof. See [Ha, Th.1.9].□

For smooth surfaces in $\mathbf{P}^4$ with normal bundle N_S there is the relation,

$$d^2 - c_2(N_S) = d^2 - 10d - 5H \cdot K - 2K^2 + 12\chi(S) = 0, \tag{0.6.}$$

which expresses the fact that S has no double points. This will be referred to as the double point formula.

Theorem (Severi) 0.7. *All smooth surfaces in $\mathbf{P}^4$, except for the Veronese surfaces, are linearly normal.*

Proof. [Se] or [Mo].□

Some classical numerical formulae for multisecant lines to a smooth surface in $\mathbf{P}^4$ have recently been studied again by Le Barz:

Multisecants 0.8. ([LB]) If $\sharp_5$ denotes the number of 5-secant lines to S which meets a general plane, and $\sharp_6$ denotes the sum of the number of 6-secants to S and the number of (-1)-lines on S, then the formulas of Le Barz yield the following numbers for a surface of degree 10.

$\pi = 9$:

	$\chi = 1$	$\chi = 2$	$\chi = 3$
$\sharp_5$	6	12	18
$\sharp_6$	7	3	3

$\pi = 10$:

	$\chi = 3$	$\chi = 4$
$\sharp_5$	2	6
$\sharp_6$	2	1

Linkage 0.9. ([PS]) Two surfaces S and S' are said to be linked (m, n) if there exist hypersurfaces V and V' of degree n and m respectively such that $V \cap V' = S \cup S'$. There are the standard sequences of linkage, namely

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_{S\cup S'}(m+n-5) \longrightarrow \mathcal{O}_{S'}(m+n-5) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_S(m+n-5) \longrightarrow \mathcal{O}_{S\cap S'}(m+n-5) \longrightarrow 0$$

The first sequence yields the relation between the Euler-Poincaré characteristics

$$\chi(S') = \chi(V \cap V') - \chi(\mathcal{O}_S(m+n-5)). \tag{0.10}$$

The corresponding sequence for linkage of curves in $\mathbf{P}^3$ yields the following relation between the sectional genera.

$$\pi(S) - \pi(S') = \frac{1}{2}(m+n-4)(d(S) - d(S')). \tag{0.11}$$

To determine the surfaces to which our surfaces are linked with, we will use:

Proposition 0.12. *If S and T are linked, then S is locally Cohen-Macaulay if and only if T is locally Cohen-Macaulay.*

Proof. See [PS, Proposition 1.3].

For a proof of existence via linkage, the following proposition will be used.

Proposition 0.13. *If T is a local complete intersection surface in $\mathbf{P}^4$, which scheme-theoretically is cut out by hypersurfaces of degree d, then T is linked to a smooth surface S in the complete intersection of two hypersurfaces of degree d.*

Proof. See [PS, Proposition 4.1].

Remark (Peskine, private communication). A slight modification of the conditions of this proposition is allowable, without changing the conclusion. Namely, at a finite set of points T need not be a local complete intersection. It suffices that it is locally Cohen-Macaulay, and that the tangent cone at that point is linked to a plane in a complete intersection.

The proof is an application of the proof of (0.13) to the strict transform of T in the blow-up of $\mathbf{P}^4$ in the points where T is not a local complete intersection.

Some useful lemmas:

Lemma 0.14. *Let E be a group of $t \leq 12$ points in $\mathbf{P}^2$, some possibly infinitely close, and assume that $h^1(\mathcal{I}_E(4)) \geq 1$, where $\mathcal{I}_E$ is the sheaf of ideals defining the scheme E. Then either E is a complete intersection of a cubic and a quartic curve, or there exists a subgroup $E' \subset E$ consisting of 6 points on a line, or of 10 or 11 points on a conic.*

Proof. (cf. [EP, Cor 2 and Remark]).□

To check whether a linear system is very ample we'll use the following lemma, which was communicated to us by J. Alexander.

Lemma 0.15. *If H has a decomposition*

$$H \equiv C + D,$$

where C and D are curves on S, such that $\dim|C| \geq 1$, and if the restriction maps $\mathrm{H}^0(\mathcal{O}_S(H)) \to \mathrm{H}^0(\mathcal{O}_D(H))$ and $\mathrm{H}^0(\mathcal{O}_S(H)) \to \mathrm{H}^0(\mathcal{O}_C(H))$ are surjective, and $|H|$ restricts to very ample linear systems on D and on every C in $|C|$, then $|H|$ is very ample on S.

Proof. We use the decomposition $H \equiv C + D$ to show that $|H|$ separates points and tangent directions on S. Let p and q be two, possibly infinitely close, points on S. By the assumptions of the lemma we may assume that $p + q$ is not contained in D or any C. In particular we may assume that $p + q$ does not meet the base locus of $|C|$. If D contains p, then we can find a curve C which does not meet $p + q$ such that $C + D$ separates p and q. If D does not meet $p + q$, then we can find a curve C which contains one of the points p or q, such that $C + D$ separates p and q. □

We deal with surfaces of degree 10 based on the following

Proposition 0.16. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ and π denotes the genus of a general hyperplane section, then*

$\pi = 6$ and S is abelian or bielliptic, or

$\pi = 8$ and S is an Enriques surface with four (-1)-lines, or a rational surface, or

$\pi = 9$ and S is a rational surface, or a blown-up $K3$ surface, or an honestly elliptic surface with $p_g = 1$, $q = 0$ and with three (-1)-lines, or a minimal surface of general type with $p_g = 2$, $q = 0$, $K^2 = 3$ and one (-2)-curve, or

$\pi = 10$ and S is a proper elliptic surface with $p_g = 2$, $q = 0$ and two (-1)-lines, or a minimal surface of general type with $p_g = 3$, $q = 0$, $K^2 = 4$ and three (-2)-curves, or

$\pi = 11$ and S is linked to an elliptic quintic scroll (S lies on a cubic hypersurface), or S is linked to a Bordiga surface (S does not lie on a cubic hypersurface), or

$\pi = 12$ and S is linked to a degenerate quadric surface, or

$\pi = 16$ and S is a complete intersection of a quadric and a quintic hypersurface.

Proof. (cf. [Ra]).□

1 6-secants and plane curves

Lemma 1.1. *If S contains a plane curve of degree d_p and $p_g = 0$, then $h^1(\mathcal{O}_S(H)) \geq \frac{1}{2}(d_p-2)(d_p-3)$, while if $p_g \geq 1$, then $h^1(\mathcal{O}_S(H)) \geq \frac{1}{2}(d_p-2)(d_p-3)+1-p_g$.*

Proof. Let C be a plane curve on S, and let $D \in |H-C|$, and consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0.$$

If $p_g = 0$, then $h^2(\mathcal{O}_S(D)) = 0$, so $h^1(\mathcal{O}_S(H)) \leq h^1(\mathcal{O}_C(H))$.
If $p_g \geq 1$, then $h^2(\mathcal{O}_S(D)) \leq p_g - 1$, so $h^1(\mathcal{O}_S(H)) \geq h^1(\mathcal{O}_C(H)) - p_g + 1$.□

Corollary 1.2. *If S has degree 10 and sectional genus 9 or 10, then any plane curve on S has degree at most 4.*

Proof. Immediate, since these surfaces have $h^1(\mathcal{O}_S(H)) \leq 2$.□

Lemma 1.3. *If S has a plane quartic curve C with $C^2 = 1$, then $\pi = 9$ and S has three 6-secants in the plane of C.*

Proof. The pencil of curves $|D| = |H-C|$ has $D^2 = 3$ base points in the plane of C. The general member is smooth of genus 4 if $\pi = 9$, and of genus 5 if $\pi = 10$. The latter is impossible by the genus bound. In the former case the lemma follows, unless the base points of $|D|$ are collinear. Now the general curve D is a complete intersection of a cubic and a quadric. Therefore the base points of $|D|$ are collinear if and only if the points $D \cap C$ are also collinear. But this means that $h^0(\mathcal{O}_C(H-D)) > 0$, while $h^0(\mathcal{O}_S(C)) = 1$ and S is regular by proposition 0.16, so this is impossible.□

Lemma 1.4. *$h^2(\mathcal{I}_S(n)) = 0$ and $h^2(\mathcal{I}_H(n)) = 0$ for any hyperplane section H of S, when $n \geq 2$.*

Proof. If $\pi = 9$ (resp. $\pi = 10$), then $1 \leq \chi \leq 3$ (resp. $3 \leq \chi \leq 4$), so $h^3(\mathcal{I}_S(1)) = 0$ and by Severi's theorem, $0 \leq h^2(\mathcal{I}_S(1)) \leq 2$ (resp. $0 \leq h^2(\mathcal{I}_S(1)) \leq 1$). Thus if $h^2(\mathcal{I}_S(2)) > 0$ and $\chi \geq 2$, then $h^2(\mathcal{I}_H(2)) > 0$ for at least a web of hyperplane sections H. If $h^2(\mathcal{I}_S(2)) > 0$ and $\chi = 1$ and $\pi = 9$, then $h^2(\mathcal{I}_H(2)) > 0$ for at least a net of hyperplane sections H. In the first case the general hyperplane section H in the web is smooth, but $2\pi - 2 < 20$ so $\mathcal{O}_H(2H)$ is non special, i.e., $h^1(\mathcal{O}_H(2)) = h^2(\mathcal{I}_H(2)) = 0$, a contradiction. In case $\chi = 1$, $\pi = 9$ and the general H in the above net is not smooth, i.e., the net has a line as a fixed curve, H decomposes as $H = L + C$, where L is a line and C is smooth. Consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_C(2H-L) \longrightarrow \mathcal{O}_H(2H) \longrightarrow \mathcal{O}_L(2H) \longrightarrow 0.$$

$h^1(\mathcal{O}_H(2H)) = h^2(\mathcal{I}_H(2)) > 0$, while $h^1(\mathcal{O}_L(2H)) = 0$, so $h^1(\mathcal{O}_C(2H-L)) > 0$. Therefore $\deg\mathcal{O}_C(2H-L) = 18 - C \cdot L = 17 + L^2 \leq 2p - 2$, where p is the arithmetic genus of C. But $2p - 2 = 2\pi - 2 + 2 - 2C \cdot L = 16 + 2L^2$, so $17 + L^2 \leq 16 + 2L^2$, i.e., $L^2 \geq 1$, which is impossible. Again since $h^3(\mathcal{I}_S(1)) = 0$ the second part of the lemma follows for $n = 2$. The argument applies inductively to the cases $n \geq 3$.□

Lemma 1.5. *If $\pi = 9$ or $\pi = 10$, then $h^0(\mathcal{I}_H(3)) \leq 1$ for any hyperplane section H of S. Furthermore $2 \leq h^1(\mathcal{I}_H(3)) \leq 3$ when $\pi = 9$, while $1 \leq h^1(\mathcal{I}_H(3)) \leq 2$ when $\pi = 10$.*

Proof. If H is contained in several cubics, then the cubics are reducible, thus either H decomposes into a line L and a curve C of degree 9 which lies on a quadric with $L \cdot C \leq 2$, or H has a plane curve component of degree at least 6. In the first case $L^2 = (H - C) \cdot L = 1 - L \cdot C \geq -1$ so $L^2 = -1$ and C has arithmetic genus 8 (resp.9), which is impossible on a quadric, while the second case contradicts lemma 1.1. The second part of the lemma now follows from Riemann-Roch and lemma 1.4. □

Let $\Delta(n)$ be the locus in $\check{\mathbf{P}}^4$ where $(h^0(\mathcal{I}_H(n))) \cdot (h^1(\mathcal{I}_H(n))) \neq 0$. This is clearly a determinantal variety. To study these degeneracy loci we use the following observations:

Lemma 1.6. *If S has a 6-secant, then $\Delta(4)$ contains a plane.*

Proof. For the general plane Π through the 6-secant $h^1(\mathcal{I}_{\Pi\cap S}(4)) > 0$, while $h^2(\mathcal{I}_H(3)) = 0$ for any hyperplane section containing $\Pi \cap S$ by lemma 1.4, whence $h^1(\mathcal{I}_H(4)) > 0$ for the general hyperplane H through the 6-secant, and the lemma follows. □

Lemma 1.7. *If $H \in \Delta(4)$, then H has a proper 6-secant or a plane curve component.*

Proof. If H does not have a plane curve component, then the sequence

$$0 \longrightarrow \mathcal{I}_H(3) \longrightarrow \mathcal{I}_H(4) \longrightarrow \mathcal{I}_{\Pi\cap H}(4) \longrightarrow 0$$

is exact for any plane Π in this hyperplane. Since $h^1(\mathcal{I}_H(3)) \leq 3$, there exists at least one plane section for which $h^1(\mathcal{I}_{\Pi\cap S}(4)) > 0$. This plane section is a scheme of length 10. If it is contained in a conic, then $h^1(\mathcal{I}_H(1)) > 0$, but S is regular so this means that $h^1(\mathcal{I}_S(1)) > 0$, which contradicts Severi's theorem. Therefore it follows from lemma 0.14 that $\Pi \cap S$ contains a subscheme of length 6 which is contained in a line, and the lemma follows. □

Lemma 1.8. *S has only finitely many plane curves.*

Proof. Since S is regular, any one-dimensional family of plane curves on S is linear. Thus the curve residual to the general plane curve in a hyperplane is again a plane curve, so the hyperplane section is contained in a quadric, impossible by Severi. □

Lemma 1.9. *Every component of $\Delta(4)$ of dimension at least 1 contains a line through each of its points. Every component of $\Delta(4)$ of dimension at least 2 contains a plane through each of its points. A plane in $\Delta(4)$ correspond to a proper 6-secant or to a line contained in S.*

Proof. By lemma 1.6, lemma 1.7 and lemma 1.8 every point in a component of dimension at least 1 lies in a plane or on a line whose corresponding hyperplane sections have a common plane component. Any component of dimension at least 2 without planes through each point gives rise to a family of plane curves, thus contradicting lemma 1.8. If the hyperplane sections corresponding to a plane have no line in common, then the general one is smooth and has a proper 6-secant by lemma 1.7. □

Lemma 1.10. $\Delta(4)$ *cannot contain a* $\mathbf{P}^3$ *or a determinantal quadric or cubic hypersurface.*

Proof. In either case $\Delta(4)$ contains at least a pencil of pairs of planes which meet along a line. Since S is not a scroll, it must contain infinitely many 6-secant lines by lemma 1.9. Furthermore, a pair of 6-secants in the pencil meet, since the corresponding planes in $\check{\mathbf{P}}^4$ meet in a line. By Bezout this means that S meets the plane spanned by a general pair of 6-secants in a curve. Thus there is a pencil of plane curves on S, impossible by lemma 1.8.□

Proposition 1.11. *Let S be a smooth surface of degree* 10 *and sectional genus* 9. *If* $\chi = 1$, *then the number N_6 of proper 6-secants is at most* 1, *and $h^1(\mathcal{I}_S(4))$ is also* 0 *or* 1. *If* $\chi = 2$, *then* $N_6 = 0$ *or* $N_6 = 1$ *and* $h^1(\mathcal{I}_S(4)) = 1$, *or* $N_6 = 3$ *and* $h^1(\mathcal{I}_S(4)) = 2$. *If* $\chi = 3$, *then* $N_6 = 3$ *and* $h^1(\mathcal{I}_S(4)) = 2$. *When the number of 6-secants is* 3, *then S cuts some plane along a quartic curve with self-intersection 1 and three points.*

Remark 1.12. *The formula of Le Barz gives the sum of the number of* 6*-secants and the number of* (-1)*-lines on S. When* $\chi = 1$ *the sum is* 7, *and when* $\chi = 2$ *and* $\chi = 3$ *the sum is* 3.

First note that $\Delta(3) \subset \Delta(4)$, since if H lies on a cubic, then it lies on at least 4 quartics, which means that $h^1(\mathcal{I}_H(4)) > 0$. Let $\Delta(3,4)$ denote the loci of hyperplanes for which $h^1(\mathcal{I}_H(3)) = 3$ and $h^1(\mathcal{I}_H(4)) = 2$, or equivalently $h^0(\mathcal{I}_H(3)) = 1$ and $h^0(\mathcal{I}_H(4)) = 5$. Now $h^1(\mathcal{I}_H(3)) = 2$ and $h^1(\mathcal{I}_H(4)) \geq 2$, or $h^1(\mathcal{I}_H(3)) = 3$ and $h^1(\mathcal{I}_H(4)) \geq 3$ are impossible like in lemma 1.10. Thus

Remark 1.13. $H \in \Delta(3,4)$ *if and only if* $h^1(\mathcal{I}_H(4)) = 2$.

Lemma 1.14. *If* $h^1(\mathcal{I}_H(4)) = 2$, *then H is reducible; it has a plane quartic component C and a connected residual component D, which is a complete intersection (2,3) or decomposes into a plane quartic component (which may coincide with C) and a conic. Furthermore, S has three 6-secants in a plane.*

Proof. First note that H is contained in a cubic and an independent quartic hypersurface. If H is linked $(3,4)$ to a curve C of degree 2 and arithmetic genus -3, then the curve C must be a double structure on a line. But any such structure on a cubic has arithmetic genus at least -2, a contradiction. So the cubic and the quartic containing H must have a common component. So H has a plane component, of degree at least 2 and at most 4, and possibly a component on an irreducible quadric. Any such component C on a quadric is of type (a,b) with $a \leq b \leq 5$, $a+b \leq 8$; for if say $b = 6$ and $a = 2$, then C has arithmetic genus $p_a(C) = 5$ and since the sectional genus is 9, C must meet the residual curve $H - C$ of degree 2 in $C \cdot H - C \geq 5$ points, which means that $H - C$ has a component on the quadric, the other values of a and b with $b \geq 6$ are impossible by similar reasoning.
If the cubic and the quartic have a common irreducible quadric, then the plane curve is a conic A, and the curve B on the quadric is of type (4,4) or (3,5), but $A^2 \leq -1$ so $A \cdot B \geq 3$, which means that $p_a(A+B) \geq 10$, impossible. Therefore the quartic and the cubic have a plane in common. The curve C of this plane is residual to a curve D in H which is on a quadric and an independent cubic. The quadric and the cubic have at most a plane in common, and any plane curve has degree at most 4 by corollary 1.2, so the curve D has

degree at most 6, and thus C is a plane quartic curve. The curve D moves in a pencil on S, whose general element is irreducible, so D is connected. If the quadric and the cubic have no common component, then D is a complete intersection (2,3). If the quadric and the cubic has a common plane, then the curve A of this plane must be a quartic, since the residual curve B is a conic lying in a plane and an independent quadric. Now $1 \leq B \cdot A \leq 2$ and $C \cdot A \leq 2$, while $B^2 \leq -1$ implies that $B \cdot (A+C) \geq 3$. If $B \cdot A = 2$ or D is a complete intersection, then $p_a(D) = 4$ and $C \cdot D = 3$ and $C^2 = H \cdot C - C \cdot D = 1$ so, by lemma 1.3, S has three 6-secants in the plane of C. If $B \cdot A = 1$, then $C \cdot A = 2$ and $B + C$ has genus 4. As above $A^2 = 1$ and there are three 6-secants in the plane of the quartic A. □

Lemma 1.15. *$h^0(\mathcal{I}_S(3)) = 0$ and $h^1(\mathcal{I}_S(3)) = \chi + 1$.*

Proof. Now $h^0(\mathcal{I}_S(3)) \leq 1$ by lemma 1.5. If $h^0(\mathcal{I}_S(3)) = 1$, then $h^1(\mathcal{I}_S(3)) = \chi + 2$, while $h^1(\mathcal{I}_S(2)) = \chi - 1$. So if $\chi = 2$ or $\chi = 3$, then $h^0(\mathcal{I}_H(3)) = 2$ for some H, contradicting lemma 1.5. If $\chi = 1$, then $h^1(\mathcal{I}_S(3)) = 3$ and $h^1(\mathcal{I}_S(4)) \geq 4$, so $h^1(\mathcal{I}_H(4)) = 2$ for at least a 2-dimensional family of hyperplane sections. By lemma 1.14, S has at least a 1-dimensional family of plane quartic curves, which is impossible by lemma 1.8. □

Proof of proposition 1.11. When $\chi = 1$, then $h^1(\mathcal{I}_S(3)) = 2$. So if $h^1(\mathcal{I}_S(4)) > 1$, then $\Delta(4)$ contains a $\mathbf{P}^3$ or a rank 4 quadric, thus contradicting lemma 1.10. When $\chi = 2$, then $h^1(\mathcal{I}_S(2)) = 1$ and $h^1(\mathcal{I}_S(3)) = 3$ by lemma 1.15, so $h^1(\mathcal{I}_H(3)) = 3$ and $h^1(\mathcal{I}_H(4)) \geq 1$ for some H. Therefore $h^1(\mathcal{I}_S(4)) \geq 1$. If $h^1(\mathcal{I}_S(4)) = 1$, then $h^1(\mathcal{I}_H(4)) \leq 1$, and $\Delta(4)$ is defined by three linear forms. So $\Delta(4)$ is a line or a plane. If it is a line, S has no 6-secants. If $\Delta(4)$ is a plane, then S has at most one 6-secant. If $h^1(\mathcal{I}_S(4)) = 2$, then $\Delta(4)$ is defined by the minors of a 2×3 matrix with linear entries. Thus $\Delta(4)$ is the union of three planes by lemma 1.9 and lemma 1.10, and $h^1(\mathcal{I}_H(4)) = 2$ for some hyperplane section H. By lemma 1.14, S has three 6-secants in the plane of a plane quartic and a residual pencil of curves of degree 6 and genus 4. If $h^1(\mathcal{I}_S(4)) \geq 3$, then S has too many plane curves. Now if $\chi = 3$, then $h^1(\mathcal{I}_S(2)) = 2$ and $h^1(\mathcal{I}_S(3)) = 4$ by lemma 1.15. Recall that $\Delta(3) \subset \Delta(4)$, hence $h^1(\mathcal{I}_S(4)) \geq 2$. Since S is minimal, it follows from remark 1.12. that S has 3 or infinitely many 6-secants. Therefore $\Delta(4)$ contains at least three planes, so if $h^1(\mathcal{I}_S(4)) = 2$, then $\Delta(3,4)$ is not empty, and lemma 1.14 applies, to show that S has three 6-secants in a plane. If $h^1(\mathcal{I}_S(4)) \geq 3$, then $\Delta(4)$ contains a determinantal cubic, thus contradicting lemma 1.10.□

Proposition 1.16. *Let S be a smooth surface of degree 10 and sectional genus 10. Then*

$$(h^0(\mathcal{I}_S(n))) \cdot (h^1(\mathcal{I}_S(n))) = 0$$

when $0 \leq n \leq 4$ and $\chi = 3$, while

$$(h^0(\mathcal{I}_S(n))) \cdot (h^1(\mathcal{I}_S(n))) = 0$$

when $0 \leq n \leq 3$, and $h^0(\mathcal{I}_S(4)) = 4$, $h^1(\mathcal{I}_S(4)) = 1$ when $\chi = 4$. In particular, S has a 6-secant when $\chi = 4$, and it has no 6-secant when $\chi = 3$.

Remark 1.17. *The formula of Le Barz gives the sum of the number of 6-secants and the number of (-1)-lines on S. When $\chi = 3$ the sum is 2, and when $\chi = 4$ the sum is 1.*

Lemma 1.18. *$h^0(\mathcal{I}_S(3)) = 0$ and $h^1(\mathcal{I}_S(3)) = \chi - 2$*

Proof. $h^0(\mathcal{I}_S(3)) \leq 1$ by lemma 1.5. If $h^0(\mathcal{I}_S(3)) = 1$, then, by lemma 1.4, $h^1(\mathcal{I}_S(3)) = \chi - 1$, while $h^1(\mathcal{I}_S(2)) = \chi - 3$. So if $\chi = 4$, then $h^1(\mathcal{I}_H(3)) = 3$ for some H, contradicting lemma 1.5. Furthermore, $h^0(\mathcal{I}_S(3)) = 1$ implies that $h^0(\mathcal{I}_S(4)) \geq 5$, so if $\chi = 3$, then $h^1(\mathcal{I}_S(4)) \geq 2$. Therefore $\Delta(4)$ contains a determinantal quadric, thus contradicting lemma 1.10. □

Proof of proposition 1.16. Now $h^1(\mathcal{I}_S(3)) = \chi - 2$, by lemma 1.4 and lemma 1.18. If $\chi = 3$, then $h^1(\mathcal{I}_S(4)) = 0$, by lemma 1.10. If $\chi = 4$, then $h^1(\mathcal{I}_S(2)) = 1$ and $h^1(\mathcal{I}_S(3)) = 2$ so $\Delta(3)$ is at least a plane. Recall that $\Delta(3) \subset \Delta(4)$, so $h^1(\mathcal{I}_S(4)) \geq 1$. But $h^1(\mathcal{I}_S(4)) \leq 1$, by lemma 1.10. Therefore $h^1(\mathcal{I}_S(4)) = 1$ and $\Delta(4)$ is a plane. By remark 1.17, S has one 6-secant since S is minimal when $\chi = 4$. When $\chi = 3$, $\Delta(4)$ is empty, so S has no 6-secants in this case.□

2 Linear systems

Proposition 2.1. *If S is a smooth rational surface of degree 10 in $\mathbf{P}^4$ with $\pi = 9$, then*

$$H \equiv 8\pi^* l - \sum_{i=1}^{12} 2E_i - \sum_{j=13}^{18} E_j,$$

or

$$H \equiv 9\pi^* l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k,$$

where $\pi : S \to \mathbf{P}^2$ is the blow-up map with exceptional curves E_i, $i = \overline{1,18}$, and l is a line in $\mathbf{P}^2$.

Proof. From proposition 1.11 and the 6-secant formula (0.8) it follows that the number of (-1)-lines on S is 6 or 7. The procedure is first to produce a number of possible candidates using adjunction, and secondly to show that only those described in the proposition are possible.

For $\pi = 9$ we obtain the following list of invariants for S, S_1 and Σ, where S_1 is the image of S under the adjunction map and Σ is the image of S_1 under the adjunction map defined by $|H_1 + K_1|$:

$S \subset \mathbf{P}^4$	$H^2 = 10$	$H \cdot K = 6$	$K^2 = -9$	$\pi = 9$
$S_1 \subset \mathbf{P}^8$	$H_1^2 = 13$	$H_1 \cdot K_1 = -3$	$K_1^2 = -9 + a$	$\pi_1 = 6$
$\Sigma \subset \mathbf{P}^5$	$H_\Sigma^2 = a - 2$	$H_\Sigma \cdot K_\Sigma = a - 12$	$K_\Sigma^2 = -9 + a + b$	$\pi_\Sigma = a - 6,$

where a is the number of (-1)-lines on S and b is the number of (-1)-lines on S_1. Σ is a surface since $4 \leq H_\Sigma^2 = a - 2 \leq 5$ when $6 \leq a \leq 7$. So the invariants for Σ make sense. When $H_\Sigma^2 = 4$, the surface Σ is a Veronese surface or a rational normal scroll, and when $H_\Sigma^2 = 5$ the surface Σ is a Del Pezzo surface. Thus H can be reconstructed via the adjunction process to get the following list of candidates:

$$1) \qquad H \equiv 5B + (6 - \frac{5}{2}e)F - \sum_{i=1}^{11} 2E_i - \sum_{j=12}^{17} E_j, \qquad e = 0 \text{ or } 2,$$

where B is a section with self-intersection $B^2 = e$ and F is a member of the ruling,

$$2) \qquad H \equiv 8\pi^* l - \sum_{i=1}^{12} 2E_i - \sum_{j=13}^{18} E_j,$$

$$3) \qquad H \equiv 9\pi^* l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k.$$

In case 1) we study curves in the linear system

$$|C| = |2B + (3-e)F - \sum_{i=1}^{11} E_i|.$$

Since $\dim|2B + (3-e)F| = 11$, there is a curve C in $|C|$. It has degree 5 and arithmetic genus 2. If C is not contained in a hyperplane, then it must be the union of a plane quartic A and a line L not meeting the plane of A. By the index theorem $A^2 \leq 1$, so $L^2 = 1 - A^2 \geq 0$, which is impossible by Riemann-Roch. Therefore C is contained in a hyperplane section H with a residual curve $C_1 = H - C$ of degree 5 and arithmetic genus 4, which is impossible by lemma 0.3.□

Proposition 2.2. *If*

$$|H| = |8\pi^* l - \sum_{i=1}^{12} 2E_i - \sum_{j=1}^{6} F_j|$$

is the linear system of hyperplane sections of a smooth surface S in $\mathbf{P}^4$, then there are three plane quartic curves on S, whose respective planes all contain the same line, which in turn is a 6-secant line for the surface. The map φ_C defined by

$$|C| = |4\pi^* l - \sum_{i=1}^{12} E_i|,$$

is of degree 4 onto $\mathbf{P}^2$, and maps a member of the linear system $|\pi^* l|$ to a plane quartic with three nodes such that each node is the image by φ_C of two of the exceptional curves F_j.

Proof. Let

$$C \equiv 4\pi^* l - \sum_{i=1}^{12} E_i,$$

$$C_{ij} \equiv C - \sum_{k=1}^{6} F_k + F_i + F_j$$

and

$$C^{ij} \equiv H - C_{ij}, \qquad \text{for} \quad 1 \leq i < j \leq 6.$$

Now $h^0(\mathcal{O}_S(C^{ij})) > 0$, so there is a curve C^{ij} in $|C^{ij}|$. It has degree 6 and arithmetic genus 3, so by lemma 0.3, it either spans a $\mathbf{P}^3$, in which case there is a residual plane quartic curve $C_{ij} \equiv H - C^{ij}$ and $h^0(\mathcal{O}_S(C^{ij})) = 2$, or C^{ij} is reducible, that is C^{ij} is the union of a plane quartic A and two skew lines L_1 and L_2, with $A \cdot L_1 = A \cdot L_2 = 1$, or a conic Q, with $A \cdot Q = 1$. One may now check that in the latter case $A \equiv C_{st}$ for some s, t and L_1 and L_2 are lines F_k and F_l.

These two possibilities for each C^{ij} fit together only if say

$$C_{12}, \quad C_{34} \quad \text{and} \quad C_{56}$$

are plane quartics and

$$|C^{12}|, \quad |C^{23}| \quad \text{and} \quad |C^{56}|$$

are their respective residual pencils.

Now $C_{12} \cdot C_{34} = C_{12} \cdot C_{56} = C_{34} \cdot C_{56} = 2$, so the planes of C_{12}, C_{34} and C_{56} meet pairwise in lines. Since the three planes span all of $\mathbf{P}^4$, they must intersect in a common line L, which is now a 6-secant for the surface S unless L lies on S.

To see that L cannot lie on S, we first note that S has no plane quintic curve by (1.2). So if L lies on S, then L is a component of the curves C_{ij} and $L \cdot (C_{ij} - L) = 3$. In this case we get $C_{12} \cdot C_{34} = (C_{12} - L) \cdot (C_{34} - L) + 6 + L^2 = 2$, but $(C_{12} - L) \cdot (C_{34} - L) \geq 0$ so $L^2 \leq -4$. Thus $C_{12} \cdot L = C_{34} \cdot L \leq -1$. Since $H \cdot L = (C_{12} + C_{34} + F_5 + F_6) \cdot L = 1$, this means that $L \cdot F_5 > 1$ or $L \cdot F_6 > 1$, which is absurd. So L cannot be contained in S.

Let $S \cap L = q_1 + \ldots + q_6$ be six, some possibly infinitely close, points such that $C_{12} \cap L = q_3 + q_4 + q_5 + q_6$, $C_{34} \cap L = q_1 + q_2 + q_5 + q_6$ and $C_{56} \cap L = q_1 + q_2 + q_3 + q_4$. Since C_{12} is a plane quartic curve on S, that is

$$\mathcal{O}_{C_{12}}(H) \cong \omega_{C_{12}} \cong \mathcal{O}_{C_{12}}(\pi^* l),$$

we see that the collinear points $q_3, \ldots, q_6$ all lie on a curve $L_0 \equiv \pi^* l$. But we get the same one for C_{34} and C_{56}, so this means that all the points q_i lie on L_0. The map φ_C maps the base points of a pencil in $|C|$ to a point, so q_1, q_2 and F_1, F_2 are mapped to the same point which is a node of $\varphi_C(L_0)$, etc. □

The quickest construction of this surface is by linkage, cf. (4.3).

Proposition 2.3. *Assume that*

$$|H| = |9\pi^* l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k|$$

is the linear system of hyperplane sections of a smooth surface S in $\mathbf{P}^4$. Let Σ be the image of $S \subset \mathbf{P}^3$ under the map φ_C defined by

$$|C| = |4\pi^* l - \sum_{i=1}^{11} E_i|.$$

There is a twisted cubic curve in this $\mathbf{P}^3$ for which the lines $\varphi_C(E_i)$, $i = \overline{1,4}$, are secants and which meets Σ in the points $\varphi_C(E_k)$, $k = \overline{12,18}$, outside these lines. Then $|H|$ is given by the linear system of quartic surfaces through the double curve of Σ, the four lines $\varphi_C(E_i)$, $i = \overline{1,4}$, and the seven points $\varphi_C(E_k)$, $k = \overline{12,18}$. Moreover, S lies on a determinantal quartic with 36 nodes, all lying on S.

Proof. First a few lemmas.

Lemma 2.4. *The linear system $|C| = |4\pi^* l - \sum_{i=1}^{11} E_i|$ has dimension 3 and has no base points, the double curve of its image is a, possibly reducible, twisted cubic curve and it has no triple points.*

Proof. If $\dim|C| > 3$, then, by lemma 0.14, the linear system has a fixed curve which will have negative degree on S, absurd, so $\dim|C| = 3$. Similarly, any fixed curve in the system will again have negative degree. If it has a basepoint, then again lemma 0.14 applies to show that there is an elliptic curve $3\pi^* l - \sum_{i=1}^{11} E_i$ of degree one on S, which is also absurd. So the first part of the lemma follows. Any smooth member of the system is a non-hyperelliptic curve of genus 3 which is mapped to a plane quintic curve by φ_C. Since the curve is not hyperelliptic, the image has only double points as singularities. Thus the double point class $7\pi^* l - \sum_{i=1}^{11} 2E_i$ contains a curve D_C on S which is mapped two to one onto its image. And the image is a curve of degree 3 which spans $\mathbf{P}^3$. It is residual to a rational quartic curve in the intersection of $\varphi_C(S)$ and a quadric surface in $\mathbf{P}^3$. Thus it is a twisted cubic curve.□

Lemma 2.5. *The linear system $|H|$ is cut out on Σ by quartic surfaces containing the twisted cubic curve $\varphi_C(D_C)$, the four secants $\varphi_C(E_i)$, $i = \overline{1,4}$, and the points $\varphi_C(E_k)$, $k = \overline{12,18}$, i.e., $|H|$ extends to a linear system on a blowup of $\mathbf{P}^3$.*

Proof. Consider the linear system of quartic surfaces in $\mathbf{P}^3$ which contains the twisted cubic curve $C_3 = \varphi_C(D_C)$ and the four secants $\varphi_C(E_i)$ $i = \overline{1,4}$. Denote by $\mathcal{M}$ the moving part of the pullback of this system to S. Clearly

$$\mathcal{M} \subset |H_0| = |9\pi^* l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j|.$$

Clearly the lemma follows if this is an equality. But the union of a twisted cubic curve and four of its secants is a curve of arithmetic genus at least 4, so it is contained in at least 10 quartic surfaces. Therefore $\dim\mathcal{M} \geq 9$. On the other hand, S is regular and $h^0(\mathcal{O}_{H_0}(H_0)) = 9$ for any smooth curve H_0, so $\dim|H_0| = 9$.□

We proceed to show that the base locus of this linear system of quartic surfaces is an arithmetically Cohen Macaulay curve of genus 11 and degree 10.

Lemma 2.6. *The general member D of the linear system $|12\pi^*l - \sum_{i=1}^4 4E_i - \sum_{j=5}^{11} 3E_j - \sum_{k=12}^{18} E_k|$ is irreducible.*

Proof. Note that D has degree 11 and arithmetic genus 10 and self intersection $D^2 = 10$. By Riemann-Roch $|D|$ is at least a pencil, so the general D is irreducible unless $|D|$ has a fixed curve. Let F denote this fixed curve and let $M = D - F$ be the moving part of D. By Riemann-Roch $|2H - D|$ is also a pencil, so let us consider the space P of quadrics in $\mathbf{P}^4$ which contain F. Since two members of D cannot be contained in the same quadric, the projective dimension of P is at least 3. This reduces the number of possibilities for F. First let us check whether F can be contained in a hyperplane. If so, we get the subcases a) F is not on a quadric in $\mathbf{P}^3$, b) F is on a quadric, c) F lies in a plane. In case a) the quadrics containing F decompose, so M lies in a pencil of hyperplanes, i.e., is a plane curve, but this means that S has a pencil of plane curves, contradicting lemma 1.8. In case b) there is at least a $\mathbf{P}^2 \subset P$ of reducible quadrics, so every M is also contained in a quadric in a $\mathbf{P}^3$, but since M moves in a pencil, this means that $A = H - M$ is a plane curve. By lemma 1.2 the degree of A is at most 4. If it is 4, then $0 \leq M^2 \leq 1$ since M moves and S has no 6-secant. But then, by (0.2) the curve M has genus 1 or 2 and degree 6 on a quadric, impossible. If A has degree 3, then M must be of type (3,4) or (2,5) on the quadric, while F has degree 4 and genus at most 1. Now M and F meet in a conic section, and by (0.2) we have $M \cdot F = 11 - p_a(M) - p_a(F) \geq 4$, with equality if M is of type (3,4) and F is elliptic. But inequality means that F has a component on the quadric of M, impossible. With equality $F^2 = D^2 - M^2 - 2M \cdot F = -2$, since by (0.2) $M^2 = 4$. But then $H - F$ has degree 6 and genus 3, and $(H - F) \cdot A = 6$, impossible. If A is a conic section, then M is of type (3,5) or (4,4), which are both impossible by (0.2). Similarly A cannot be a line. In case c) a similar argument to the one of case b) leads to a contradiction.
If F is not contained in a hyperplane, then the quadrics containing it cut out a curve, and F has degree 6 and genus 2, or degree 5 and genus 0 or 1, or degree 4 and genus at most 0, or degree 3 and genus at most -1. In either case we can write $M = \alpha\pi^*l - \sum_{i=1}^4 c_i E_i - \sum_{j=5}^{11} b_j E_j - \sum_{k=12}^{18} a_k E_k$. Note that a_k is 0 or 1 for each i. Furthermore, if $a_{12} + ... + a_{18} < 7$, then one member of the pencil $|M|$ decomposes into a line and a curve M_0, where M and M_0 have the same arithmetic genus. Thus

$$M \cdot F = M_0 \cdot F + 1 \leq H \cdot F + 1,$$

where

$$M \cdot F = 11 - p_a(M) - p_a(F)$$

from (0.2). Furthermore, if M_2 is the image of M by the second adjunction, i.e., on the Del Pezzo, then

$$H_2 \cdot M_2 = -MK + \sum_{j=5}^{11} b_j + \sum_{k=12}^{18} a_k = H \cdot M + 4p_a(M) - 4 - 2M^2 - \sum_{k=12}^{18} a_k.$$

We go now case by case. If $p_a(F) = 2$, then M has degree 5 and genus at most 2 since it is not a plane curve, so $H_2 \cdot M_2 \leq 5 + 4p_a(M) - 4 - \sum_{k=12}^{18} a_k \leq 5 + 4p_a(M) - 4$ means that

M has genus 1 or 2. If $p_a(M) = 2$, then M spans a hyperplane so $F \cdot M = 7 \leq F \cdot H = 6$, impossible. If $p_a(M) = 1$, then $a_{12} + ... + a_{18} < 7$, so there is an elliptic curve M_0 of degree 4 which spans a hyperplane, and by the above $F \cdot M_0 = 7 \leq F \cdot H = 6$, impossible. This argument works similarly when $p_a(F) \leq 1$.□

Lemma 2.7. *The general member D of the linear system $|12\pi^* l - \sum_{i=1}^{4} 4E_i - \sum_{j=5}^{11} 3E_j - \sum_{k=12}^{18} E_k|$ lies on a Del Pezzo surface S_D and is a member of the linear system $|7\pi^* l - \sum_{i=1}^{5} 2E_i|$ on the Del Pezzo.*

Proof. D has degree 11 and arithmetic genus 10. By Riemann-Roch D lies on at least two quadrics. If it lies on three quadrics, then it must lie on a cubic scroll. Since it lies on an independent quintic, it is linked to four skew lines on the scroll, so it is of type $6\pi^* l - e$ on the scroll. The map to $\mathbf{P}^3$ is a g^3_{11} which is not a projection of the map into $\mathbf{P}^4$. So it is defined by a system $3\pi^* l - \sum_{i=1}^{7} p_i$ on the cubic scroll, where the $p_i's$ are assigned base points. To have dimension 3 at least 5 of the $p_i's$ lie on a line, which means that the image in $\mathbf{P}^3$ lies on a quadric, absurd. So D lies on two quadrics which defines a cone over an elliptic curve or a Del Pezzo. In either case it lies on an independent cubic, so it is linked to a line in the complete intersection of the quadrics and the cubic. On a cone over an elliptic curve, the curve would have three branches through the vertex, imposing a singularity at this point of the surface S. So D lies on a Del Pezzo, and is linked to a line in the complete intersection of the Del Pezzo surface and a cubic hypersurface.□

Lemma 2.8. *The linear system $|C_{|D}|$ extends to a linear system $|3\pi^* l - \sum_{i=1}^{4} E_i|$ on S_D with exactly two base points on D.*

Proof. The adjoint linear system to D on S_D is $|4\pi^* l - \sum_{i=1}^{5} E_i|$. Let $p_1 + ... + p_7$ be a general member of the pencil residual to $|C_{|D}|$ in $|K_D|$, blow up S_D in these points and let F_i, $i = \overline{1,7}$ be the exceptional curves. Then $|4\pi^* l - \sum_{i=1}^{5} E_i - \sum_{i=1}^{7} F_i|$ has dimension 3. By lemma 0.14 this is possible only if there is a curve $3\pi^* l - \sum_{i=1}^{5} E_i - \sum_{i=1}^{7} F_i$, a curve $2\pi^* l - \sum_{i=1}^{j} E_{k_i} - \sum_{i=1}^{10-j} F_{k_i}$ or a curve $\pi^* l - \sum_{i=1}^{j} E_{k_i} - \sum_{i=1}^{6-j} F_{k_i}$ on the blown up surface. The first case is impossible since the map $\varphi_{C|D}$ would have a quadruple point, the second is impossible since then the image of D would lie on a quadric. In the third case the fixed curve must meet one of the $E_i's$, otherwise it would intersect D and impose an unassigned basepoint, impossible. Thus the fixed curve meets one of the $E_i's$, say E_5 and the lemma follows.□

It follows immediately from this lemma that $\varphi_C(D)$ lies on a cubic and has arithmetic genus 11. The pencil of these cubic surfaces, coming from the pencil $|D|$ has a base curve which consists of the four lines $\varphi_C(E_i)$ $i = \overline{1,4}$, and their two transversals, and a twisted cubic curve for which the four lines are secants. Together with the twisted cubic which is the double curve of $\Sigma = \varphi_C(S)$, the four lines and this twisted cubic curve form a curve of degree 10 and genus 11, which does not lie on a cubic (the cubic would belong to the above pencil). Therefore it is defined by the maximal minors of 4×5-matrix with linear entries. These minors restrict to $\varphi_C(S)$ to give the embedding of S in $\mathbf{P}^4$, i.e., the points $\varphi_C(F_i)$, $i = \overline{1,7}$, are the intersection of $\varphi_C(S)$ with a twisted cubic outside the lines $\varphi_C(E_i)$, $i = \overline{1,4}$. The image of $\mathbf{P}^3$ by this map is a determinantal quartic with nodes corresponding to the 4-secants and the singular points of the base curve of degree 10 and genus 11: Clearly a 4-secant is contracted, it is a (-2)-curve on a general quartic in the linear system containing it, so the image is a quadratic singularity. For the singularities on the base curve we have

Lemma 2.9. *Let C_1 and C_2 be two curves intersecting transversally in a point p in $\mathbf{P}^3$. Let $V \to \mathbf{P}^3$ be the blowup along the ideal of $C_1 \cup C_2$. Then V has a quadratic singularity at a point q over p. The strict transform on V of any surface with multiplicity a along C_1 and multiplicity b along C_2 is locally Cohen-Macaulay at q if and only if $|a-b| \leq 1$. If the branches have normal crossings and are transverse to the plane spanned by the two branches, then the strict transform is smooth at q if $|a-b| = 1$, and does not meet q if $a = b$.*

Proof. This is a purely local calculation which can be worked out by blowing up two concurrent lines in $\mathbf{P}^3$ and looking at the singular quadric in $\mathbf{P}^4$ defined by the quadrics through the two lines. The details are left to the reader.□

Now, the number of 4-secants is 20 and each of them intersect Σ once outside the base curve. The number of singular points is 16, and at each point the difference between the number of branches of Σ through the two branches of the base curve is 1, so the last statement of proposition 2.3 follows.□

2.10. Construction of B. To prove the existence it is clear how to go the other way. Start with 11 general points in the plane, or with a general quintic with a twisted cubic double curve. In both cases the surface is smooth outside the twisted cubic curve and each point on the twisted cubic is a double point of the surface. Starting with the plane the result follows from a study of the linear system $|4\pi^*l - \sum_{i=1}^{11} E_i|$. Starting with four lines in $\mathbf{P}^3$ not all on a quadric, pick two disjoint general pencils in the linear system of cubics through these lines. Each pencil defines a twisted cubic curve residual to the four lines and its two transversals. There is a pencil of quartic surfaces double along the twisted cubic, and containing the four lines, in fact the complete intersection is exactly the four lines plus a quadruple structure on the twisted cubic curve. Thus it is easy to see that the general quintic Σ, that we are looking for, is smooth outside the twisted cubic curve.
Clearly the net of quadrics containing the twisted cubic defines a rational map of degree one to $\mathbf{P}^2$, and the linear system on $\mathbf{P}^2$ can easily be deduced from this map. The union of the two twisted cubics and the four lines form a curve of degree 10 and genus 11, which is defined by the maximal minors of 4×5-matrix with linear entries. The five maximal minors of this matrix define a rational map of $\mathbf{P}^3$ into $\mathbf{P}^4$, the image is a determinantal quartic. The following lemma gives a description of the singularities of this map.

Lemma 2.11. *Let $C \subset \mathbf{P}^3$ be a reduced curve defined by the maximal minors of a (4×5)-matrix M with linear entries, let $\pi : V \to \mathbf{P}^3$ be the blow up of $\mathbf{P}^3$ along the ideal of C, and let $\varphi_M : V \to \mathbf{P}^4$ be the morphism defined by the maximal minors of M. Then any scheme Z of length 2 on V which is mapped by φ_M to a point is mapped by π into a 4-secant line to C.*

Proof. Any 4-secant line to C is contracted by φ_M, so we need to show the converse. If $\pi(Z)$ does not have support on C, then it follows from lemma 0.14 that in the general plane

P through $\pi(Z)$, there is a 4-secant line to C through $\pi(Z)$, or conic through 8 points on C or a plane cubic through 10 points on C. But no plane section of C is contained in a cubic so the latter two is impossible. If $\pi(Z)$ has length 2 and support on C, then the same argument works to show that Z is embedded by φ_M since C has no 5-secant.□

Let S be the strict transform of Σ in V. Then S is smooth and the map φ_M embeds S into $\mathbf{P}^4$: S is normalized and at each singular point of the base curve Σ is of type $(0,1)$ or $(1,2)$, so by lemma 2.9, the strict transform is smooth over these points. On the other hand, by lemma 2.11, any double point of the map comes from points on a 4-secant, and each 4-secant meets the quintic surface Σ once outside the base curve. Clearly S is a surface of type B.

We proceed to the surfaces with $\chi = 2$. From proposition 1.11 we know that S has three 6-secants in a plane, or one or no 6-secant.

Proposition 2.12. *If $\chi = 2$ and S has three 6-secants in a plane, then S is birationally a K3-surface. It is the image of a complete intersection (2,3) in $\mathbf{P}^4$ with one quadratic singularity under a map defined by quadrics through the singular point and tangent to the surface at three points which form a theta characteristic on a hyperplane section.*

Proof. Since S has three 6-secants, $K^2 = -3$ and $H \cdot K = 6$ it follows from the 6-secant formula that S has three (-1)-conics, which we denote by E_1, E_2 and E_3. Thus S is birationally K3 and $K = E_1+E_2+E_3$. By (1.11) the plane of the three 6-secants intersects the surface along a quartic curve C with self-intersection $C^2 = 1$. Let $|D| = |H-C|$ denote the residual pencil.

Lemma 2.13. $C \cdot E_i = D \cdot E_i = 1$, *for* $i = \overline{1,3}$.

Proof. Now, $C \cdot E_1 \leq 2$ since E_1 is a conic and cannot lie in the plane of C. Assume that $C \cdot E_1 = 2$. Then $C \cup E_1$ spans a hyperplane, and the pencil $|D|$ contains a member $D = D_0 + E_1$. We get that $D_0 \cdot E_1 = 1$, while the arithmetic genus of D is 4, so the arithmetic genus of D_0 is also 4. But D_0 has degree 5, so this is impossible by lemma 0.3. Similarly $C \cdot E_i \leq 1$ for $i = 2,3$. Now, by adjunction, $C \cdot K = C \cdot (E_1 + E_2 + E_3) = 3$, so the lemma follows.□

Let S_1 be the minimal model of S and let C_1 and D_1 be the image on S_1 of C and D respectively under the blowing down map. Since $C \cdot D = 3$ it follows from the lemma that $C_1 \cdot D_1 = 6$ and that $C_1^2 = 4$ and $D_1^2 = 6$. D is canonically embedded in $\mathbf{P}^3$, it is a complete intersection (2,3), so the linear system $|D_1|$ defines a map $\varphi_{D_1} : S_1 \to \mathbf{P}^4$, whose image is a complete intersection $(2,3)$. The image of C_1 is a curve of degree 6 and genus at least 3, so the image of C_1 is a hyperplane section. Thus $D_1 - C_1$ is effective, it is a (-2)-curve, which we denote by A. The image $\varphi_{D_1}(A)$ is a quadratic singularity. Furthermore, since $C + A$ and D are linearly equivalent and the general D does not meet A, the restrictions $\mathcal{O}_D(2D)$ and $\mathcal{O}_D(C+D)$ are isomorphic. But $\mathcal{O}_D(C+D)$ is the canonical bundle on D, so by adjunction $\mathcal{O}_D(C+D) \cong \mathcal{O}_D(D+E_1+E_2+E_3)$. Thus $\mathcal{O}_D(D) \cong \mathcal{O}_D(E_1+E_2+E_3)$ and $\omega_D \cong \mathcal{O}_D(2(E_1 + E_2 + E_3))$. Therefore E_1, E_2, E_3 are blown down to the points p_1, p_2, p_3 on S_1, which form a theta characteristic on the general D_1 through these points.□

2.14. Construction of C. For existence, let V_3 be a cubic hypersurface in $\mathbf{P}^4$ with exactly one isolated quadratic singularity at a point x, and let V_2 be a smooth quadric hypersurface which contains the double point of V_3 such that the complete intersection $S_0 = V_2 \cap V_3$ has a quadratic singularity at x and is smooth elsewhere. Let

$$\pi_0 : X \to \mathbf{P}^4$$

be the blowing-up of $\mathbf{P}^4$ in the point x, and let S_1 be the strict transform of S_0 on X with a curve A lying over the point x. Then S_1 is smooth, and the curve A is an irreducible rational curve with self-intersection $A^2 = -2$. Next, let D_1 be a general hyperplane section of S_0, and let Π be a plane in $\mathbf{P}^4$ which is tangent to D_1 in three distinct points p_1, p_2, p_3. More specifically we require that the intersections $V_2 \cap \Pi$ and $V_3 \cap \Pi$ are, respectively, an irreducible plane conic and a plane cubic curve which both go through and have a common tangent at the points p_1, p_2, p_3. Denote the preimage of the points p_1, p_2, p_3 on S_1 also by p_1, p_2, p_3, and blow them up with a map $\pi_1 : S \to S_1$ to get a smooth surface S with exceptional divisors E_1, E_2, E_3.
Let D_1 and A also denote the pullback (total transform) on S of the hyperplane divisor on S_0 and of the (-2)-curve over the node x on S_0, respectively. Consider the linear system of curves

$$|H| = |2D_1 - A - \sum_{i=1}^{3} 2E_i|$$

on S.

Proposition 2.15. *The above data V_2, V_3 and Π has been chosen such that the linear system of curves $|H|$ on S is very ample and embeds S as a surface of degree 10 in $\mathbf{P}^4$.*

Proof. The proof amounts to exploiting the decomposition of the divisor H given by $H \equiv C + D$, where

$$|D| = |D_1 - \sum_{i=1}^{3} E_i|$$

and

$$|C| = |D - A|.$$

Note that $|D|$ is a pencil of curves on S, while $|C|$ contains exactly one curve, call it C, namely the strict transform of the hyperplane section of S_0 which contains the points p_1, p_2, p_3 and x.

Lemma 2.16. *The linear system $|H|$ restricts to the canonical linear series on both the curve C and any smooth curve in $|D|$. Furthermore the two restriction maps*

$$\mathrm{H}^0(\mathcal{O}_S(H)) \to \mathrm{H}^0(\mathcal{O}_D(H))$$

and

$$\mathrm{H}^0(\mathcal{O}_S(H)) \to \mathrm{H}^0(\mathcal{O}_C(H))$$

are both surjective.

Proof. For any smooth curve $D \in |D|$ the divisor $(p_1 + p_2 + p_3) = (\sum_{i=1}^{3} E_i)_{|D}$ is a theta-characteristic on D, while $|D_1|$ is the adjoint linear system to D, so

$$|K_D| = |D_{1|D}| = |(\sum_{i=1}^{3} 2E_i)_{|D}|.$$

Therefore

$$|H_{|D}| = |(2D_1 - A - \sum_{i=1}^{3} 2E_i)_{|D}| = |(D_1 - A)_{|D}| = |K_D|,$$

where the latter equality holds since A does not meet D at all.
Next, consider the cohomology of the exact sequences

$$0 \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_D(H) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0$$

of sheaves on S. As noted above, $h^0(\mathcal{O}_S(C)) = 1$, so by duality and Riemann-Roch,

$$h^1(\mathcal{O}_S(C)) = h^2(\mathcal{O}_S(C)) = 0.$$

Therefore, in the first sequence,

$$h^1(\mathcal{O}_D(H)) = h^1(\mathcal{O}_D(K_D)) = 1$$

implies that $h^1(\mathcal{O}_S(H)) = 1$ and, by Riemann-Roch again, that $h^0(\mathcal{O}_S(H)) = 5$. In the second sequence $h^1(\mathcal{O}_S(D)) = h^2(\mathcal{O}_S(D)) = 0$. Thus $h^1(\mathcal{O}_S(H)) = 1$ implies that $h^1(\mathcal{O}_C(H)) = 1$ and $h^0(\mathcal{O}_C(H)) = 3$. Since $p_a(C) = 3$ and $H \cdot C = 4$, the linear series $|H|_C|$ is the canonical linear series on C. Note also that the restriction maps are both surjective.□

For the very ampleness of $|H|$, it suffices, by lemma 0.15, to check that the linear series $|H_{|D}|$ is very ample for every curve $D \in |D|$.

Lemma 2.17. *$|H|$ restricts to a very ample linear series on the curve $C + A$ on S.*

Proof. First check that C is not hyperelliptic. As noted above, C is the strict transform on S of the hyperplane section, call it L_0, of S_0 which contains the points p_1, p_2, p_3 and x. Since L_0 has a double point at x, we see that the morphism defined by the canonical linear series on C is the projection of L_0 from the point x into a plane. Thus C is hyperelliptic if and only if this map is 2 to 1, or geometrically, any line in $\mathbf{P}^4$ through x which meets L_0 in a point away from x will meet L_0 in two (possibly infinitely close) points away from x. But by Bezout and our choice of V_2 and V_3, this means that any such line is contained in the surface S_0, which is absurd. Therefore $|H_{|C}|$ is very ample.
For A, note that $A \cdot H = 2$, so $|H_{|A}|$ is very ample. If $\mathrm{H}^0(\mathcal{O}_S(H)) \to \mathrm{H}^0(\mathcal{O}_A(H))$ is not surjective, or if $|H|$ does not separate points $p \in C \setminus A$ and $q \in A \setminus C$, then since $C \cdot A = 2$, the curve A is mapped into the plane of C, which is absurd since by lemma 2.16 the image of any curve in $|D|$ spans a $\mathbf{P}^3$.□

Lemma 2.18. *$|H|$ restricts to a very ample linear series on the curves in $|D|$ which have one of the E_i's as a component.*

Proof. If $D = G_1 + E_1$ say, then $G_1 \cdot E_1 = 2$, and G_1 is a plane quartic curve, so if $|H|$ does not separate points $p \in G_1 \setminus E_1$ and $q \in E_1 \setminus G_1$, then $|H_{|(G_1+E_1)}|$ must have dimension 2, impossible. For G_1 we get

$$|(C+D)_{|G_1}| = |(2D_1 - \sum_{i=1}^{3} 2E_i)_{|G_1}| = |(D_1 - E_1)_{|G_1}| = |K_{G_1}|.$$

So $|H_{|G_1}|$ is very ample unless G_1 is hyperelliptic. But the hyperplane cutting $\pi(G_1)$ cuts the quadric V_2 transversely at the point $\pi(E_1)$ since $\Pi \cap V_2$ is smooth, so the projection from $\pi(E_1)$ is an isomorphism. $|H_{|E_1}|$ is very ample since it has degree 2. So it remains to show that the restrictions of sections are surjective in both cases. For a general D we know that the restriction is surjective, so $G_1 + E_1$ spans a $\mathbf{P}^3$ and one of the restrictions is surjective. But if G_1 or E_1 is mapped to a line, then $G_1 + E_1$ lies in a plane since $E_1 \cdot G_1 = 2$, contradiction.□

For the rest of the curves D, $|H_{|D}|$ is very ample since the map $\pi : S \to S_0$ restricts to an isomorphism, D_1 is the pullback of the hyperplane divisor on S_0, and

$$|(C+D)_{|D}| = |2D_{|D}| = |(2D_1 - \sum_{i=1}^{3} 2E_i)_{|D}| = |D_{1|D}|.$$

This concludes the proof of proposition 2.15.□

Proposition 2.19. *If $\chi = 2$ and S has one 6-secant, then S is birationally K3. It has two (-1)-lines E_1 and E_2, and a (-1)-curve E_0 of degree 4, and two linear systems of curves $|A_0|$ and $|B_0|$ of degree 6 and genus 4. S is embedded into $\mathbf{P}^4$ by the linear system of curves in $|A_0 + B_0 - 4E_0 - E_1 - E_2|$ such that the product of the pencils of curves $|A_0 - 2E_0|$ and $|B_0 - 2E_0|$ defines a map onto a quadric surface, through which E_0 acquires one double point at a point which coincides with the image of E_1 and E_2.*

Proof. By the 6-secant formula S has two (-1)-lines E_1 and E_2, and since $K^2 = -3$ and $H \cdot K = 6$ the surface has a (-1)-rational quartic curve E_0. In fact the minimal model has a canonical divisor K_0 with $K_0^2 \geq 0$ and $H \cdot K_0 = 0$ or $H \cdot K_0 \geq 3$. Only $H \cdot K_0 = 0$ fits in our case, so S is birationally K3 and K consists of the three (-1)-curves.

Lemma 2.20. *Any plane quartic curve C on S with self-intersection $C^2 = 0$ will satisfy:*

$$C \cdot E_0 = 2, \quad C \cdot E_1 = C \cdot E_2 = 1.$$

Proof. Let $D = H - C$. Then D has degree $H \cdot D = 6$, arithmetic genus 3 and self-intersection $D^2 = 2$. Assume that $D \cdot E_i = e_i$ for $i = \overline{0,2}$, then $D_0 = D + \sum_{i=0}^{2} e_i E_i$ has self-intersection $D_0^2 = 2 + \sum_{i=0}^{2} e_i^2$. Furthermore D_0 has arithmetic genus $p_a(D_0) =$

$3+\sum_{i=0}^{2}\binom{e_i}{2}$, and since $D_0\cdot E_i=0$, adjunction gives $2p_a(D_0)-2=D_0^2$. This leads to the relation

$$e_0+e_1+e_2=2.$$

Note that $e_0>0$ since $H\cdot E_0=(C+D)\cdot E_0=4$ and E_0 spans all of $\mathbf{P}^4$; if not there would be a curve $H-E_0$ on S of degree 6 and arithmetic genus 5 which would necessarily have a plane quintic curve as a component, impossible by corollary 1.2.
Furthermore $e_i\geq 0$, for $i=1,2$, since D moves in a pencil with only isolated base points by corollary 1.2. Therefore the only possible cases are $e_0=1$ and $\{e_1,e_2\}=\{0,1\}$, or $e_0=2$ and $e_1=e_2=0$. The second case leads to the conclusion of the lemma. In the first case assume that $e_1=1$, the other case being analogous. Then there is a curve D_2 in the pencil $|D|$ which contains E_2 as a component. Thus $D_2=A+E_2$ where A is a curve of degree 5 and genus 3. This curve again must decompose into a plane quartic curve A_0 and a line L cutting it. So $A_0\cdot L=1$. By the index theorem $A_0^2\leq 1$, and by lemma 1.3 $A_0^2\neq 1$, so $A_0^2\leq 0$. Since $A^2=1$ this means that $L^2\geq -1$. Thus $L^2=-1$ and $L\cdot A=0$. But this holds only if $L=E_1$, which leads to $E_1\cdot E_2=E_1\cdot(E_2+A)=E_1\cdot D=e_1=1$, impossible.□

Lemma 2.21. *The hyperplane section of S which contains the exceptional lines decomposes into two plane quartic curves A and B with $A\cdot B=2$ and $A^2=B^2=0$. The 6-secant to S is the line of intersection between the planes of A and B.*

Proof. First assume that the exceptional lines E_1 and E_2 meet the 6-secant line, and let $C=H-E_1-E_2$, where H is the hyperplane section containing E_1 and E_2. By Riemann-Roch, C lies on at least two cubic surfaces, and the 6-secant line of S intersect C in a scheme of length at least 4. So if the C lies on an irreducible cubic, it is linked to the 6-secant line in the intersection of two cubics. Then the pencil of planes through this line defines a pencil of degree 4 on C. The sum of this pencil and a hyperplane section of C form a canonical divisor on C. But, on the $K3$ surface, $H+E_0$ restricts to C to form a canonical divisor, which means that E_0 intersect C in a scheme of length 4 in a plane, impossible. Therefore any cubic containing C is reducible, and they all have a component in common.
If this fixed component is a plane, then C decomposes into a plane quartic A and an elliptic quartic curve B. The arithmetic genus of C is 7, so $A\cdot B=4$. Since the 6-secant line is at least a 4-secant to C it must lie in the plane of A and the pencil $|H-A|$ must have two base points in the plane. Thus $(H-A)^2=2$ and $A^2=0$. But, by lemma 2.20, this means that $A\cdot(B+E_1+E_2)=A\cdot(H-A)=6$, a contradiction.
If this fixed component is an irreducible quadric, then C decomposes into a line L and a curve A on the quadric of arithmetic genus 6, with $L\cdot A=2$. Again, the 6-secant line is a 4-secant to C so it is a line on the quadric, which is a 4-secant to A. This is possible only if the intersection of $L+E_1+E_2$ with the quadric has a scheme of length 2 residual to the intersection with A, i.e., that $A\cdot(E_1+E_2)=2$. But this means that $L\cdot(E_1+E_2)=2$, so $(H-L)\cdot L=4$ and $(L+E_1+E_2)^2=-1$, which is impossible.
So the fixed component must be two planes. One of the planes must intersect the surface along a quartic curve A and the 6-secant must lie in this plane, so as above $A^2=0$ and $A\cdot E_i=1$. Thus $(H-A-E_1-E_2)\cdot A=2$, which means that the curve $B=H-A-E_1-E_2$

must have arithmetic genus 3, so it is also a plane quartic curve with self-intersection $B^2 = 0$, and the line of intersection of the two planes is the 6-secant to the surface.
Now, assume that one of the exceptional lines, say E_1, does not meet the 6-secant. The 6-secant and E_1 span a hyperplane which cuts the surface in E_1 and a curve C_1 of degree 9 and arithmetic genus 8. Since any plane section of C_1 through the 6-secant lie on 3 cubic curves, and $h^1(\mathcal{I}_{C_1}(2)) = 1$, C_1 must lie on a pencil of cubics. This is possible only if the cubics have a fixed component. The fixed component cannot be a plane since S does not contain any plane quintic curve. So the fixed component is a quadric, and C_1 decomposes into a line E and a curve C_0, such that C_0 is on a quadric and $E \cdot C_0 \leq 2$. This means that E is a (-1)-line, i.e., $E = E_2$, and C_0 has arithmetic genus 7. Thus C_0 cannot lie on an irreducible quadric, and it is the union of two plane quartic curves A and B which meet in two points. Since $A \cdot E_1 = A \cdot E_2 = 1$, it follows that $B^2 = 0$, and similarly $A^2 = 0$.□

By (2.20) we have $A \cdot E_0 = B \cdot E_0 = 2$. Consider the two pencils of curves $|A + E_1 + E_2|$ and $|B + E_1 + E_2|$, they have two base points each, so the product of the pencils define a rational map of S onto a quadric, this map has degree 4 outside the base points, and the curve E_0 is mapped to an elliptic quartic curve, so it acquires a double point. Proposition 2.19 now follows from

Lemma 2.22. *The inverse image of the double point is two points on E_0 and the two (-1)-curves E_1 and E_2.*

Proof. Since $A + E_1 + E_2 = H - B$ and $B + E_1 + E_2 = H - A$, the base points of the pencils $|A + E_1 + E_2|$ and $|B + E_1 + E_2|$ lie in the planes of B and A respectively, in fact since S has only one 6-secant, the base points all lie on this 6-secant. We denote the base points of $|A + E_1 + E_2|$ by a_1 and a_2 and the base points of $|B + E_1 + E_2|$ by b_1 and b_2. There are two more points on the 6-secant which lie on the surface, they form $A \cap B$ and we denote them by c_1 and c_2.
The curves A and B are canonically embedded on S into $\mathbf{P}^4$. So $a_1 + a_2 + c_1 + c_2$ is a canonical divisor on A. But by adjunction the canonical line bundle on A is $\mathcal{O}_A((A + E_1 + E_2) + E_0)$, so $\mathcal{O}_A(A + E_1 + E_2) = \mathcal{O}_A(a_1 + a_2)$ means that $A \cap E_0 = c_1 + c_2$. Similarly $B \cap E_0 = c_1 + c_2$. Now there is a net of curves in $|A + B + 2E_1 + 2E_2|$ through the points $a_1, a_2, b_1, b_2, c_1, c_2$, and in fact all these curves contain E_1 and E_2 so the lemma follows.□

For existence cf.(4.8).

Proposition 2.23. *If $\chi = 2$ and S has no 6-secants, then S is a smooth elliptic surface, the canonical curve has four components: a plane cubic curve C and three (-1)-lines, E_1, E_2 and E_3. The linear system*

$$|D| = |H + E_1 + E_2 + E_3 - C|$$

defines a birational morphism of S into $\mathbf{P}^3$ whose image is a surface of degree 7 with a quadruple point at a point q, a double curve of degree 9 and arithmetic genus 9 lying on the cone with vertex at q over a plane cubic curve isomorphic to C, with 6 branches through q along the 6 lines of intersection of 4 planes through q. Furthermore the surface S lies on a rational quartic with a double plane, the plane of the curve C.

Proof. Since S has no 6-secant, it has three (-1)-lines and the canonical curve K has the components

$$K \equiv C + \sum_{i=1}^{3} E_i,$$

where C is an elliptic curve of degree 3, i.e., a plane cubic curve, and the E_i are the (-1)-lines. Thus S is birationally an elliptic surface.
Consider the pencil

$$|D_0| = |H - C|$$

and the linear system

$$|H_1| = |H + \sum_{i=1}^{3} E_i|$$

of curves on S.

Lemma 2.24. *The general member of $|D_0|$ is trigonal, and $|H_1|$ defines a morphism which is the composition of the blowing down of the (-1)-lines and an embedding into $\mathbf{P}^6$.*

Proof. First note that the pencil $|D_0|$ has no fixed component, since a fixed component of $|D_0|$ would be contained in the plane of C; now $D_0 \cdot C = 3$, so this fixed component would be a line, call it L. Thus $L \cdot C = 3$ and $L \cdot 2C = 6 = H \cdot 2C$. But $|2C|$ is a pencil of elliptic curves whose general member is irreducible, so L would be a 6-secant to such a curve, which is absurd. Thus the general member $D_0 \in |D_0|$ is irreducible. It has degree $H \cdot D_0 = 7$ and arithmetic genus $p_a(D_0) = 6$, so it is linked to a line in the complete intersection of a quadric and a quartic surface. One of the rulings on the quadric surface will sweep out a g^1_3 on D_0, i.e., D_0 is trigonal.
Furthermore $|H_1| = |D_0 + C + \sum_{i=1}^{3} E_i| = |D_0 + K|$ is the adjoint linear system to D_0 on S. From the global sections of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_S(H_1) \longrightarrow \mathcal{O}_{D_0}(H_1) \longrightarrow 0$$

of sheaves of S, it follows, since S is regular, that the restriction map $\mathrm{H}^0(\mathcal{O}_S(H_1)) \longrightarrow \mathrm{H}^0(\mathcal{O}_{D_0}(H_1))$ is surjective and that $h^0(\mathcal{O}_S(H_1)) = 7$. Thus $|H_1|$ is very ample on the general curve in $|D_0|$ and blows down the E_i's. Since $|H|$ is very ample the lemma follows.□

Corollary 2.25. *The embedding of S into $\mathbf{P}^4$ is the projection of a smooth surface in $\mathbf{P}^6$ from the linear span of 3 collinear points x_1, x_2, x_3.*

Proof. Denote the image of S in $\mathbf{P}^6$ by S_1. Let $x_i = \varphi_{H_1}(E_i)$, for $i = 1, 2, 3$. Since $H \equiv H_1 - \sum_{i=1}^{3} E_i$, the points x_i must be collinear. Moreover, they form a member of the g^1_3 on the general D_0.□

Denote by L the line spanned by the points x_1, x_2 and x_3. Denote by C_1 the image of C on S_1. The curve C_1 is a plane cubic curve on S_1 whose plane we denote by Π. Consider the linear system

$$|D| = |H_1 - C_1|$$

of curves on S_1. It has projective dimension $\dim|D| = 3$. Denote the subpencil of $|D|$ which corresponds to the pencil $|D_0|$ on S_1 by $\mathcal{P}$; i.e., the pencil of curves in $|D|$ which meet the points x_i. Although all of the curves in $\mathcal{P}$ are trigonal, this is not necessarily the case for all the curves in $|D|$.

Lemma 2.26. *There is a net of trigonal curves in $|D|$.*

Proof. To see this, consider the base locus $Z_{\mathcal{P}} \subset S_1$ of the pencil $\mathcal{P}$. We may write $Z_{\mathcal{P}} = Z + Z_L$, where Z_L is of length three and is contained in L, while Z is of length four and has support outside L. For a general member $D_p \in \mathcal{P}$, the scheme Z is a divisor which by duality on D_p spans a $\mathbf{P}^3$ together with the plane Π. In fact

$$Z + (\Pi \cap D_p) \equiv K_{D_p} - Z_L$$

as divisors on D_p. We denote this $\mathbf{P}^3 \subset \mathbf{P}^6$ by $V_{\mathcal{P}}$. Now there is a net of curves D in $|D|$ which contain Z, and, by duality again, all of these curves are trigonal.□

Denote this net by $\mathcal{N}$. Every member D of $\mathcal{N}$ is a trigonal curve canonically embedded by $|H_1|$, therefore it lies on a rational scroll denoted by S_D, whose ruling restricts to the trigonal linear series on D. For the general member $D_p \in \mathcal{P}$ the line L must be a member of this ruling. L cannot meet any other member of the ruling, otherwise the projection of D_p from the line L into $\mathbf{P}^3$ would be three to one, therefore the scroll S_D must be smooth.

Let V_0 be the intersection of the quadric hypersurfaces containing S_1. We proceed now to study V, the irreducible component of V_0 which contains S_1, and the projection of V to $\mathbf{P}^3$ from the plane Π. Note that all rational normal scrolls in the net $\{S_D | D \in \mathcal{N}\}$ are contained in V_0 since their rulings are 3-secants to S_1, and that Π is also contained in V_0.

First, let us consider the possibilities for V_0. Since $\mathcal{O}_{H_1}(2H_1)$ is non-special on any smooth curve H_1, it follows from Riemann-Roch and the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(H_1) \longrightarrow \mathcal{O}_{S_1}(2H_1) \longrightarrow \mathcal{O}_{H_1}(2H_1) \longrightarrow 0$$

of sheaves on S_1 that $h^0(\mathcal{I}_{S_1}(2)) = 3$.

Lemma 2.27. *$\dim V_0 < 4$.*

Proof. If V_0 is four-dimensional, then it is of degree three and irreducible, since it lies on three linearly independent quadrics. Codimension two varieties of degree three are ruled by a pencil of linear spaces of codimension three. In our case V_0 is ruled by a pencil of $\mathbf{P}^3$s. This pencil must clearly restrict to the ruling of the scrolls S_D. Thus L is contained in one of the $\mathbf{P}^3$s. On the other hand, these $\mathbf{P}^3$s must sweep out a pencil of curves on S_1. Projecting from L, it follows that the member of this pencil which belongs to the $\mathbf{P}^3$ of L, is mapped onto a line. Thus the curves of the pencil must all be rational, which is absurd for an elliptic surface of Kodaira dimension 1.□

Thus V_0 and V are three-dimensional and are contained in the complete intersection of three quadric hypersurfaces. The scrolls S_D are parts of hyperplane sections of V: The net $\mathcal{N}$ of divisors on S_1 is the restriction to S_1 of the net of hyperplanes which contains the linear space $V_{\mathcal{P}}$. The restriction of the same net to V has the scrolls S_D as members of the moving part, since the general such member must be irreducible. The fixed part of this net is $T = V \cap V_{\mathcal{P}}$.

Lemma 2.28. *T is a cubic surface.*

Proof. Let D_p be a general irreducible member of $\mathcal{N}$, and let S_D be the corresponding scroll. First note that $S_D \cap V_{\mathcal{P}}$ is a twisted cubic curve; in fact if E_0 is a section with $E_0^2 = 0$ on S_D and F is a member of the ruling, then D_p is of type $3E_0 + 4F$ on S_D, the adjoint linear system is $|E_0 + 2F|$, so $|E_0 + F|$ restricts to the g_7^3 residual to the trigonal linear series defined by the ruling $|F|$. Now $D_p \cap V_{\mathcal{P}} = Z + (\Pi \cap D_p) \equiv K_{D_p} - Z_L$, is a member of this g_7^3, so it lies on a curve in $|E_0 + F|$ which is a twisted cubic curve.

Lemma 2.29. *For a general member of $\mathcal{N}$ the corresponding twisted cubic curve is irreducible.*

Proof. Assume that every twisted cubic is reducible. Then the twisted cubics would have a line as a component which is a trisecant to the corresponding D_p. Consider the plane of C in $\mathbf{P}^4$. The curves $|D_0|$ are of type $(3,4)$ on a quadric and they have 4 base points in the plane. Every trisecant to D_p which is contained in $V_{\mathcal{P}}$ is a trisecant to the corresponding curve in $|D_0|$ on S and would lie in this plane and in the corresponding quadric. Thus all the quadrics meet the plane in two lines, and one of these lines is fixed for the pencil of quadrics. It would contain at least 3 base points so it would be a 6-secant to S, impossible.□

Corollary 2.30. *D_p and S_D intersect Π only on C_1, and the scheme Z spans $V_{\mathcal{P}}$.*

Proof of 2.28 *continued.* The surface swept out by the twisted cubics contains C_1, so it has a component different from Π which has degree at least 3. Now, if $V_{\mathcal{P}}$ is not contained in V_0, then $V \cap V_{\mathcal{P}}$ is at most a quadric surface. It would have the plane Π as a component, impossible by lemma 2.29. Thus $V_{\mathcal{P}}$ is a component of V_0 which by linkage intersects the other components in a cubic surface. But V intersects $V_{\mathcal{P}}$ in a surface of degree at least 3, so lemma 2.28 follows.□.

By (2.30), the plane Π intersects each scroll in the scheme $C_1 \cap S_D$ of length 3, hence the linear system $|D|$ has no base points. The morphism

$$\varphi_D : S_1 \to \mathbf{P}^3$$

defined by $|D|$ is birational since $D^2 = 7$ and the image spans $\mathbf{P}^3$. The image $\Sigma = \varphi_D(S_1)$ is a surface of degree seven. The image of a curve D_p of the net $\mathcal{N}$ must be a plane curve of degree seven, and therefore the image of the scroll S_D by the projection from the plane Π is a plane in $\mathbf{P}^3$. The base locus Z of $\mathcal{N}$ is mapped onto a point

$$q = \varphi_D(Z).$$

Therefore $\varphi_D(D_p)$ acquires a quadruple point at q. Additionally, $\varphi_D(D_p)$ acquires three double points from the members of the ruling of S_D which meet C_1. That there are no other singularities can be checked by the genus formula of a plane curve.

The morphism φ_D extends to a rational map

$$proj_\Pi : V \dashrightarrow \mathbf{P}^3$$

which is generically finite. Since V is contained in the complete intersection of three quadrics, it is birational. In fact, if P is a $\mathbf{P}^3 \subset \mathbf{P}^6$ which contains the plane Π, then the three quadrics will restrict to P as the union of Π and three other planes. If the intersection of the other planes is finite, then it is one point. The family of scrolls $\{S_D\}$ is mapped by $proj_\Pi$ onto the net of planes through the point q, while the curves in $\mathcal{N}$ are mapped to plane septic curves in these planes, with a quadruple point at q and three extra double points.

The image $C_0 = \varphi_D(C_1)$ of the curve C_1 is a plane cubic curve; it lies on a cubic cone with vertex at q, which we denote by S_3. This cone is the image in $\mathbf{P}^3$ of the exceptional divisor coming from the blowing up of V along C_1.
The inverse rational map

$$\rho : \mathbf{P}^3 \dashrightarrow V \subset \mathbf{P}^6,$$

restricts to a plane $proj_\Pi(S_D)$ as the map defined by a linear system of plane quartic curves with a triple point at q and three simple points outside q as assigned base points, since the curve D_p is canonically embedded in S_D. Therefore ρ is defined by a linear system $|d_0|$ of quartic surfaces with a triple point, as assigned basepoint, at q, and with an assigned base curve, denoted by C_B, which meets a general plane through q in three points outside q.
Let

$$\pi : U \to U_0 \to \mathbf{P}^3$$

be the composition of blowing up first q to get U_0 and then the strict transform of C_B on U_0 to get U. Let $E_{q,0}$ be the exceptional divisor on U_0, let E_q be its strict transform on U, and let E_B be the exceptional divisor over C_B. Then ρ extends to a morphism

$$\rho_U : U \to V \subset \mathbf{P}^6$$

which is defined by the linear system

$$|d| = |\pi^* d_0 - 3E_q - E_B|$$

of divisors on U. The canonical divisor on U is

$$K_U \equiv -\pi^* d_0 + 2E_q + E_B.$$

The image by the map ρ_U of the strict transform E_q of $E_{q,0} \cong \mathbf{P}^2$ is the cubic surface $T = V \cap V_{\mathcal{P}}$. Now, E_q is the projective plane $E_{q,0}$ blown up in the points q_i, $i = \overline{1,n}$ of the intersection with the strict transform of C_B on U_0. The restriction to E_q of the linear system $|d|$ is the linear system of plane cubic curves on $E_{q,0}$ with assigned base points at these points of intersection. Therefore T is a Del Pezzo surface, $n = 6$, and C_B has six branches at q. This means that C_B is a curve of degree nine in $\mathbf{P}^3$.
Now, $\rho_U(U) = V \subset \mathbf{P}^6$ has degree 7. On the other hand, the restriction of $|d|$ to the strict transform of a general plane in $\mathbf{P}^3$ is the linear system of quartic curves with 9 assigned base points, which is a linear system of degree 7 and projective dimension 5, so its image

in V must be a hyperplane section of V. But this means that the base curve C_B must be contained in S_3.

Summing up: The surface $\Sigma = \varphi_D(S_1)$ must have a quadruple point at q and must have C_B as a double curve. If Σ_U is the strict transform of Σ on U, then Σ_U must meet E_q in a curve which is the strict transform of a quartic curve with double points at the points q_i, $i = \overline{1,6}$, on $E_{q,0}$. Thus the points q_i lie on a conic, or are the points of intersection of four lines in the plane E_q, where no three lines meet in a point. Let us exclude the first of these cases. In fact the strict transform on U of each plane through q intersects E_q in the pullback of a line whose image in V is a twisted cubic curve. If the 6 points lie on a conic, the image of the line in V would be a plane cubic curve and S_1 would be singular, contradiction.
Thus we may consider the map of Σ into $\mathbf{P}^4$ as defined by the quartics containing the curve C_B and a line through q. Together, the two curves form a curve C_g of degree 10 and genus 11, which is linked in the complete intersection of two quartics to a non-hyperelliptic curve of genus 3 and degree 6. Therefore C_g is defined by the maximal minors of a 4×5-matrix with linear entries. The 5 minors define the map to $\mathbf{P}^4$ and the image of $\mathbf{P}^3$ is a determinantal quartic. Moreover, the exceptional surface over q is mapped two to one onto the plane spanned by C, so the quartic is singular along this plane.□

2.31. Construction of E. It is clear from the above description how to construct such a surface. Start with a smooth plane cubic through the 6 points of intersection of four lines. On the cone over this curve with vertex q, look at curves linear equivalent to the plane curve plus the lines through the 6 special points. Such a curve is linked on the cone via a quartic with triple point at q to three lines through the vertex, which span $\mathbf{P}^3$, so it is clear from Bertini that one can find a curve C_B in the above linear system which is smooth outside the vertex q. Let U denote the blow up, first of $\mathbf{P}^3$ at q and then of the strict transform of C_B. Let $S_{3,U}$ and E_q denote the strict transforms on W of the cubic cone and of the exceptional divisor over q, respectively. If h denotes the pullback of a plane, and E_B denotes the exceptional divisor over C_B, then $S_{3,U}$ belongs to the linear system $|3h - E_B - 3E_q|$ of divisors on U. The linear system

$$|d| = |h + S_{3,U}| = |4h - E_B - 3E_q|$$

of divisors on U defines the map $\varphi_{|d|} : U \to \mathbf{P}^6$. Consider the linear system of divisors

$$|\Sigma_U| = |7h - 2E_B - 4E_q| = |d + S_{3,U} + 2E_q|$$

on W. Observe that

$$(\Sigma_U)_{|E_q} \equiv L_1 + L_2 + L_3 + L_4.$$

Furthermore, the linear system $|\Sigma_U|$ has base points only on $S_{3,U}$. In fact restricting to this scroll one can easily show that there are no base points outside E_q. Therefore, by Bertini, one can choose a smooth member $\widetilde{S_1} \in |\Sigma_U|$, making sure that

$$(\widetilde{S_1} \cap S_{3,U}) \cap (\widetilde{S_1} \cap E_q) = \emptyset.$$

The canonical divisor of $\widetilde{S_1}$ is given by adjunction:

$$K_U \equiv -4h + 2E_q + E_B,$$

so

$$K_{\widetilde{S_1}} \equiv (K_U + \widetilde{S_1})_{|\widetilde{S_1}} \equiv (S_{3,U} + E_q)_{|\widetilde{S_1}} \equiv (\widetilde{S_1} \cap S_{3,U}) + L_1 + L_2 + L_3 + L_4.$$

Therefore $L_1 + \ldots + L_4$ is part of the canonical divisor on $\widetilde{S_1}$. Now if $1 \leq i \leq 4$, then

$$K_{\widetilde{S_1}} \cdot L_i = L_i^2$$

since L_i does not meet any of the other components of $K_{\widetilde{S_1}}$. Thus, the curves L_i, $i = \overline{1,4}$, are (-1)-curves on $\widetilde{S_1}$, which are blown down on $S_1 = \varphi_{|d|}(\widetilde{S_1})$.

Pick a general line through q. Together with C_B it forms a curve of degree 10, genus 11, which is defined by the 4-minors of a 4×5 matrix with linear entries. The line is mapped to a line L in $\mathbf{P}^6$, which is a trisecant of S_1. The composition of $\varphi_{|d|}$ and the projection from L into $\mathbf{P}^4$ is given by the quartic minors above, and maps S_1 onto a surface S. As in the case of the rational surface (cf. 2.10) the map defined by these quartics would get double points only from the 4-secants to the base curve. Therefore it is easy to check that S is embedded in $\mathbf{P}^4$.

Proposition 2.32. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi = 9$ and $\chi = 3$, then $K^2 = 3$, $p_g = 2$, $q = 0$, and S has exactly one (-2)-curve A such that S is embedded in $\mathbf{P}^4$ by the linear system $|2K - A|$.*

Proof. We first study the pencil of canonical curves $|K|$. Now, $H \cdot K = 6$ and $p_a(K) = K^2 + 1 = 4$, so a general integral curve in the pencil must be canonically embedded by $|H|$. Thus, if K is a general canonical curve and we consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(K - H) \longrightarrow \mathcal{O}_S(2K - H) \longrightarrow \mathcal{O}_K(2K - H) \longrightarrow 0$$

of sheaves on S, then we must have that $h^0(\mathcal{O}_K(2K - H)) = 1$. On the other hand $h^1(\mathcal{O}_S(K - H)) = h^1(\mathcal{O}_S(H)) = 0$ by the Severi's theorem and Riemann-Roch, so we get that $h^0(\mathcal{O}_S(2K - H)) = 1$. Let A be the curve of $|2K - H|$. Then $H \cdot A = 2$ and $K \cdot A = 0$ and $A^2 = -2$, so A is a (possibly reducible) (-2)-curve of degree two on S.□

2.33. Construction of F. For existence consider the image S_0 of S by the bicanonical map. Then $S_0 \subset \mathbf{P}^5$. If x, y are linearly independent sections of $\mathcal{O}_S(K)$, then there is a quadratic relation between the sections x^2, xy, y^2 of $\mathcal{O}_S(2K)$, therefore S_0 lies on a quadric Q of rank 3. Let $\rho : X \to Q$ be the desingularization map. $X = \mathbf{P}(\mathbf{E})$ is a $\mathbf{P}^3$-bundle over a $\mathbf{P}^1$. On X there is a divisor corresponding to the section of the bundle $E \otimes \mathcal{O}_{\mathbf{P}^1}(-2)$, call it B, and let F denote a fiber of the natural projection $\pi : X \to \mathbf{P}^1$. B is contracted to the vertex plane of Q by ρ, in fact the map ρ is defined by the linear system of divisors $|B + 2F|$. We will construct a surface S_1 on X which is mapped onto S_0 in Q by ρ. In fact S_1 is linked in the complete intersection of two divisors $D_1 \in |2B + 6F|$ and $D_2 \in |3B + 6F|$ to three quadric surfaces $Q_0+Q_1+Q_2$, which are fibers of the projection $\pi : D_1 \to \mathbf{P}^1$. Now, S_0 has one quadratic singularity; this is imposed by a quadratic singularity on D_2, which is cut transversally by D_1, and which does not meet B or any of the quadric surfaces Q_i. A Bertini argument assures the smoothness of S_0 outside the singularity, and that S_0 has a quadratic singularity. The projection from the singular point to $\mathbf{P}^4$ induces an embedding as soon the fiber Q_p of D_1 which meets the singularity is smooth.

Remark 2.33. *An upshot of this is that the projection of $D_2 \subset Q$ to $\mathbf{P}^4$ is a quartic hypersurface with a plane, which contains S.*

In fact the pencil of surfaces $D_2 \cap F$ are mapped to cubic surfaces which are residual to a plane in hyperplane sections of the quartic.

Proposition 2.34. *If S is a smooth surface of degree 10 with $\pi = 10$ in $\mathbf{P}^4$, then S is a regular elliptic surface with two (-1)-lines and $p_g = 2$; it is the projection from a point p in $\mathbf{P}^5$ of a surface S_0 which is linked $(2,3,3)$ to a Del Pezzo surface of degree 6, such that both S_0 and the Del Pezzo has an improper node at p, or S is a minimal regular surface of general type with three (-2)-curves A_1, A_2 and A_3 embedded as conics and $p_g = 3$, such that S is embedded by the linear system $|2K - A_1 - A_2 - A_3|$ in $\mathbf{P}^4$.*

Proof. By proposition 1.16 and remark 1.17, it follows that if $\chi = 3$, then S has no 6-secant and two (-1)-lines, and if $\chi = 4$ then S has one 6-secant and no (-1)-lines. For the case $\chi = 3$, consider the map defined by the linear system $|H_0| = |H + E_1 + E_2|$, where E_1 and E_2 are the (-1)-lines on S. Clearly, this linear system defines a birational morphism of S into $\mathbf{P}^5$ which sends the exceptional lines to a point p. We denote by S_0 the image. Then S_0 has an improper node at p and the embedding of S into $\mathbf{P}^4$ is the projection from this point.

Lemma 2.35. *S_0 lies on one quadric and on three independent cubic hypersurfaces.*

Proof. Let C_H be a general hyperplane section through p, and consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_{S_0}(1) \longrightarrow \mathcal{I}_{S_0}(2) \longrightarrow \mathcal{I}_{C_H}(2) \longrightarrow 0.$$

Since S_0 is linearly normal in $\mathbf{P}^5$ the surface S_0 lies on a quadric as soon as C_H does. But $\mathcal{O}_{C_H}(2)$ is non-special, so C_H lies on a quadric by Riemann-Roch. On the other hand, if S_0 lies on more then one quadric then the projection of a complete intersection of two quadrics through p yields a quadric or a cubic hypersurface containing S, impossible. A similar argument with the above exact sequence, using also a general section not through p, gives the result for cubics.□

Lemma 2.36. *S_0 is contained in a complete intersection $(2,3,3)$.*

Proof. Let V be the intersection of all the cubics containing S_0. If V is a threefold, then V has degree at most 4, since it is contained in four cubics which are independent of the quadric Q. But then the projection of V from the point p has degree at most 3 and contains S, impossible.□

Let T be the surface which is linked to S_0 in a general pencil of cubics containing S on the quadric Q. Since $p_g(S_0) = 2$ and the general canonical curve is an elliptic normal curve of degree 6, the surface T lies on a pencil of irreducible quadric hypersurfaces. Since T has degree 6 it is linked $(2,2,2)$ to a surface U of degree 2. Using a formula analogous to (0.11) on Q it follows that T has sectional genus 1 and U has sectional genus -1. Locally at p, both T and U must have an improper node, so U must be two planes which meet at p. From the minimal free resolution of U, we get a minimal free resolution of S_0. In particular we see that S_0 is cut out by the quadric and the 4 cubic hypersurfaces. Therefore, by Bertini, the general T is smooth outside the node. It is the projection of a Del Pezzo in $\mathbf{P}^6$ from a point on a secant line, such that the image T in $\mathbf{P}^5$ acquires an improper node.

For the case $\chi = 4$, consider a general member $C_K \in |K|$. Since $H \cdot C_K = 8$ and $p_a(C_K) = K^2 + 1 = 5$, the linear series $|H_{|C_K}|$ is not canonical on C_K, only if C_K spans only a hyperplane in $\mathbf{P}^4$. But $p_g = h^0(\mathcal{O}_S(K)) \geq 3$, so the residual curve $D \equiv H - C_K$ has degree $H \cdot D = 2$ and is contained in a line, which is absurd. Therefore C_K is canonically embedded in $\mathbf{P}^4$ by $|H_{|C_K}|$. Consider now the cohomology associated to the exact sequence

$$0 \longrightarrow \mathcal{O}_S(K-H) \longrightarrow \mathcal{O}_S(2K-H) \longrightarrow \mathcal{O}_{C_K}(2K-H) \longrightarrow 0$$

of sheaves on S. By Severi's theorem and the Riemann-Roch theorem $h^1(\mathcal{O}_S(K-H)) = h^1(\mathcal{O}_S(H)) = 0$ and $h^0(\mathcal{O}_S(K-H)) = 0$, and by the above $\mathcal{O}_{C_K}(2K) \cong \mathcal{O}_{C_K}(H)$, so $h^0(\mathcal{O}_{C_K}(2K-H)) = 1$. Therefore $h^0(\mathcal{O}_S(2K-H)) = 1$. Let A be the curve in $|2K-H|$. Since $K \cdot A = 0$, A must be the union of (-2)-curves on S. Now $A^2 = -6$ and $p_a(A) = -2$, so A must be the union of three numerically disjoint (-2)-curves which we denote by A_1, A_2 and A_3. Thus $A_i \cdot A = A_i \cdot (2K-H) = -2$, for $i = \overline{1,3}$. But $K \cdot A_i = 0$, so we get that $H \cdot A_i = 2$, which means that the (-2)-curves A_1, A_2 and A_3 are embedded as conics in $\mathbf{P}^4$.

Therefore S is a regular surface embedded by the linear system

$$|2K - A_1 - A_2 - A_3|,$$

in $\mathbf{P}^4$. This concludes the proof of Proposition 2.34.□

For existence, the elliptic surface can be reconstructed via the linkage in $\mathbf{P}^5$, but we will give a different proof of existence using linkage in $\mathbf{P}^4$ (cf. 4.15).

For the surface of general type the easiest proof of existence is by using linkage in $\mathbf{P}^4$, (cf. 4.20). Alternatively, consider the image S_1 of S by the bicanonical map. Since the sections of $\mathcal{O}_S(2K)$ contain the square of the sections of $\mathcal{O}_S(K)$, the surface S_1 must lie on a four-dimensional cone X over a Veronese surface in $\mathbf{P}^7$. Let X_0 be the natural desingularization of X. Then

$$X_0 = \mathbf{P}(E) = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(2) \oplus \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}).$$

Let $\rho : X_0 \to X$ be the desingularization map. On X_0 there is a divisor corresponding to the section of the bundle $E \otimes \mathcal{O}_{\mathbf{P}^2}(-2)$, call it B_0, and let F denote the pullback of a line by the natural projection $\pi : X_0 \to \mathbf{P}^2$. B_0 is contracted to the vertex line of X

by ρ, and in fact the map ρ is defined by the linear system of divisors $|B| = |B_0 + 2F|$. X_0 has a canonical divisor $K_0 \equiv -3B - F$, and we compute for the intersection numbers $B^4 = 4$ and $B^3 \cdot F = 2$ and $B^2 \cdot F^2 = 1$. For two general members G_1 and G_2 of $|2B|$, the complete intersection $G_1 \cap G_2$ is a smooth surface Σ. By adjunction, $B - F$ restricts to the canonical divisor K_Σ on Σ. Thus Σ is a regular surface with $\chi(\Sigma) = 4$ and $K_\Sigma^2 = 4$. Since $B^3 \cdot B_0 = 0$, Σ is embedded by ρ in $\mathbf{P}^7$ as the complete intersection of X and two quadric hypersurfaces Q_1 and Q_2. The surface S_1 is such a complete intersection with three linearly independent quadratic singularities. The map to $\mathbf{P}^4$ is the projection from the plane spanned by these nodes.

3 Syzygies

To describe the minimal free resolution of the ideal sheaf of a surface S in $\mathbf{P}^4 = \mathbf{P}(V)$ we use Beilinson's spectral sequence [Bei] whose E_1-terms are

$$E_1^{pq} = H^q(\mathbf{P}(V), F(p)) \otimes \Omega^{-p}(-p)$$

and which converges to

$$E^m = \begin{cases} F, & \text{for } m{=}0; \\ 0, & \text{otherwise;} \end{cases}$$

where $F = \mathcal{I}_S(m)$ for a suitable m. In the sequel, the proper twist m will always be 4. Furthermore for the differentials we have the following:

Lemma [Bei] 3.1. *There are canonical isomorphisms $Hom(\Omega^i(i), \Omega^j(j)) \cong \Lambda^{i-j}V$ defined by contraction and the composition of morphisms coincides with multiplication in ΛV.*

Lemma [D] 3.2. *Let A be a $s \times t$ matrix with entries in V defining the morphism*

$$A : \bigoplus_t \Omega^i(i) \longrightarrow \bigoplus_s \Omega^{i-1}(i-1).$$

If A is pointwise surjective then for any nontrivial linear combination of the rows of A $(a_1, ..., a_t)$ we have that $dim_k\ span_V(a_1, \ldots, a_t) \geq i+1$.

The irreducibility claims in the table at the beginning of the paper will be consequences of the following statement:

Lemma 3.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be two vector bundles on $\mathbf{P}^4$ of ranks r and respectively $r+1$ and let φ_1 and φ_2 be two injective morphisms between $\mathcal{E}$ and $\mathcal{F}$. Assume also that both determinantal loci $S_i = \{\, p \in \mathbf{P}^4 \mid \operatorname{rk} \varphi_i(p) < r \,\}$, $i = 1, 2$ have the expected codimension two. Then S_1 and S_2 lie in the same irreducible component of the Hilbert scheme.*

Proof. Let $X = \operatorname{Spec}(k[t])$ and let $\varphi = (1-t)\varphi_1 + t\varphi_2$. One deduces easily the existence of an open subset of X containing 0 and 1 and parameterizing a flat deformation from S_1 to S_2. See also [K] or [BoM].□

The results of this section will recover some of the constructions via the Eagon-Northcott complex performed in [DES].

Proposition 3.4. *The ideal sheaf of a smooth rational surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$, embedded by*

$$H \equiv 8\pi^* l - \sum_{i=1}^{12} 2E_i - \sum_{j=13}^{18} E_j$$

has a minimal free resolution:

$$\begin{array}{ccccccccc}
 & & 2\mathcal{O}(-4) & & & & & & \\
 & & \oplus & & & & & & \\
0 \leftarrow \mathcal{I}_S \leftarrow & & 5\mathcal{O}(-5) & \nwarrow & 9\mathcal{O}(-6) & & 3\mathcal{O}(-7) & & \\
 & & \oplus & & \oplus & \longleftarrow & \oplus & \nwarrow & \\
 & & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & & \mathcal{O}(-9) \leftarrow 0.
\end{array}$$

Proof. The surface has one 6-secant and $\Delta(4)$ is the plane corresponding to all hyperplanes through the unique 6-secant. Lemmas 1.4, 1.15 and proposition 1.11 give also the cohomology diagram:

i ↑

	2			
			2	1
				2

→ p

$h^i(\mathcal{I}_S(p))$

where empty boxes are zeroes. Beilinson's spectral sequence provides then a presentation of the ideal sheaf:

$$0 \longrightarrow 2\Omega^3_{\mathbf{P}^4}(3) \longrightarrow \mathcal{F} \oplus 2\mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0 \qquad (*)$$

with $\mathcal{F}$ the kernel of the differential d_1:

$$0 \longrightarrow \mathcal{F} \longrightarrow 2\Omega^1_{\mathbf{P}^4}(1) \xrightarrow{d_1} \mathcal{O} \longrightarrow 0$$

See [DES] for a construction of the surface S using this presentation. From lemma 3.2 it follows that the morphism d_1 is defined by two linearly independent elements of V so an easy computation yields a minimal free resolution for the vector bundle $\mathcal{F}$:

$$\begin{array}{ccccccccc}
 & & 15\mathcal{O}(-1) & & 11\mathcal{O}(-2) & & 3\mathcal{O}(-3) & & \\
0 \leftarrow \mathcal{F} \longleftarrow & & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \nwarrow & \\
 & & \mathcal{O}(-2) & & 3\mathcal{O}(-3) & & 3\mathcal{O}(-4) & & \mathcal{O}(-5) \leftarrow 0
\end{array} \qquad (**)$$

The claim of the proposition follows now from $(*)$ and $(**)$. The three linear forms in the last syzygy above, or equivalently part of the last syzygy in the minimal resolution of $\mathcal{I}_S$, annihilate the module $I_S/(I_S)_{\leq 5}$, where I_S denotes the homogeneous ideal of S. Hence this module is supported on a line, which thus is the unique 6-secant line to S, the dual of the variety $\Delta(4)$.□

Proposition 3.5. *The ideal sheaf of a smooth rational surface of degree 10 in* $\mathbf{P}^4$, *with* $\pi = 9$, *embedded by*

$$H \equiv 9\pi^* l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k$$

has a minimal free resolution:

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} \mathcal{O}(-4) \\ \oplus \\ 10\mathcal{O}(-5) \end{array} \nwarrow 18\mathcal{O}(-6) \leftarrow 10\mathcal{O}(-7) \leftarrow 2\mathcal{O}(-8) \leftarrow 0.$$

Proof. From proposition 1.11, remark 1.12 and the proposition 2.1 it follows that S has no 6-secant line, and that $h^1(\mathcal{I}_S(4)) \leq 1$. If $h^1(\mathcal{I}_S(4)) = 1$, then $\Delta(4)$ is a plane, namely the set of hyperplanes through a line L, lying this time on S since the surface has no 6-secants. But then the argument in proposition 3.4 would show that $I_S/(I_S)_{\leq 5}$ is supported on L, which in particular means that L is not on S, a contradiction. Together with lemmas 1.4 and 1.15 we obtain therefore the following cohomology table:

i

	2			
			2	
				1

p

$h^i(\mathcal{I}_S(p))$

Beilinson's spectral sequence recovers a presentation of the ideal sheaf as in [DES]:

$$0 \longrightarrow 2\Omega^3_{\mathbf{P}^4}(3) \longrightarrow 2\Omega^1_{\mathbf{P}^4}(1) \oplus \mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0$$

and consequently the announced resolution. $\square$

Proposition 3.6. *The ideal sheaf of a smooth elliptic surface* S *of degree 10 in* $\mathbf{P}^4$, *with* $\pi = 9$ *has a minimal free resolution:*

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} \mathcal{O}(-4) \\ \oplus \\ 9\mathcal{O}(-5) \end{array} \nwarrow \begin{array}{c} 14\mathcal{O}(-6) \\ \oplus \\ \mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 5\mathcal{O}(-7) \\ \oplus \\ 2\mathcal{O}(-8) \end{array} \nwarrow \mathcal{O}(-9) \leftarrow 0\,.$$

Proof. It follows from lemmas 1.4, 1.15 and proposition 1.11 that S has no 6-secants and that $\Delta(4)$ is a line, namely the set of all hyperplanes containing the plane P of the elliptic cubic curve which is part of the canonical divisor. Moreover, lemma 1.4, 1.14 and proposition 1.15 give the following cohomology table:

i ↑

1				
	1			
		1	3	1
				1

→ p

$h^i(\mathcal{I}_S(p))$

whence Beilinson's spectral sequence provides the presentation:

$$0 \longrightarrow \mathcal{O}(-1) \oplus \Omega^3_{\mathbf{P}^4}(3) \oplus \Omega^2_{\mathbf{P}^4}(2) \longrightarrow \mathcal{F} \oplus \mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0 \qquad (*)$$

where $\mathcal{F}$ is the kernel of d_1:

$$0 \longrightarrow \mathcal{F} \longrightarrow 3\Omega^1_{\mathbf{P}^4}(1) \xrightarrow{d_1} \mathcal{O} \longrightarrow 0.$$

Since $\Delta(4)$ is a line, d_1 is given by three linearly independent elements of V so we get the minimal resolution of $\mathcal{F}$:

$$\begin{array}{ccccccc} & & & 20\mathcal{O}(-2) & & 6\mathcal{O}(-3) & \\ 0 \leftarrow \mathcal{F} \leftarrow 25\mathcal{O}(-1) & \nwarrow & & \oplus & \longleftarrow & \oplus & \nwarrow \\ & & & \mathcal{O}(-3) & & 2\mathcal{O}(-4) & \quad \mathcal{O}(-5) \leftarrow 0\,. \end{array} \qquad (**)$$

where the two linear forms of the last syzygy above are defining the plane P. From $(*)$ and $(**)$ we get easily the minimal resolution of the ideal sheaf of S.□

Proposition 3.7. *The ideal sheaf of a smooth $K3$ surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$ and one 6-secant has a minimal free resolution:*

$$\begin{array}{ccccccccc} & \mathcal{O}(-4) & & & & & & & \\ & \oplus & & & & & & & \\ 0 \leftarrow \mathcal{I}_S \leftarrow & 9\mathcal{O}(-5) & \nwarrow & 15\mathcal{O}(-6) & & 7\mathcal{O}(-7) & & \mathcal{O}(-8) & \\ & \oplus & & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \leftarrow 0 \\ & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & & \mathcal{O}(-9) & . \end{array}$$

Proof. We proceed as for the elliptic surface. In this case the cohomology diagram is the same but $\Delta(4)$ is now a plane, namely the set of all hyperplanes through the 6-secant line L. Beilinson's spectral sequence gives then a presentation:

$$0 \longrightarrow \mathcal{O}(-1) \oplus \Omega^3_{\mathbf{P}^4}(3) \oplus \Omega^2_{\mathbf{P}^4}(2) \longrightarrow \mathcal{F} \oplus \mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0 \qquad (*)$$

with $\mathcal{F}$ the kernel of d_1:

$$0 \longrightarrow \mathcal{F} \longrightarrow 3\Omega^1_{\mathbf{P}^4}(1) \xrightarrow{d_1} \mathcal{O} \longrightarrow 0.$$

The morphism d_1 is given by a triple $(v_1, v_2, v_3) \in V^3$, but since $\Delta(4)$ is a plane, v_1, v_2 and v_3 are linearly dependent so we can assume that $v_3 = 0$ and v_1, v_2 are linearly independent. Therefore $\mathcal{F} = \mathcal{F}' \oplus \Omega^1_{\mathbf{P}^4}(1)$ where

$$\mathcal{F}' = \ker(2\Omega^1_{\mathbf{P}^4}(1) \overset{(v_1,v_2)}{\twoheadrightarrow} \mathcal{O}).$$

Moreover, using the minimal free resolution of $\mathcal{F}'$ we computed in proposition 3.4, one gets for $\mathcal{F}$:

$$0 \leftarrow \mathcal{F} \longleftarrow \begin{array}{c} 25\mathcal{O}(-1) \\ \oplus \\ \mathcal{O}(-2) \end{array} \longleftarrow \begin{array}{c} 21\mathcal{O}(-2) \\ \oplus \\ 3\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} 8\mathcal{O}(-3) \\ \oplus \\ 3\mathcal{O}(-4) \end{array} \longleftarrow \begin{array}{c} \mathcal{O}(-4) \\ \oplus \\ \mathcal{O}(-5) \end{array} \leftarrow 0$$

and hence from $(*)$ the desired resolution of the ideal sheaf of S.□

As a consequence of the above two propositions we get that, for both elliptic and $K3$ surfaces, the $H^1(\mathcal{I}_S(*))$ -module is a monogeneous artinian module with Hilbert function $(1,3,1)$ over $R = k[x_0, \dots x_4]$, but the module structure is different. We use the above description to sketch a construction of such surfaces. We produce first a vector bundle $\mathcal{G}$ of rank 6 with the property that $H^1(\mathcal{G}(*)) = H^1(\mathcal{I}_S(* + 4))$ and $H^i(\mathcal{G}(*)) = 0$ for $i = 2, 3$. Let then $\mathcal{F} = \mathcal{O}(-1) \oplus \Omega^3_{\mathbf{P}^4}(3)$. In both cases the degeneracy locus of a generic $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ will be a surface with the desired invariants. Also the Eagon-Northcott complex of φ

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0.$$

will lead to the above computed resolutions. In both cases we take

$$\mathcal{G} = \ker(5\mathcal{O} \oplus 2\mathcal{O}(1) \overset{\psi}{\twoheadrightarrow} \mathcal{O}(2)),$$

where the linear part of ψ is given, without loss of generality, by x_0 and x_1 and the quadratic by $q_1, \dots q_5 \in k[x_2, x_3, x_4]$ quadrics in three variables without common zeroes. Therefore the choice of ψ is equivalent to that of a hyperplane section of the Veronese surface in $\mathbf{P}^5$. But there are only two possible types: an irreducible hyperplane section (generic one) whose corresponding quadrics define a $\mathcal{G}$ leading to the elliptic surface, or a reducible hyperplane section - two conics with a common point - whose quadrics define a $\mathcal{G}$ leading to the $K3$ surface. In the first case $\dim_k \mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 35$, while in the second case it is 36, whence we easily deduce that both families of surfaces have dimension 44.
We remark also that a variation of the hyperplane section of the Veronese surface gives a deformation of elliptic surfaces of degree 10, $\pi = 9$ to a scheme belonging to the irreducible component of the Hilbert scheme containing the $K3$ surfaces of the above type.

Proposition 3.8. *The ideal sheaf of a smooth $K3$ surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$ and with three 6-secants has a minimal free resolution:*

$$0 \leftarrow \mathcal{I}_S \leftarrow \begin{array}{c} 2\mathcal{O}(-4) \\ \oplus \\ 4\mathcal{O}(-5) \\ \oplus \\ 3\mathcal{O}(-6) \end{array} \nwarrow \begin{array}{c} 7\mathcal{O}(-6) \\ \oplus \\ 8\mathcal{O}(-7) \end{array} \longleftarrow \begin{array}{c} 2\mathcal{O}(-7) \\ \oplus \\ 7\mathcal{O}(-8) \end{array} \nwarrow 2\mathcal{O}(-9) \leftarrow 0\,.$$

Proof. Lemmas 1.4, 1.15 and proposition 1.11 give the following cohomology table:

i

1				
	1			
		1	3	2
				2

p

$h^i(\mathcal{I}_S(p))$

so Beilinson's spectral sequence gives then a presentation of $\mathcal{I}_S(4)$:

$$0 \longrightarrow \mathcal{O}(-1) \oplus \Omega^3_{\mathbf{P}^4}(3) \oplus \Omega^2_{\mathbf{P}^4}(2) \longrightarrow \mathcal{F} \oplus 2\mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0 \qquad (*)$$

where $\mathcal{F}$ is the kernel of d_1:

$$0 \longrightarrow \mathcal{F} \longrightarrow 3\Omega^1_{\mathbf{P}^4}(1) \xrightarrow{d_1} 2\mathcal{O} \longrightarrow 0.$$

Moreover, by lemma 1.6 and proposition 1.11, $\Delta(4)$ is a degenerated cubic scroll: three planes with a common line. By lemma 3.1, d_1 is defined by a matrix $D = \begin{pmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$ with entries in V whose 2×2 minors, when we regard it as a matrix of linear forms in $\check{\mathbf{P}}^4$, are cutting out the scroll $\Delta(4)$. But $\Delta(4)$ is degenerated so making, if necessary, a change of coordinates D can be brought, by lemma 3.2, to the form $D = \begin{pmatrix} u & v & 0 \\ 0 & v & w \end{pmatrix}$ with u, v, w linearly independent elements of V. Hence we can compute the minimal free resolution of $\mathcal{F}$:

$$0 \leftarrow \mathcal{F} \longleftarrow \begin{array}{c} 20\mathcal{O}(-1) \\ \oplus \\ 3\mathcal{O}(-2) \end{array} \longleftarrow \begin{array}{c} 13\mathcal{O}(-2) \\ \oplus \\ 8\mathcal{O}(-3) \end{array} \longleftarrow \begin{array}{c} 3\mathcal{O}(-3) \\ \oplus \\ 7\mathcal{O}(-4) \end{array} \nwarrow 2\mathcal{O}(-5) \leftarrow 0\,.$$

The proposition follows now easily. We remark also that the plane spanned by the 6-secant lines to S is $\mathbf{P}(span_k(u, v, w))$ and, more precisely, that the three 6-secants to S are in fact $\mathbf{P}(\overline{uv}), \mathbf{P}(\overline{uw})$ and $\mathbf{P}(\overline{vw})$.$\square$

Remark 3.9. Conversely, to construct such a surface one takes

$$\mathcal{G} = \mathcal{O} \oplus \ker(4\mathcal{O} \oplus 2\mathcal{O}(1) \xrightarrow{\psi} \mathcal{O}(2))$$

with ψ generic and $\mathcal{F} = \mathcal{O}(-1) \oplus \Omega^3_{\mathbf{P}^4}(3)$. For a generic $\psi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ the degeneracy locus $S = \{\, x \in \mathbf{P}^4 \mid \mathrm{rk}\varphi(x) < \mathrm{rk}\mathcal{F} \,\}$ is a $K3$ surface of the considered type. See also [DES] for this construction.

Proposition 3.10. *The ideal sheaf of a smooth general type surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$ has a minimal free resolution:*

$$\begin{array}{ccccccccccc}
 & & & \mathcal{O}(-4) & & & & & & & \\
 & & & \oplus & & & & & & & \\
0 \leftarrow \mathcal{I}_S & \leftarrow & & 8\mathcal{O}(-5) & \nwarrow & 13\mathcal{O}(-6) & & 6\mathcal{O}(-7) & & \mathcal{O}(-8) & \\
 & & & \oplus & & \oplus & \longleftarrow & \oplus & \longleftarrow & \oplus & \leftarrow 0 \\
 & & & 3\mathcal{O}(-6) & & 8\mathcal{O}(-7) & & 7\mathcal{O}(-8) & & 2\mathcal{O}(-9) & .
\end{array}$$

Proof. From lemma 1.4, 1.15 and proposition 1.11 we get the cohomology table:

i

2				
		2	4	2
				1

p

$h^i(\mathcal{I}_S(p))$

and that $\Delta(4)$ is a degenerated cubic scroll: three planes having a common line, corresponding to the three 6-secants of S. As in the above proposition we obtain the presentation:

$$0 \longrightarrow 2\mathcal{O}(-1) \oplus 2\Omega^2_{\mathbf{P}^4}(2) \longrightarrow \mathcal{F} \oplus \mathcal{O} \oplus \Omega^1_{\mathbf{P}^4}(1) \longrightarrow \mathcal{I}_S(4) \longrightarrow 0$$

with $\mathcal{F}$ the kernel of the map given by the matrix D in the proof of proposition 3.8:

$$0 \longrightarrow \mathcal{F} \longrightarrow 3\Omega^1_{\mathbf{P}^4}(1) \xrightarrow{D} 2\mathcal{O} \longrightarrow 0.$$

The claim of the proposition follows, and we get a similar description of the 6-secant lines.$\square$

Remark 3.11. Conversely to construct such a surface we take

$$\mathcal{G} = \ker(\mathcal{O} \oplus 6\mathcal{O}(1) \xrightarrow{\psi} 2\mathcal{O}(2))$$

for a special morphism : the 2×6 matrix which is the linear part of ψ should drop rank in exactly three points. Let $\mathcal{F} = 2\mathcal{O}(-1) \oplus 2\mathcal{O}$. Then for a generic $\varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$ one obtains a surface of general type of degree 10, with $\pi = 9$. We remark also that the three 6-secant lines to the surface are exactly the lines joining the points where the linear part of ψ is dropping rank.

Proposition 3.12. *The ideal sheaf of a smooth elliptic surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 10$ has a minimal free resolution:*

$$\begin{array}{ccccccc} & & 3\mathcal{O}(-4) & & & & \\ 0 \leftarrow \mathcal{I}_S \longleftarrow & & \oplus & \nwarrow & & & \\ & & 3\mathcal{O}(-5) & & 9\mathcal{O}(-6) \longleftarrow 5\mathcal{O}(-7) \longleftarrow \mathcal{O}(-8) \leftarrow 0. \end{array}$$

Proof. In proposition 1.16 we determined the following cohomology table:

i

2				
	1			
			1	
				3

p

$h^i(\mathcal{I}_S(p))$

Therefore we obtain a presentation of the ideal sheaf as cokernel of:

$$0 \longrightarrow 2\mathcal{O}(-1) \oplus \Omega^3_{\mathbf{P}^4}(3) \longrightarrow 3\mathcal{O} \oplus \Omega^1_{\mathbf{P}^4}(1) \longrightarrow \mathcal{I}_S(4) \longrightarrow 0 \qquad (*)$$

hence the claimed resolution. See [DES] for a construction, using $(*)$, of the surface S .□

Proposition 3.13. *The ideal sheaf of a smooth, general type surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 10$ has a minimal free resolution:*

$$\begin{array}{ccccccc} & 3\mathcal{O}(-4) & & \mathcal{O}(-5) & & & \\ & \oplus & & \oplus & & & \\ 0 \leftarrow \mathcal{I}_S \leftarrow & 3\mathcal{O}(-5) & \longleftarrow & 6\mathcal{O}(-6) & \nwarrow & 2\mathcal{O}(-7) & \\ & \oplus & & \oplus & & \oplus & \nwarrow \\ & \mathcal{O}(-6) & & 3\mathcal{O}(-7) & & 3\mathcal{O}(-8) & \mathcal{O}(-9) \leftarrow 0\,. \end{array}$$

Proof. From proposition 1.16 it follows that S has one 6-secant, that $\Delta(4)$ is formed by all hyperplanes containing the 6-secant and the following cohomology table:

i

3				
		1	2	1
				3

p

$h^i(\mathcal{I}_S(p))$

Beilinson's spectral sequence gives then a presentation of the ideal sheaf of S as cokernel of:

$$0 \longrightarrow 3\mathcal{O}(-1) \oplus \Omega^2_{\mathbf{P}^4}(2) \longrightarrow \mathcal{F} \oplus 3\mathcal{O} \longrightarrow \mathcal{I}_S(4) \longrightarrow 0 \qquad (*)$$

with $\mathcal{F}$, as in 3.4, the kernel of the differential d_1:

$$0 \longrightarrow \mathcal{F} \longrightarrow 2\Omega^1_{\mathbf{P}^4}(1) \xrightarrow{d_1} \mathcal{O} \longrightarrow 0 \qquad (**)$$

Moreover the natural map

$$H^1(\mathcal{I}_S(2)) \otimes H^0(\mathcal{O}_{\mathbf{P}^4}(1)) \longrightarrow H^1(\mathcal{I}_S(3))$$

is surjective and $\Delta(3) = \Delta(4)$. The claimed resolution follows now easily. From the description of $\Delta(3)$ we get also that the linear syzygy of the three quartic generators of the ideal sheaf of S is formed by the equations of the unique 6-secant line. The same remark applies for the linear part of the last syzygy.□

Conversely, as above, one uses $(*)$ and $(**)$ to reconstruct the surface.

4 Linkage

In this section we are going to describe minimal elements of the liaison classes to which the above studied surfaces belong and, therefore, to provide alternative constructions for some of the surfaces. For general facts concerning even liaison classes of codimension 2, locally Cohen-Macaulay subschemes of $\mathbf{P}^n$ see [PS], [LR], [BoM] and [MDP]. We'll need also in the sequel the following version of a lemma from [LR]:

Lemma 4.1. *Let Z be a codimension two, locally Cohen-Macaulay subscheme of $\mathbf{P}^n$, and define the speciality of Z as $e(Z)$:=max$\{ t \mid h^{n-2}(\mathcal{O}_Z(t)) \neq 0 \}$.*
a) If $h^0(\mathcal{I}_Z(e(Z)+n)) = 0$, then Z is a minimal element in its even liaison class.
b) If, moreover, $h^0(\mathcal{I}_Z(e(Z)+n+1)) = 0$, then Z is the unique minimal element in its even liaison class.

Proof. The lemma is stated in [LR] only for space curves but the proof works also in the general case.□

Corollary 4.2. *A smooth rational surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$, embedded by*

$$H \equiv 9\pi^* l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k$$

is minimal in its even liaison class.

Proposition 4.3. *A smooth rational surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$, embedded by*

$$H \equiv 8\pi^* l - \sum_{i=1}^{12} 2E_i - \sum_{j=13}^{18} E_j$$

is minimal in its even liaison class and is linked (4,4) to a reducible surface $Z = T_1 \cup T_2$ where T_2 is a smooth cubic scroll and T_1 is a degenerated cubic scroll, union of three planes having in common the directrix line of T_2 and such that there are no further intersection points with T_2. Moreover Z is the unique minimal scheme in its even liaison class.

Proof. Associated with the surface S, we got in proposition 2.2 three plane quartic curves C_{12}, C_{34} and C_{56} such that their corresponding planes Π_{12}, Π_{34}, resp. Π_{56} have in common a line L, the unique 6-secant line to S. Moreover, S meets each of the planes Π_{ij} in two points on L outside the curve C_{ij} so we get that each such plane has two pencils of 5-secant lines to S. Now the family of 5-secants to S is one dimensional since S is contained in a pencil of irreducible quartics. Therefore, by Le Barz's formula, we obtain that

$$T_1 = \Pi_{12} \cup \Pi_{34} \cup \Pi_{56}$$

is the union of the 5-secants to S.
S can be linked (4,4) to a surface Z of degree 6 with $\pi = 1$. T_1 is contained in any quartic hypersurface containing S so it will necessarily be a component of Z. Consider now the blowing up $p : \tilde{S} \to S$ of S in the points $\{q_1, \dots q_6\} = S \cap L$. We denote by $G_1, \dots, G_6$ the exceptional divisors. The projection of S along L defines a morphism of degree 4 $\varphi : \tilde{S} \to \mathbf{P}^2$ which is in fact the morphism induced by $\tilde{D} \equiv p^*H - \sum_{i=1}^{6} G_i$. Recall now from the end of the proof of proposition 2.2 the existence of a curve $L_1 \in \mid \pi^* l \mid$ passing through all the points q_i , $i = \overline{1,6}$ and consider then $\tilde{L_1} = (p \circ \pi)^* l - \sum_{i=1}^{6} G_i$. $\tilde{L_1}$ is mapped to a plane conic by φ since $\tilde{D} \cdot \tilde{L_1} = 2$ and obviously $\mid \tilde{D} - \tilde{L_1} \mid = p^* \mid H - \pi^* l \mid = \emptyset$. Therefore there exists a curve $\tilde{C} \equiv 2\tilde{D} - \tilde{L_1}$ on $\tilde{S}$ and we denote by C its image on S; $\deg C = 12$ and $p_a(C) = 12$.

Lemma 4.4. *C lies on a smooth cubic scroll T_2 containing L as directrix.*

Proof. We have $h^2(\mathcal{O}_S(2H - C)) = h^0(\mathcal{O}_S(K + C - 2H)) = 0$ so Riemann Roch on C and the cohomology of the exact sequence:

$$0 \longrightarrow \mathcal{I}_C(2) \longrightarrow \mathcal{O}_S(2) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0$$

give that $h^0(\mathcal{I}_C(2)) \geq 2$. Since $h^0(\mathcal{I}_C(1)) = 0$ it follows that also $h^0(\mathcal{I}_{C \cap H}(2)) \geq h^0(\mathcal{I}_C(2)) \geq 2$ for a generic hyperplane H. Let $D = p(\tilde{D})$ for a general $\tilde{D} \in \mid \tilde{D} \mid$. Then D meets C in six points on L and in six points $x_1, \dots x_6$ on S outside L.
We work for a moment on $\tilde{S}$. φ restricts to a map of degree 3 on $\tilde{C}$, so we may group the x_i's on S into two sets such that, say x_1, x_2, x_3 span a plane which contains L and x_4, x_5, x_6 span another plane which contains L. Since $\varphi(\tilde{C})$ is a conic we get that $x_1 + x_2 + x_3$ and $x_4 + x_5 + x_6$ belong to the same linear series on C. Now $L \cup \{x_1, \dots x_6\}$ is contained in at least two quadrics so we get that either $\{x_1, x_2, x_3\}$ or $\{x_1, x_2, x_3\}$ is contained in a line. But they belong, as divisors on C, to the same linear series so they are both contained in a line. Varying D we see that the lines L_1 and L_2 are describing the ruling of a scroll T_2 containing the curve C and the line L. Moreover T_2 is contained in all quadrics containing C and is smooth since L_1 and L_2 cannot meet. The scroll T_2 is rational and has a hyperplane section divisor $H_{T_2} \equiv 2l - E$, viewed as $\mathbf{P}^2$ blown up in one point. The line

L equals the directrix E, while C meets L in six points, so $C \cdot E = 6$. Since deg $C = 12$ we get on T_2: $C \equiv 9l - 6E$. $\square$

From the above lemma it follows that any conic l on T_2 cuts C in 9 points and therefore any quartic hypersurface that contains S must contain also T_2. It follows that T_2 is a component of Z whence

$$Z = T_1 \cup T_2 = \Pi_{12} \cup \Pi_{34} \cup \Pi_{56} \cup T_2$$

as claimed. For the rest of the assertions one computes $e(S) = -1$, $e(Z) = -2$ and uses lemma 4.1.$\square$

4.5. Construction of surfaces A. We start now with two cubic scrolls as above, T_1 and T_2 such that T_1 is the union of three planes Π_1, Π_2 and Π_3 through a common line L, and T_2 is smooth having L as directrix and meeting T_1 only along L. The surface $Z = T_1 \cup T_2$ has degree 6 and sectional genus $\pi = 1$.

Lemma 4.6. *The scheme Z is cut out by* 11 *quartic hypersurfaces and the generic quartic in I_Z is singular along L containing it with multiplicity* 2.

Proof. Consider the residual exact sequences

$$0 \longrightarrow \mathcal{I}_{\Pi_3 \cup T_2}(3) \longrightarrow \mathcal{I}_Z(4) \longrightarrow \mathcal{I}_{\Pi_1 \cup \Pi_2 \cup L' \cup L'', H_1}(4) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{I}_{T_2}(2) \longrightarrow \mathcal{I}_{\Pi_3 \cup T_2}(3) \longrightarrow \mathcal{I}_{\Pi_3 \cup M' \cup M'', H_2}(3) \longrightarrow 0,$$

where H_1 is the hyperplane spanned by $\Pi_1 \cup \Pi_2$, L' and L'' are rulings of T_2 such that $L \cup L' \cup L'' = H_1 \cap T_2$, while H_2 is a general hyperplane through Π_3 and M', M'' are other two rulings of T_2 defined by $H_2 \cap T_2 = L \cup M' \cup M''$. The lemma follows since $h^i(\mathcal{I}_{T_2}(2)) = 0$, for $i = 1, 2$, T_2 is defined by the minors of a 2×3 matrix with linear entries, $\mathcal{I}_{\Pi_3 \cup M' \cup M'', H_2}(3) \simeq \mathcal{I}_{M' \cup M'', H_2}(2)$ and $\mathcal{I}_{\Pi_1 \cup \Pi_2 \cup L' \cup L'', H_1}(4) \simeq \mathcal{I}_{L' \cup L'', H_1}(2)$ are globally generated, while $h^1(\mathcal{I}_{M' \cup M'', H_2}(2)) = 0$.$\square$

As a consequence of the above lemma, Z can be linked in the complete intersection of two general quartic hypersurfaces to an irreducible surface S with the desired invariants: deg $S = 10$, $\pi = 9$, $p_g = q = 0$. It is easily seen that, for general choices in the linkage, S is smooth outside the line L. To see the behavior at the intersection points with L we'll work out explicitly this linkage.

Consider the blowing-up

$$\sigma : \widetilde{\mathbf{P}^4} = \mathbf{P}(2\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(1)) \longrightarrow \mathbf{P}^4$$

of $\mathbf{P}^4$ along the line L, with exceptional divisor $E = \mathbf{P}(2\mathcal{O}_{\mathbf{P}^2}) = \mathbf{P}(3\mathcal{O}_{\mathbf{L}}) = \mathbf{P}^2 \times \mathbf{L}$. Let B be a divisor of E corresponding to a section of $3\mathcal{O}_L(1)$ and F corresponding to a fibre of the projection $\sigma : \mathbf{P}^2 \times L \to L$. If a hypersurface V of degree v contains the line L with multiplicity m, then its strict transform $\widetilde{V}$ will meet E along $\overline{V}$, numerically of type

$$\overline{V} \equiv mB + (v - 2m)F.$$

Let now V_1 and V_2 be two general quartic hypersurfaces through Z. By lemma 4.6, they have multiplicity two along L, hence $\overline{V_i} \equiv 2B$, $i = \overline{1,2}$. On the other hand the strict transforms of the scroll T_2 and of a plane Π_i cut E along $\overline{T_2} \equiv B^2$, since the cubic scroll is linked to a plane in the complete intersection of two hyperquadric cones simple along L, and along $\overline{\Pi_i} \equiv (B - F)(B - F) \equiv B^2 - 2BF$, respectively. Therefore $\overline{T_1} \equiv 3(B^2 - 2BF)$, and for a general $(4,4)$ linkage the strict transform of S on $\widetilde{\mathbf{P}^4}$ meets E in a curve equivalent to $(2B)(2B) - (B^2 + 3B^2 - 6BF) \equiv 6BF$. A Bertini argument shows now that for a general choice of the linkage, the surface S is smooth, and thus it is a rational surface as claimed. Moreover, since a curve of type BF is blown down on S, it follows that L is a 6-secant line to S. We remark also that the quartics containing S are singular along the 6-secant L.

Proposition 4.7. *A smooth $K3$ surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$ and three 6-secants is linked $(4,4)$ to a reducible surface $Z = P_1 \cup P_2 \cup P_3 \cup T$, where T is a multiplicity three structure on a plane P given, after a suitable change of coordinates, by the homogeneous ideal*

$$I_T = (\, x_0^2, x_0x_1, x_1^3, ax_1^2 + b_1b_2b_3x_0 \,)$$

with $a, b_i \in k[x_2, x_3, x_4], i = \overline{1,3}$, homogeneous forms of degree 2, resp. 1, without common factors when assuming $P = \{\, x_0 = x_1 = 0 \,\}$, and where P_1, P_2 and P_3 are planes which pairwise span all of $\mathbf{P}^4$ and cut the scheme T along multiplicity two structures on the lines $L_i = \{\, x_0 = x_1 = b_i = 0 \,\}$, for $i = \overline{1.3}$. Moreover, the scheme Z is minimal in its even liaison class and is linked $(3,4)$ to a scheme Y of degree 6, with $\pi = 1$, which is minimal in the even liaison class of S.

Proof. We shall use in the sequel notations from proposition 2.12: C is the plane quartic curve and P its plane, $D \equiv H - C$ the residual pencil, $\pi : S \to S_1$ the blowing down of the three (-1)-conics E_1, E_2 and E_3 to the points p_1, p_2 and resp. p_3. D_1 is the image of D on S_1, and corresponding $\varphi_{|D_1|} : S_1 \to \mathbf{P}^4$ the induced morphism whose image S_0 is a $(2,3)$ complete intersection with one quadratic singularity. By proposition 2.12, the points p_1, p_2, p_3 form a theta-characteristic on the general D_1 through these points. Since D_1 is canonically embedded, it follows that p_1, p_2, p_3 and the tangent plane to S_0 at p_i span only a hyperplane in $\mathbf{P}^4$, for all $i = \overline{1,3}$. Therefore we obtain the existence of a curve $G_i \in | \, D - E_i \, |$, for all $i = \overline{1,3}$ (cf. also 2.14). It has degree 4 and arithmetic genus 3 so it is a plane curve. We denote by P_i the plane spanned by G_i. Let q_1, q_2, q_3 be the base points of the pencil D in the plane P and let $L_{ij} = \overline{q_iq_j}$ be the three 6-secant lines to S. Now $G_i \cdot C = 2$, $E_i \cap C = \{q_i\}$ and $D \cap C = \{\, q_1, q_2, q_3 \,\}$, hence P_i cuts P along the line L_{jk}, for all triples $\{\, i, j, k \,\} = \{\, 1, 2, 3 \,\}$. Since S meets each of the planes P_i in two points on L_{jk} outside the curve G_i, for $\{\, i, j, k \,\} = \{\, 1, 2, 3 \,\}$, we get that each plane P_i has two pencils of 5-secants lines to S and therefore is contained in any quartic hypersurface containing S.

Now the surface S is linked $(4,4)$ to a surface Z of degree 6, with $\pi = 1$ which must have as components the planes P_i, $i = \overline{1,3}$. We consider further the scheme $X = S \cup P_1 \cup P_2 \cup P_3$. It has degree 13 and sectional genus 18 but it is not locally Cohen-Macaulay, having exactly six bad points $m_1, n_1, m_2, n_2, m_3, n_3$, where $L_{ij} \cap C = \{ q_i, q_j, m_k, n_k \}$ for all triples $\{ i, j, k \} = \{ 1, 2, 3 \}$. Therefore X is linked in the complete intersection of the above two quartic hypersurfaces to a scheme T of degree 3, with $\pi = -2$. The generic hyperplane section of T is a locally CM, connected curve of degree 3 and arithmetical genus -2, hence it is a triple structure on a line, hence T also is a multiplicity three structure on a plane. But, by Bezout's theorem, the plane P is contained in the complete intersection of the two quartic hypersurfaces, whence T must be a triple structure on P. Moreover, the liaison exact sequences give that each plane P_i cuts the scheme T along a doubling of the 6-secant line L_{jk}, for all $\{ i, j, k \} = \{ 1, 2, 3 \}$. Now a similar computation to [Ma] yields that, may be after a suitable change of coordinates, we may assume, when $P = \{ x_0 = x_1 = 0 \}$, that T is given by the homogeneous ideal $I_T = (x_0^2, x_0 x_1, x_1^3, a x_1^2 + b x_0)$ with $a, b \in k[x_2, x_3, x_4]$ homogeneous forms of degree 2, resp. 3 such that $\mathrm{codim}_P(V(a) \cap V(b)) = 2$. T is not Cohen-Macaulay exactly in the points of this finite set, therefore we must have $V(a) \cap V(b) = \{ m_1, n_1, m_2, n_2, m_3, n_3 \}$. It follows that there exist linear forms b_1, b_2 and $b_3 \in k[x_2, x_3, x_4]$ such that $b = b_1 b_2 b_3$ and such that the 6-secant lines are given by $L_i = \{ x_0 = x_1 = b_i = 0 \}$. For the last statement of the proposition one computes $e(Z) = e(Y) = -2$ and uses lemma 4.1, since neither Z nor Y are contained in a quadric hypersurface.□

Remark 4.8. The scheme Z above is the union of the 5-secant lines to S and this fits with the corresponding formula of Le Barz.

Proposition 4.9. *A smooth $K3$ surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$ and one 6-secant can be bilinked $(4,5)$ and $(4,4)$ to a reducible surface $Z = P \cup Q \cup T$, where T is a smooth cubic scroll, Q is a smooth quadric surface cutting T along two rulings and P is a plane cutting the scroll T along its directrix, which in turn is the 6-secant of the surface S. Moreover, Z is the unique minimal element in its even liaison class.*

Proof. Let V denote the unique quartic hypersurface containing $S \subset \mathbf{P}^4$, cf. proposition 3.7. Let also H_{12} denote the hyperplane which contains the exceptional lines E_1 and E_2. By proposition 2.21, the residual curve $H_{12} \cap S - E_1 - E_2$ decomposes into two plane quartic curves A and B, with $A^2 = B^2 = 0$ and $A \cdot B = 2$. The planes π_1 and π_2 spanned by A and B, respectively, meet along the unique 6-secant L to the surface S. Thus $\pi_i \subset V$, for $i = 1, 2$, and residual to them in $V \cap H_{12}$ there exists a smooth quadric surface Q meeting S along $E_1 \cup E_2$. By proposition 3.7, the quintic hypersurfaces containing S cut out the surface outside the 6-secant L, so S can be linked in the complete intersection of V and a general quintic hypersurface W to an irreducible surface Y of degree 10, sectional genus 9. From the liaison exact sequences we get $h^0(\mathcal{I}_Y(4)) = 2$ this time, so we can link Y on V to a locally Cohen-Macaulay scheme Z, with $\deg Z = 6$ and $\pi(Z) = 1$. We identify in the sequel the components of Z. First we remark that, since $Q \cap Y = W \cap Q - E_1 - E_2$ is a curve of type $(3,5)$ on the quadric, Q is contained in all quartic hypersurfaces containing Y, hence Q is a component of Z. On another side, from lemma 1.4 and proposition 3.7, we deduce that $\Delta(3)$ is a line, thus there exists a plane P such that all the hyperplanes $H \supset P$ have the property that $h^0(\mathcal{I}_{H\cap S}(3)) = 1$. Any quartic containing $H \cap S$ depends on the unique cubic through $H \cap S$, therefore P lies inside V.

Claim. *P meets the surface S along a zero-dimensional scheme.*

Proof of the claim. Assume that $P \cap S$ contains a curve C. The general residual curve $H \cap S - C$, for $H \supset P$, is contained then in an irreducible quadric surface inside V, since again any quartic surface in H containing $H \cap S$ depends on the unique cubic through $H \cap S$. Thus V is singular along P, and by linkage $P \cap Y$ is a curve of degree 5. It follows that P is contained in all quartic hypersurfaces containing Y, and thus it is a component of Z. But the liaison exact sequences yield $Z \cap S = K = E_0 \cup E_1 \cup E_2$, hence the intersection curve $C \subset P \cap S$ must then satisfy $C \subset E_0 \cup E_1 \cup E_2$, which is a contradiction.□

Since P meets S only in points it follows that $P \cap Y$ contains a plane quintic, and thus as above, by Bezout, P must be a component of the scheme Z. The plane P contains the 6-secant line L since $\Delta(3) \subset \Delta(4) = L^\vee = \mathbf{P}^2$, but it doesn't lie in $H_{12} = \mathrm{span}_k(E_1, E_2)$. Therefore $P \cap Q = L \cap Q = \{p_1, p_2\}$ and residual to $P \cup Q$ in Z there is an irreducible surface T of degree 3, which contains the rational normal curve E_0. Moreover, since $Q \cap Y$ is a curve of type $(3,5)$ on the quadric, it follows that T intersects Q along two skew lines, and hence T is a smooth cubic scroll meeting the quadric surface along two rulings. The other claims of the proposition follow from lemma 4.1.□

4.10. Construction of surfaces D. Let, as above, T be a rational cubic scroll in $\mathbf{P}^4$ and let Q be a smooth quadric surface cutting T along two lines in its ruling, say L_1 and L_2 and consider next a plane P passing through the directrix L of T, cutting the scroll only along this line, and not contained in the hyperplane spanned by the quadric surface. Let again $Z = P \cup T \cup Q$. $\deg Z = 6$ and $\pi(Z) = 1$ and Z is locally Cohen-Macaulay and a local complete intersection except for the points $\{p_i\} = L \cap L_i$, $i = \overline{1,2}$. We prove in the sequel that Z can be backwards linked $(4,4)$ and $(4,5)$ to a smooth $K3$ surface of type D. First a lemma.

Lemma 4.11.
a) The scheme $X = T \cup Q$ is a degenerated elliptic quintic scroll in $\mathbf{P}^4$ and, in particular, the homogeneous ideal I_X is generated by 5 cubics.
b) The homogeneous ideal I_Z is generated by 10 quartics and one quintic. Moreover the quartics cut out the scheme Z outside L and the generic quartic is singular along and contains L with multiplicity two.

Proof. Let H be the hyperplane spanned by the quadric surface Q. The residual exact sequence

$$0 \longrightarrow \mathcal{I}_T(2) \longrightarrow \mathcal{I}_X(3) \longrightarrow \mathcal{I}_{H\cap X,H}(3) \longrightarrow 0$$

remains exact after taking global sections since $h^1(\mathcal{I}_T(2)) = 0$. Therefore $h^0(\mathcal{I}_X(3)) = 5$ and it suffices to check whether T is cut out by quadrics and $H \cap X$ by cubics. The former is clear since T is defined by the minors of a 2×3 matrix with linear entries, while $\mathcal{I}_{H\cap X,H}(3) = \mathcal{I}_{Q\cup L,H}(3) \simeq \mathcal{I}_{L,H}(1)$ is clearly global generated. Moreover, taking cohomology in the above sequence we get

$$h^1(\mathcal{I}_X(k)) = 0 \quad \text{for all } k, \quad h^2(\mathcal{I}_X(k)) = 0 \quad \text{for} \quad k \neq 0, \quad \text{and} \quad h^2(\mathcal{I}_X) = h^2(\mathcal{I}_{Q\cup L}) = 1,$$

thus Beilinson's spectral sequence gives a resolution of the form

$$0 \longrightarrow \Omega^3_{\mathbf{P}^4}(3) \longrightarrow 5\mathcal{O}_{\mathbf{P}^4} \longrightarrow \mathcal{I}_X(3) \longrightarrow 0$$

and hence X is a degenerated elliptic quintic scroll. Consider now the exact sequence

$$0 \longrightarrow \mathcal{I}_X(k-1) \longrightarrow \mathcal{I}_Z(k) \longrightarrow \mathcal{I}_{Z\cap H',H'}(k) \longrightarrow 0$$

where H' is a general hyperplane containing P. It remains also exact after taking global sections since $h^1(\mathcal{I}_X(k)) = 0$, for all k. Now $Z \cap H' = P \cup D \cup f_1 \cup f_2$, where $D = Q \cap H'$ is a smooth conic and f_1, f_2 are rulings of T, thus $h^0(\mathcal{I}_{Z\cap H'}(k)) = h^0(\mathcal{I}_{D\cup f_1\cup f_2}(k-1))$ and we compute $h^0(\mathcal{I}_{D\cup f_1\cup f_2}(2)) = 0$, $h^0(\mathcal{I}_{D\cup f_1\cup f_2}(3)) = h^0(\mathcal{O}_{\mathbf{P}^3}(3)) - 2h^0(\mathcal{O}_{\mathbf{P}^1}(3)) - h^0(\mathcal{O}_{\mathbf{P}^1}(6)) = 5$ and $h^0(\mathcal{I}_{D\cup f_1\cup f_2}(4)) = 16$. Moreover, the homogeneous ideal $I_{D\cup f_1\cup f_2}$ is generated by the 5 cubics and one extra quartic and, since by *a*) I_X is generated by 5 cubics, the first assertion of *b*) follows. For the second part it is enough to observe that the cubics in $H^0(\mathcal{I}_{D\cup f_1\cup f_2}(3))$ vanishing on L and cut out, scheme-theoretically in fact, $D \cup f_1 \cup f_2 \cup L$.□

As a consequence of the above lemma Z can be linked in the complete intersection of two quartic hypersurfaces to an irreducible surface Y, with deg $Y = 10$, $\pi(Y) = 9$ which contains and is singular along L and which is smooth outside this line. Y can be further linked $(4,5)$ to a surface S with the desired invariants: deg$S = 10$, $\pi(S) = 9$, and from the liaison exact sequences $p_g = 1$, $q = 0$. It is easily seen that S is smooth outside L, for a general choice of the linkages. In order to see the behavior at the intersection with L we'll work on the blow-up $\widetilde{\mathbf{P}^4}$ of $\mathbf{P}^4$ along L, like in the proof of (4.5).

Namely, keeping the same notations for the blow-up, if V_1 and V_2 are two general quartic hypersurfaces containing Z, then, by lemma 4.10, they have multiplicity two along L, thus $\overline{V_i} \equiv 2B$, $i = \overline{1,2}$. On the other hand the strict transforms of P and Q cut E along $\overline{P} \equiv (B-F)(B-F) \equiv B^2 - 2BF$ and $\overline{Q} \equiv (B-F)2F \equiv 2BF$ respectively. Also, as argued in (4.5), the strict transform of T cuts E along $\overline{T} \equiv B^2$. It follows that, for a general $(4,4)$ linkage, the strict transform of Y on $\widetilde{\mathbf{P}^4}$ meets E in a curve equivalent to $(2B)(2B) - (B^2 + (B^2 - 2BF) + 2BF) \equiv 2B^2$. A local computation shows that the general quintic hypersurface containing Y has multiplicity one along L. Therefore, for a general choice of the $(4,5)$ linkage, the strict transform of S on $\widetilde{\mathbf{P}^4}$ will meet E in a curve equivalent to $2B(B+3F) - 2B^2 \equiv 6BF$. A Bertini argument shows now that for a general choice of the linkage, the surface S residual to Y is smooth. Moreover, since a curve of type BF is blown down on S, it follows that L is a 6-secant line to S.

To show that S is indeed a $K3$ surface of type D we determine the one dimensional components in the intersections $S \cap Q$ and $S \cap T$. The liaison exact sequence for Y gives

$P \cap Y \equiv 3H_P - K_P - L \equiv 5H_P$, $Q \cap Y \equiv 3H_Q - K_Q - [Q \cap T] \equiv 5l_1 + 3l_2$, where l_1 and l_2 denote the classes of the two rulings of the quadric, and $T \cap Y \equiv 3H_T - K_T - [T \cap (P \cup Q)] \equiv 3(C_0 + 2f) - (-2C_0 - 3f) - C_0 - 2f \equiv 4C_0 + 7f$, with C_0 denoting the numerical class of the directrix L on the scroll T and f the class of a ruling. The one-dimensional components of the intersection scheme $S \cap Z$ are the residuals (in term of conductor ideals) of the above curves in the complete intersection of Z with the quintic hypersurface used in the linkage of Y with S. Therefore, for a general choice of the linkage, P cuts S only in points, T cuts S along a scheme whose one-dimensional part K_1 is equivalent to $5H_C - [T \cap Y] \equiv C_0 + 3f$ and Q cuts S along a curve K_2 equivalent to $5H_Q - [Q \cap T] \equiv 2l_2$ plus a zero dimensional scheme. On the other hand, the scheme $K_1 \cup K_2$ is exactly the canonical divisor of S. Now the liaison exact sequence

$$0 \longrightarrow \mathcal{I}_{Y \cup Z}(5) \longrightarrow \mathcal{I}_{Y \cup P \cup T}(5) \longrightarrow \mathcal{O}_Q \longrightarrow 0$$

remains also exact after taking global sections, hence the quintics in $H^0(\mathcal{I}_Y(5))$ cut on Q a linear system whose fixed part is exactly $Q \cap Y$. Therefore, for a general choice of the $(4,5)$ linkage, the curve $K_2 \subset Z \cap S$ is reduced and hence it is the union of two skew lines, say E_2 and E_3, in the ruling of Q containing L_1 and L_2. Eventually, the adjunction formula on S yields $E_i^2 + KE_i = 2E_i^2 = 2p_a(E_i) - 2 = -2$, $i = \overline{2,3}$ and thus E_2 and E_3 are exceptional lines on S. It follows that $E_1 := K_1$ is a (-1) quartic on S and hence S is, as claimed, a $K3$ surface of type D; i.e., embedded by a linear system of type

$$H = H_{min} - 4E_1 - E_2 - E_3.$$

Proposition 4.12. *A smooth elliptic surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$ can be bilinked $(4,5)$ and $(4,4)$ to a reducible surface $Z = M \cup Q$ where M is a locally Cohen-Macaulay multiplicity four structure on a plane P given, after a suitable change of coordinates and assuming $P = \{x_0 = x_1 = 0\}$, by the homogeneous ideal*

$$I_M = (x_0, x_1)^3 + (gx_0^2 - fx_0x_1, hx_0^2 - fx_1^2, hx_0x_1 - gx_1^2)$$

with $g, h, f \in k[x_2, x_3, x_4]$, homogeneous forms of degree 2 without common factors, and where Q is a smooth quadric surface in a hyperplane non containing P. Moreover, Z is the unique minimal element in its even liaison class.

Proof. We keep the notations made in the proof of proposition 2.23 and the lemmas thereafter. Namely, on W the blow up, first of $\mathbf{P}^3$ at the vertex q, then of the strict transform of C_B and then of the strict transform of the line $L \subset \mathbf{P}^6$, h will denote the pullback of a plane, E_L the exceptional divisor over L, E_q the strict transform of the exceptional divisor over q, and E_B the strict transform of the exceptional divisor over C_B. Then, for the strict transforms $\widetilde{S_1}$ and $\widetilde{H}$ of the elliptic surface and of a hyperplane in $\mathbf{P}^4$ respectively, we have

$$\widetilde{S_1} \in |7h - 2E_B - 4E_q|$$

and

$$\widetilde{H} \in |4h - E_B - 3E_q - E_L|.$$

By proposition 3.6, the quintic hypersurfaces containing S cut out the surface, so S can be linked in the complete intersection of the unique quartic X containing it and a general quintic hypersurface to an irreducible surface Y of degree 10 and sectional genus 9. Now, as in proposition 4.9, $h^0(\mathcal{I}_Y(4)) = 2$ so we can link further Y in the complete intersection of X with a general quartic hypersurface to a locally Cohen-Macaulay scheme Z with $\deg Z = 6$, $\pi(Z) = 1$. We determine in the sequel the components of Z. In terms of linear equivalence, for the strict transform $\widetilde{Z}$ of Z, we have

$$\widetilde{Z} \equiv \widetilde{S_1} - \widetilde{H} \equiv 3h - E_B - E_q + E_L.$$

But a divisor in $|3h - E_B - E_q|$ is mapped down to $\mathbf{P}^3$ onto a cubic surface containing the curve C_B, which thus necessarily coincides with the cone S_3. It follows that in fact $\widetilde{Z}$ splits as

$$\widetilde{Z} \equiv (3h - E_B - 3E_q) + 2E_q + E_L,$$

and since $S_{3,W} \equiv 3h - E_B - 3E_q$ gets contracted via the map to $\mathbf{P}^4$ onto the plane cubic C, while E_q is mapped $2:1$ onto the plane P spanned by C, we deduce that Z is the union of a multiplicity 4 structure on the plane P and of a quadric Q. The intersection between S and Q is exactly the union of the 3 exceptional lines E_i, $i = \overline{1,3}$, so in particular the quadric surface Q is smooth. We identify in the sequel the multiplicity four structure M on the plane P. Let Q' be the residual quadric surface of Q in the intersection of the quartic hypersurface X with the hyperplane spanned by Q. It is easily seen that the union $Y \cup Q$ is linked in the complete intersection of X and a sextic hypersurface to the union $S \cup Q'$. The scheme $S \cup Q'$ is locally Cohen-Macaulay since any point on this scheme which is not of this type would lie necessarily also on Y and thus would be a base point of the whole linear system where Y is moving. But $|Y|$ has no base points on the surface S since this last one is cut out by quintic hypersurfaces on X. We deduce that $S \cup Q'$ is locally Cohen-Macaulay, and then by linkage $Y \cup Q$ is also of this type. Finally, the above linkage between Y and Z implies that the multiplicity four structure M is a locally Cohen-Macaulay scheme. Moreover, the above linkages yield $h^0(\mathcal{I}_M(2)) = h^0(\mathcal{O}_{Y\cup Q}(K-H)) = h^0(\mathcal{I}_{S\cup Q'}(4)) - 1 = 0$, whence M is a multiple structure of type I_{X_9} in the list of [Ma], i.e., of the type claimed in the statement of the proposition, since all the other multiplicity four Cohen-Macaulay structures in the list lie on at least one hyperquadric.□

Proposition 4.13. *A smooth general type surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 9$ can be bilinked $(4,5)$ and $(4,4)$ to a reducible surface $Z = T \cup P_1 \cup P_2 \cup P_3$, where T is a smooth cubic Del Pezzo surface and P_1, P_2 and P_3 are three planes in general position, cutting T along the three coplanar 6-secants of the surface S. Moreover Z is a minimal element in the even liaison class of S.*

Proof. We study $\Delta(3)$. Recall from proposition 1.11 that $\Delta(4)$ is the union of three planes meeting along a line L, corresponding to the pencil of hyperplanes through the $\mathbf{P}^2$ spanned by the 6-secants. On the other side, $\Delta(3) \subset \Delta(4)$ is defined by the minors of a 2×4 matrix with linear entries, and moreover it contains the line L by lemma 1.14 and the remark before. Therefore, either $\Delta(3)$ contains a plane, or $\Delta(3)$ is a degenerated rational quartic curve, namely a union of lines $L \cup L_1 \cup L_2 \cup L_3$ (some might coincide), such that no 3 lines of type L_i are coplanar (if they would be coplanar, the plane they span lies inside $\Delta(3)$). Assume first that there exists a plane Π such that $\Pi \subset \Delta(3)$, and let H be a hyperplane section of S corresponding to a general point of Π. Then $h^0(\mathcal{I}_H(3)) = 1$, $h^1(\mathcal{I}_H(3)) = 3$ and $h^0(\mathcal{I}_H(4)) = 4$, because otherwise $H \in \Delta(3,4)$ and thus, by lemma 1.14 H, would be reducible, which is impossible. It follows that the unique quartic hypersurface V containing S (cf. 1.11) splits off a plane in each hyperplane parameterized by Π. Hence V is a cone with vertex the line L. But this is impossible since, by lemma 1.8, S has no pencils of plane curves. Therefore $\Delta(3)$ is a curve, and we may write $\Delta(3) = L \cup L_1 \cup L_2 \cup L_3$, where $L \cap L_i \neq \emptyset$, for all i. We observe also that two lines L_i and L_j are necessarily disjoint. Otherwise, the two corresponding dual planes $L_i^\vee$ and $L_j^\vee$ would span only a hyperplane in $\mathbf{P}^4$, and in that hyperplane V would decompose as $V = L_i^\vee \cup L_j^\vee \cup Q$, where Q is a quadric surface such that $Q \supset \operatorname{span}(L_i, L_j) \cap S$, absurd. Therefore the planes $P_i := L_i^\vee$ meet pairwise only in points. As in proposition 4.9 one shows that they meet S only along zero-dimensional schemes, thus a Bezout argument shows that the 3 planes P_i are components of the bilinked scheme Z. Residual to them, there exist a scheme T of degree 3, which contains the intersection curve $D = S \cap Z \in |K_S|$. For a general choice of the linkage in the statement of the proposition, D is an integral canonical curve of degree 6 (cf. proof of 2.32), and one deduces easily that T is a smooth cubic Del Pezzo surface. The last statement in the proposition follows from lemma 4.1.□.

Proposition 4.14. *A smooth elliptic surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 10$ can be linked $(4,4)$ to a reducible surface $Z = T_1 \cup T_2$, union of a smooth Del Pezzo surface T_1 of degree 4 and a smooth quadric surface T_2 such that the hyperplane of the quadric cuts T_1 along $G_1 + G_2 + F_1 + F_2$, consisting of four lines with $G_1 \cdot G_2 = F_1 \cdot F_2 = 0$ and such that F_1 and F_2 are members of one of the rulings of T_2 and G_1 and G_2 meet transversally the quadric. Furthermore the scheme Z is minimal in its even liaison class and can be linked $(3,4)$ to a scheme Y of degree 6, with $\pi = 2$ which is minimal in the even liaison class of the surface S.*

Proof. We recall from proposition 2.25 that S has only two exceptional lines E_1 and E_2 and that $K \equiv F + E_1 + E_2$, with $|F|$ a base point free pencil defining the elliptic fibration. Consider the residual curve $D \equiv H - E_1 - E_2$ in the intersection of S with the hyperplane spanned by the two exceptional lines. Then D is a curve of degree 8 and arithmetic genus 8, hence Riemann-Roch gives $\chi(\mathcal{O}_D(2)) = 9$. The cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(2H - D) \longrightarrow \mathcal{O}_S(2H) \longrightarrow \mathcal{O}_D(2H) \longrightarrow 0$$

gives $h^1(\mathcal{O}_D(2)) = 0$ since, by lemma 1.4, $h^1(\mathcal{O}_S(2)) = 0$ and also $h^2(\mathcal{O}_S(2H - D)) = h^0(\mathcal{O}_S(D - 2H + K)) = 0$. Therefore taking global sections in the exact sequence

$$0 \longrightarrow \mathcal{I}_D(2) \longrightarrow \mathcal{O}_{\mathbf{P}^3}(2) \longrightarrow \mathcal{O}_D(2) \longrightarrow 0$$

shows that D lies on a quadric surface T_2. It can't be a cone since then D would have genus 9. Also T_2 is not reducible since S has no 6-secants and, by corollary 1.2, no plane curves of degree ≥ 5. It follows that T_2 is a smooth quadric in the hyperplane spanned

by E_1 and E_2 and D is a curve of type $(3,5)$ on it. Therefore, since S is contained in a net of irreducible quartics, and by Le Barz's formula there are two 5-secants to S meeting a general plane in $\mathbf{P}^4$, we may characterize T_2 as the union of the 5-secants to S. Any quartic hypersurface containing S must also contain T_2.
S can be linked $(4,4)$ to a scheme Z of degree 6, with $\pi = 2$ which, by the above discussion, contains T_2 as a component. Therefore we get $Z = T_1 \cup T_2$, with T_1 a scheme of degree 4 such that $S \cap T_1 \in \mid 2H - F \mid$. On another side T_1 is the projection from the improper node of the Del Pezzo surface $T \subset \mathbf{P}^5$ in the proof of the proposition 2.34. So T_1 is smooth and $T_1 \cap T_2$ is of type $(2,0)$ on T_2. The configuration of lines in the proposition is now clear, and the statements about linkage follow from lemma 4.1.□

4.15. Construction of surfaces G. We start as above. Let T_1 be a smooth Del Pezzo surface of degree 4, and let $G_1 + G_2 + F_1 + F_2$ be one of its hyperplane sections, which consists of four exceptional lines, such that say $G_1 \cdot G_2 = F_1 \cdot F_2 = 0$. Let now T_2 be a smooth quadric surface in this hyperplane such that F_1 and F_2 are members of one its rulings, while G_1 and G_2 meet transversally T_2.

Lemma 4.16. *$T = T_1 \cup T_2$ is a local complete intersection scheme, it lies on a pencil of reducible cubic hypersurfaces, and its homogeneous ideal I_T is generated by quartics.*

Proof. One uses the residual exact sequence

$$0 \longrightarrow \mathcal{I}_{T_1}(3) \longrightarrow \mathcal{I}_T(4) \longrightarrow \mathcal{I}_{T\cap H}(4) \longrightarrow 0,$$

where H is the hyperplane spanned by T_2 and argues as in lemma 4.11.□

As a consequence of the previous lemma, we may link T in the complete intersection of two general quartic hypersurfaces with a smooth surface S, with invariants: deg $S = 10$, $\pi = 10$, and from the liaison exact sequences, $p_g = 2$, $q = 0$. In particular, S is by proposition 2.34 a non-minimal proper elliptic surface with two exceptional lines. A closer look to the linkage, as in the construction 4.10, shows in fact that the elliptic pencil of S is given by the moving part of the trace on S of the pencil of quadrics defining T_2.

Remark 4.17. *The scheme T_2 may be also characterized, in accordance with the corresponding formula of Le Barz, as the union of the 5-secant lines to S.*

Proposition 4.18. *A smooth general type surface S of degree 10 in $\mathbf{P}^4$, with $\pi = 10$ can be linked $(4,4)$ to a reducible surface $Z = T_1 \cup T_2$ where T_1 is a cubic Del Pezzo surface and T_2 is a degenerated cubic scroll, union of three planes having a line in common and such that T_2 cuts T_1 along a doubling of this line. Moreover the scheme Z is minimal in its liaison class and can be linked $(3,4)$ to a scheme Y of degree 6, with $\pi = 2$, which is minimal in the even liaison class of S.*

Proof. It follows from proposition 2.34 that S is embedded by $H \equiv 2K - A_1 - A_2 - A_3$, with A_i, for $i = \overline{1,3}$, disjoint (-2)-conics. Since $p_g = 3$ and $K \cdot A_i = 0$, $i = \overline{1,3}$, by taking global sections in the exact sequence

$$0 \longrightarrow \mathcal{O}_S(K - A_i - A_j) \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_{A_i}(K) \oplus \mathcal{O}_{A_j}(K) \longrightarrow 0$$

we get that there exists a curve $D_k \in \mid K - A_i - A_j \mid$, for all triples $\{i,j,k\} = \{1,2,3\}$. It has degree 4 and genus 3, hence it is a plane curve. Now $D_1 \cdot D_2 = D_1 \cdot D_3 = D_2 \cdot D_3 = 2$ so the planes Π_1, Π_2 and Π_3 of D_1, D_2 and D_3 resp. meet pairwise in lines. Since the three planes span all of $\mathbf{P}^4$ they must intersect along a common line L. If L would lie on S then, by corollary 1.2, it needs to be a component of all curves D_i, $i = \overline{1,3}$, whence the splitting $D_i = L + M_i$. But then $L \cdot M_i = 3$ and $L^2 + L \cdot K = -2$ give $K \cdot L - L \cdot A_j - L \cdot A_k = 1$, for all $\{i,j,k\} = \{1,2,3\}$, and hence, summing up, $6K \cdot L - 2(L \cdot A_1 + L \cdot A_2 + L \cdot A_3) = 3$, which is absurd. Therefore L doesn't lie on S so it must be the unique 6-secant for the surface. Furthermore, since S meets each of the planes Π_i in two points on L outside the curve D_i, $i = \overline{1,3}$, we get that each plane Π_i has two pencils of 5-secant lines to S. Altogether we have counted six 5-secants to S meeting a general plane in $\mathbf{P}^4$, and this fits with Le Barz's formula. Therefore $T_2 = \Pi_1 \cup \Pi_2 \cup \Pi_3$ is the union of the 5-secant lines to S.
By proposition 1.16, S can be linked $(4,4)$ to a surface Z of degree 6, with $\pi = 2$. T_2 is contained in any quartic hypersurface containing S so the planes Π_1, Π_2 and Π_3 will necessarily be components of Z. Let T_1 be the residual component:

$$Z = T_1 \cup \Pi_1 \cup \Pi_2 \cup \Pi_3.$$

Thus T_1 has degree 3. The liaison exact sequence

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_S(3) \longrightarrow \mathcal{O}_{S \cap Z}(3) \longrightarrow 0$$

gives $S \cap Z \equiv 3H - K \equiv H + D_1 + D_2 + D_3 \equiv H + \sum_{i=1}^{3} \Pi_i \cap S$, hence $T_1 \cap S \equiv H$ on S and T_1 is contained in a hyperplane. Now $T_1 \cap (S \cup \Pi_1 \cup \Pi_2 \cup \Pi_3) \equiv 3H_{T_1} - K_{T_1} \equiv 4H_{T_1}$, hence $T_1 \cap S$ is linked in the intersection of a quartic and the cubic T_1 to a curve F of degree 2 and arithmetic genus $p_a(F) = -2$. This means that F is a double structure on a line L' on the cubic T_1. In fact the line L' must coincide with the line L since T_2 has no components in the hyperplane of T_1 and the doubling of L' is contained in $\Pi_1 \cup \Pi_2 \cup \Pi_3$. It is also easily seen that, for a general choice of the linkage, T_1 is a smooth cubic Del Pezzo surface meeting T_2 as stated in the proposition. The rest follows from lemma 4.1.□

Remark 4.19. *Using, for instance, the above description of Z one can show that, for generic choices in the second linkage, Y is also a reducible surface $F_1 \cup F_2$, where F_1 is a smooth symmetric Castelnuovo surface and F_2 is a general plane cutting F_1 along the singular line of the rank 3 hyperquadric containing the Castelnuovo surface. This line is exactly the unique 6-secant to the surface S.*

4.20. Construction of surfaces H. As for surfaces of type G, we start with a Del Pezzo surface of degree 3 and three planes forming a degenerate cubic scroll, such that they all intersect the hyperplane of the Del Pezzo along the same line L on the Del Pezzo. The union of these four surfaces is easily seen to be cut out by quartics, and furthermore performing linkage on the blow up of $\mathbf{P}^4$ along the line L, like in proposition 4.5, one shows that the union of the four surfaces is linked $(4,4)$ to a smooth surface of degree 10, for which L becomes a 6-secant line.

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