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VECTOR FIELDS AND DEFORMATIONS OF ISOTROPIC SUPER-GRASSMANNIANS OF MAXIMAL TYPE

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Abstract. One determines the holomorphic vector fields and the deformations of the isotropic super-Grassmannians of maximal type $I^\circ \mathrm{Gr}_{2r|2s,r|s}$ associated with the complex orthosymplectic Lie superalgebras.

1. Preliminaries

In [2,6,7] the holomorphic vector fields and the deformations of complex super-Grassmannians were studied. It was proved, in particular, that, for a wide class of super-Grassmannians, all holomorphic vector fields are induced by linear transformations and the tangent sheaf 1-cohomology vanishes. Here we want to apply the same methods in order to get similar results for isotropic super-Grassmannians of maximal type associated with orthosymplectic Lie superalgebras. It turns out that the super-Grassmannian of maximal type associated with the Lie superalgebra $\mathfrak{osp}_{2r-1|2s}(\mathbb{C})$ is isomorphic to a connected component of that associated with $\mathfrak{osp}_{2r|2s}(\mathbb{C})$ (which is well known in the classical situation), and so we shall study only the latter case.

Let us denote by $\mathrm{IGr}_{2r|2s,r|s}$ the isotropic super-Grassmannian of maximal type associated with the classical Lie superalgebra $\mathfrak{osp}_{2r|2s}(\mathbb{C})$ (see [4]). Its reduction is the product of two isotropic complex Grassmannians $\mathrm{IGr}^s_{2r,r} \times \mathrm{IGr}^a_{2s,s}$, where the first factor is the Grassmannian of isotropic r-planes in the vector space $\mathbb{C}^{2r}$ endowed with a non-degenerate symmetric bilinear form, while the second one is that of isotropic s-planes in $\mathbb{C}^{2s}$ endowed with a non-degenerate skew-symmetric bilinear form. The supermanifold $\mathrm{IGr}_{2r|2s,r|s}$ admits a natural transitive action of the orthosymplectic Lie supergroup $\mathrm{OSp}_{2r|2s}(\mathbb{C})$, inducing on its reduction the standard transitive action of the Lie group $\mathrm{O}_{2r}(\mathbb{C}) \times \mathrm{Sp}_{2s}(\mathbb{C})$.

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Let $(e_1, \dots, e_{2r})$, $(f_1, \dots, f_{2s})$ be the standard bases of $\mathbb{C}^{2r}$, $\mathbb{C}^{2s}$ respectively. We suppose that the orthosymplectic Lie supergroup leaves invariant the bilinear form in $\mathbb{C}^{2r|2s}$ given in the basis $(e_1, \dots, e_{2r}, f_1, \dots, f_{2s})$ by the matrix

$$\begin{pmatrix} 0 & 1_r & 0 & 0 \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_s \\ 0 & 0 & -1_s & 0 \end{pmatrix}$$

We denote by o the graded isotropy subspace of maximal dimension

$$o = \langle e_{r+1}, \dots, e_{2r}, f_{s+1}, \dots, f_{2s} \rangle$$

of $\mathbb{C}^{2r|2s}$.It is well known that the manifold $\mathrm{IGr}^s_{2r,r}$ has two connected components, while $\mathrm{IGr}^a_{2s,s}$ is connected. We choose the connected component

$$M = \mathrm{I}^\circ\mathrm{Gr}^s_{2r,r} \times \mathrm{IGr}^a_{2s,s}$$

of $\mathrm{IGr}^s_{2r,r} \times \mathrm{IGr}^a_{2s,s}$, containing the point o, and denote by $\mathrm{I}^\circ\mathrm{Gr}_{2r|2s,r|s}$ the corresponding connected component of $\mathrm{IGr}_{2r|2s,r|s}$. Sometimes we will denote this supermanifold by $(M, \mathcal{O})$, where $\mathcal{O}$ is its structure sheaf.

The natural action of the Lie supergroup $\mathrm{OSp}_{2r|2s}(\mathbb{C})$ induces the transitive action of its identity component $\mathrm{SOSp}_{2r|2s}(\mathbb{C})$ on $(M, \mathcal{O})$. The reduction of the latter supergroup is

$$G = G_0 \times G_1,$$

where

$$G_0 = \mathrm{SO}_{2r}(\mathbb{C}),\ G_1 = \mathrm{Sp}_{2s}(\mathbb{C}).$$

Let P denote the stabilizer G_o of the point $o \in M$ in G; we have

$$P = P_0 \times P_1,$$

where $P_0 \subset G_0, P_1 \subset G_1$. The subgroup

$$R = R_0 \times R_1,$$

where

$$R_0 \simeq \mathrm{GL}_r(\mathbb{C}),\ R_1 \simeq \mathrm{GL}_s(\mathbb{C}),$$

leaving invariant the subspaces

$$\langle e_1, \dots, e_r \rangle,\ \langle e_{r+1}, \dots, e_{2r} \rangle,\ \langle f_1, \dots, f_s \rangle,\ \langle f_{s+1}, \dots, f_{2s} \rangle,$$

is the reductive part of P. The matrices from R are of the form

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & (A^t)^{-1} & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & (B^t)^{-1} \end{pmatrix},$$

where $A \in \mathrm{GL}_r(\mathbb{C})$, $B \in \mathrm{GL}_s(\mathbb{C})$, while those from P have the form

$$\begin{pmatrix} A & 0 & 0 & 0 \\ U & (A^t)^{-1} & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & V & (B^t)^{-1} \end{pmatrix}.$$

The tangent Lie algebras and Lie superalgebras of Lie groups and Lie supergroups will be denoted, as usually, by the corresponding Gothic lower case letters. We have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \ \mathfrak{g}_0 = \mathfrak{so}_{2r}(\mathbb{C}), \ \mathfrak{g}_1 = \mathfrak{sp}_{2s}(\mathbb{C}).$$

The Lie algebra $\mathfrak{p}$ of P admits the semi-direct decomposition

$$\mathfrak{p} = \mathfrak{r} + \mathfrak{n},$$

where $\mathfrak{n}$ is the nil-radical of $\mathfrak{p}$. We have

$$\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1,$$

where $\mathfrak{n}_0 \subset \mathfrak{g}_0$, $\mathfrak{n}_1 \subset \mathfrak{g}_1$ consist of the matrices

(1) $$u = \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix}, \ v = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix},$$

U and V being a skew-symmetric $r \times r$ and a symmetric $s \times s$-matrix respectively. The subalgebra $\mathfrak{n}$ is commutative.

We shall use the standard coordinate system on $\mathrm{IGr}_{2r|r,2s|s}$ in a neighborhood of o introduced in [4, Ch.5, Sec.6] , changing slightly the notation; more precisely, transposing the coordinate matrix. This matrix will have the form

(2) $$Z = \begin{pmatrix} X & \Xi \\ 1_r & 0 \\ -\Xi^t & Y \\ 0 & 1_s \end{pmatrix},$$

where $X = (x_{\alpha\beta})$ and $Y = (y_{ij})$ are a $r \times r$-matrix and a $s \times s$-matrix of even coordinates, $X^t = -X$, $Y^t = -Y$, and $\Xi = (\xi_{\alpha s})$ is a $r \times s$-matrix of odd ones. At the point o we have $x_{\alpha\beta} = y_{ij} = 0$. The natural action of $\mathrm{OSp}_{2r|2s}(\mathbb{C})$ on $\mathrm{IGr}_{2r|2s,r|s}$ is given by the matrix multiplication from the left.

Let ρ_0, ρ_1 be the standard representations of $\mathrm{GL}_r(\mathbb{C}), \mathrm{GL}_s(\mathbb{C})$ and σ_0, σ_1 their adjoint representations in the corresponding derived algebras $\mathfrak{sl}_p(\mathbb{C})$, $p = r, s$. The trivial 1-dimensional representation of any group will be denoted by 1. In what follows, we shall omit for simplicity the trivial factors 1 in the notation of the representations.

As in [6], we exploit the theory of homogeneous vector bundles. Let $E = E_\psi$ be a finite-dimensional P-module determined by a holomorphic linear representation ψ of P. We denote by $\mathbf{E} = \mathbf{E}_\psi$ the corresponding homogeneous vector bundle over M and by $\mathcal{E} = \mathcal{E}_\psi$ the sheaf of its holomorphic sections. As is well known, the tangent sheaf Θ on M is isomorphic to $\mathcal{E}_\tau$, where the isotropy representation τ of P is completely reducible and satisfies the condition

$$\tau|R = \bigwedge^2 \rho_0 + \mathrm{S}^2\,\rho_1. \tag{3}$$

The supermanifold $(M, \mathcal{O})$ is, in general, non-split. As usually, we associate with it the split supermanifold $(M, \operatorname{gr}\mathcal{O})$. Its structure sheaf is the graded sheaf associated with the filtration

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \dots, \tag{4}$$

where $\mathcal{J} = (\mathcal{O}_{\bar 1})$. We have $\operatorname{gr}\mathcal{O} \simeq \bigwedge \mathcal{E}$, where $\mathcal{E} = \mathcal{J}/\mathcal{J}^2$. The holomorphic vector bundle $\mathbf{E}$ over M associated with $\mathcal{E}$ has the fibers $\mathbf{E}_x = \mathcal{J}_x/m_x\mathcal{J}_x$, $x \in M$, where m_x is the maximal ideal of $\mathcal{O}_x$.

Clearly, the action of $\mathrm{OSp}_{2r|2s}(\mathbb{C})$ on the super-Grassmannian induces actions of G on the sheaves $\mathcal{O}$, $\mathcal{J}$, $\mathcal{E}$ and on the vector bundle $\mathbf{E}$, covering the standard action of G on M. Thus, $\mathbf{E}$ is a homogeneous vector bundle over M.

Proposition 1. *We have*

$$\operatorname{gr}\mathcal{O} \simeq \bigwedge \mathcal{E}_\varphi,$$

where φ is the irreducible representation of P such that

$$\varphi|R = \rho_0^* \otimes \rho_1^*.$$

Proof. Clearly, $\mathcal{J}/\mathcal{J}^2 = \mathcal{E}_\varphi$, where φ is the representation of P induced in the fibre $\mathbf{E}_o = \mathcal{J}_o/m_o\mathcal{J}_o$. To calculate it, we use the coordinate matrix (2). The action of P on $(M, \mathcal{O})$ is expressed by means of the coordinates in the following way:

$$\begin{aligned}\tilde Z &= \begin{pmatrix} A & 0 & 0 & 0 \\ U & (A^t)^{-1} & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & V & (B^t)^{-1} \end{pmatrix} \begin{pmatrix} X & \Xi \\ 1_r & 0 \\ -\Xi^t & Y \\ 0 & 1_s \end{pmatrix} \\ &= \begin{pmatrix} AX & A\Xi \\ (A^t)^{-1} + UX & U\Xi \\ -B\Xi^t & BY \\ -V\Xi^t & (B^t)^{-1} + VY \end{pmatrix}.\end{aligned} \tag{5}$$

We must reduce the result to the form (2) by multiplying from the right by the matrix $\begin{pmatrix} (A^t)^{-1} + UX & U\Xi \\ -V\Xi^t & (B^t)^{-1} + VY \end{pmatrix}^{-1}$. We may set $X = 0, Y = 0$ which simplifies the calculation. Then

$$\begin{pmatrix} (A^t)^{-1} & U\Xi \\ -V\Xi^t & (B^t)^{-1} \end{pmatrix}^{-1} \equiv \begin{pmatrix} A^t & -A^tU\Xi B^t \\ B^tV\Xi^tA^t & B^t \end{pmatrix}$$

modulo $\mathcal{J}_o^2$. Hence,

$$\tilde{Z} \equiv \begin{pmatrix} 0 & A\Xi B^t \\ 1_r & 0 \\ -B\Xi^t A^t & 0 \\ 0 & 1_s \end{pmatrix}$$

modulo $m_o\mathcal{J}_o^2$. Since the entries of Ξ determine a basis of $\mathbf{E}_o$, this implies our assertion.

Our goal is to calculate the 0- and 1-cohomology of the tangent sheaf $\mathcal{T} = \mathcal{D}er\ \mathcal{O}$ of $\mathrm{IGr}_{2r|2s,r|s}$. As in [6], we consider first the $\mathbb{Z}$-graded sheaf $\tilde{\mathcal{T}} = \mathcal{D}er\ \mathrm{gr}\,\mathcal{O}$. It is known (see [4]) that for any $q \geq -1$ there exists a natural exact sequence of sheaves

$$0 \to \mathcal{T}_{(q+1)} \to \mathcal{T}_{(q)} \to \tilde{\mathcal{T}}_q \to 0, \tag{6}$$

where $\mathcal{T}_{(q)}$ are the subsheaves of $\mathcal{T}$ forming a filtration of this sheaf and defined by

$$\begin{aligned} &\mathcal{T}_{(-1)} = \mathcal{T}, \\ &\mathcal{T}_{(q)} = \{\delta \in \mathcal{T} | \delta\mathcal{O} \subset \mathcal{J}^q, \delta\mathcal{J} \subset \mathcal{J}^{q+1}\},\ q \geq 0. \end{aligned} \tag{7}$$

The sequence (6) will permit us to relate the cohomology of $\mathcal{T}$ to that of $\tilde{\mathcal{T}}$. To calculate the cohomology of the latter sheaf, one uses the exact sequence

$$0 \to \mathcal{A}_{q+1} \xrightarrow{\alpha} \tilde{\mathcal{T}}_q \xrightarrow{\beta} \mathcal{B}_q \to 0. \tag{8}$$

Here

$$\mathcal{A}_q = \mathcal{E}_\varphi^* \otimes \bigwedge^q \mathcal{E}_\varphi = \mathcal{E}_{\Phi_q}$$

with

$$\Phi_q = \varphi^* \otimes \bigwedge^q \varphi, \tag{9}$$

and

$$\mathcal{B}_q = \Theta \otimes \bigwedge^q \mathcal{E}_\varphi = \mathcal{E}_{\mathrm{T}_q}$$

with

$$\mathrm{T}_q = \tau \otimes \bigwedge^q \varphi. \tag{10}$$

The mapping β is the restriction of a derivation of degree q onto the structure sheaf $\mathcal{F}$ of M, and α identifies any sheaf homomorphism $\mathcal{E}_\varphi \to \bigwedge^{p+1} \mathcal{E}_\varphi$ with its extension which is a derivation of degree q and is zero on $\mathcal{F}$. In particular,

$$\mathcal{T}_{(-1)} \simeq \mathcal{A}_0 = \mathcal{E}_\varphi^* = \mathcal{E}_{\varphi^*}.$$

Now we make some remarks concerning the action of the group G on the sheaves involved. Clearly, the action of G on the structure sheaf $\mathcal{O}$ induces an action of

G on $\mathcal{T}$, preserving the parities. It follows that G preserves the filtrations (4) and (7), inducing an action on the sheaf $\tilde{\mathcal{T}}$. Thus, $\tilde{\mathcal{T}}_q$ for any q is a locally free analytic sheaf on M which is homogeneous with respect to G. One sees easily that the homomorphisms in the exact sequences (6) and (8) are G-equivariant.

To conclude these preliminaries, we shall write explicitly certain fundamental vector fields on $(M, \mathcal{O})$ associated with the action of G, using the local coordinates from (2). Let us denote by $X \rightsquigarrow X^*$ the Lie superalgebra homomorphism $\mathfrak{osp}_{2r|2s}(\mathbb{C}) \to H^0(M, \mathcal{T})$ induced by the action of $\mathrm{SOSp}_{2r|2s}(\mathbb{C})$ on $(M, \mathcal{O})$.

Let

$$H = \mathrm{diag}(\lambda_1, \dots, \lambda_r, -\lambda_1, \dots, -\lambda_r, \mu_1, \dots, \mu_s, -\mu_1, \dots, -\mu_s)$$

be the general diagonal matrix lying in $\mathfrak{g}$. Using (5), we get

$$H^* = \sum_{\alpha<\beta}(\lambda_\alpha + \lambda_\beta)x_{\alpha\beta}\frac{\partial}{\partial x_{\alpha\beta}} + \sum_{i\le j}(\mu_i + \mu_j)y_{ij}\frac{\partial}{\partial y_{ij}} + \sum_{\alpha,i}(\lambda_\alpha + \mu_i)\xi_{\alpha i}\frac{\partial}{\partial \xi_{\alpha i}}. \tag{11}$$

Now, for the elements u, $v \in \mathfrak{n}$ given by (1), we get, using (5) again:

$$\begin{aligned} u^* &= \sum_{\alpha,\beta}(XUX)_{\alpha\beta}\frac{\partial}{\partial x_{\alpha\beta}} - \sum_{i,j}(\Xi^t U \Xi)_{ij}\frac{\partial}{\partial y_{ij}} + \sum_{\alpha,k}(XU\Xi)_{\alpha k}\frac{\partial}{\partial \xi_{\alpha k}}, \\ v^* &= -\sum_{\alpha,\beta}(\Xi V \Xi^t)_{\alpha\beta}\frac{\partial}{\partial x_{\alpha\beta}} + \sum_{i,j}(YVY)_{ij}\frac{\partial}{\partial y_{ij}} + \sum_{\alpha,k}(\Xi V Y)_{\alpha k}\frac{\partial}{\partial \xi_{\alpha k}}. \end{aligned}$$

Let us choose the basis $X_{\alpha\beta}\,(\alpha < \beta)$, $Y_{ij}\,(i \le j)$ of $\mathfrak{n}$ given by

$$\begin{aligned} X_{\alpha\beta} &= \frac{1}{2}(E_{\alpha\beta} - E_{\beta\alpha}), \\ Y_{ij} &= \frac{1}{2}(F_{ij} + F_{ji})\ (i \ne j), \\ Y_{ii} &= F_{ii}, \end{aligned} \tag{12}$$

where $E_{\alpha\beta}$ and F_{ij} are the natural bases of the vector spaces of matrices $\mathrm{M}_r(\mathbb{C})$ and $\mathrm{M}_s(\mathbb{C})$ respectively. Then, in particular, we have

$$\begin{aligned} X^*_{\alpha\beta} &= \sum_{\gamma,\delta} x_{\gamma\alpha}x_{\beta\delta}\frac{\partial}{\partial x_{\gamma\delta}} - \sum_{i,j}\xi_{\alpha i}\xi_{\beta j}\frac{\partial}{\partial y_{ij}} \\ &\quad + \frac{1}{2}\sum_{\gamma,k}(x_{\gamma\alpha}\xi_{\beta k} - x_{\gamma\beta}\xi_{\alpha k})\frac{\partial}{\partial \xi_{\gamma k}}, \\ Y^*_{ij} &= -\sum_{\alpha,\beta}\xi_{\alpha i}\xi_{\beta j}\frac{\partial}{\partial x_{\alpha\beta}} + \sum_{k,l} y_{ki}y_{jl}\frac{\partial}{\partial y_{kl}} \\ &\quad + \frac{1}{2}\sum_{\gamma,k}(y_{jk}\xi_{\gamma i} + y_{ik}\xi_{\gamma j})\frac{\partial}{\partial \xi_{\gamma k}}\ (i \ne j), \\ Y^*_{ii} &= -\sum_{\alpha,\beta}\xi_{\alpha i}\xi_{\beta i}\frac{\partial}{\partial x_{\alpha\beta}} + \sum_{k,l} y_{ki}y_{il}\frac{\partial}{\partial y_{kl}} \\ &\quad + \sum_{\gamma,k} y_{ik}\xi_{\gamma i}\frac{\partial}{\partial \xi_{\gamma k}}. \end{aligned} \tag{13}$$

Let now $\mathfrak{n}^-$ be the nilpotent subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{p}$; it has the form $\mathfrak{n}^- = \mathfrak{n}_0^- + \mathfrak{n}_1^-$, where $\mathfrak{n}_0^- \subset \mathfrak{g}_0$, $\mathfrak{n}_1^- \subset \mathfrak{g}_1$ consist of the matrices

$$u = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \; v = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

U and V being a skew-symmetric $r \times r-$ and a symmetric $s \times s-$matrix respectively (cf. (1)). Consider the basis of $\mathfrak{n}^-$ formed by the elements $U_{\alpha\beta}\,(\alpha < \beta)$, $V_{ij}\,(i < j)$, V_{ii} corresponding to the matrices $U = E_{\alpha\beta} - E_{\beta\alpha}$, $V = E_{ij} + E_{ji}\,(i < j)$; E_{ii} respectively. One sees easily that

$$U^*_{\alpha\beta} = \frac{\partial}{\partial x_{\alpha\beta}}, \; V^*_{ij} = \frac{\partial}{\partial y_{ij}}. \tag{14}$$

2. The cohomology of $\mathcal{A}_q$ and $\mathcal{B}_q$

In this section we shall calculate the 0- and 1-cohomology of the sheaves $\mathcal{A}_q$ and $\mathcal{B}_q$. As in [6,7], we use the theorem of Bott (see [1], Theorem IV′) permitting to calculate the cohomology of the homogeneous sheaf $\mathcal{E}_\psi$ on M defined by a completely reducible representation ψ of P. More precisely, this theorem gives an algorithm for determining the highest weights of the G-modules $H^p(M, \mathcal{E}_\psi)$ in terms of the highest weights of ψ. To apply it, we have to introduce some notation related to weights and roots of G.

We choose the Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ in the tangent Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of G such that $\mathfrak{t}_0$ and $\mathfrak{t}_1$ are the Cartan subalgebras of $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively, formed by all diagonal matrices

$$H_0 = \mathrm{diag}(\lambda_1, \dots, \lambda_r, -\lambda_1, \dots, -\lambda_r),$$
$$H_1 = \mathrm{diag}(\mu_1, \dots, \mu_s, -\mu_1, \dots, -\mu_s).$$

We consider the following system of positive roots:

$$\Delta^+ = \Delta_0^+ \cup \Delta_1^+,$$

where

$$\Delta_0^+ = \{\lambda_i - \lambda_j, \; \lambda_i + \lambda_j \, (i < j)\},$$
$$\Delta_1^+ = \{\mu_p - \mu_q \, (p < q), \; \mu_p + \mu_q \, (p \leq q\}.$$

The half of the sum of all positive roots of $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}$ will be denoted by γ_0, γ_1, γ respectively; we have $\gamma = \gamma_0 + \gamma_1$. The corresponding system of simple roots of $\mathfrak{g}$ is

$$\Pi = \Pi_0 \cup \Pi_1,$$

where

$$\Pi_0 = \{\alpha_1, \dots, \alpha_r\}, \; \Pi_1 = \{\beta_1, \dots, \beta_s\}$$

are the systems of simple roots of $\mathfrak{g}_0$, $\mathfrak{g}_1$ respectively; here we denote

$$\alpha_1 = \lambda_1 - \lambda_2, \dots, \alpha_{r-1} = \lambda_{r-1} - \lambda_r,\ \alpha_r = \lambda_{r-1} + \lambda_r;$$
$$\beta_1 = \mu_1 - \mu_2, \dots, \beta_{s-1} = \mu_{s-1} - \mu_s,\ \beta_s = 2\mu_s.$$

We denote by $\mathfrak{t}^*(\mathbb{R})$ the real subspace of $\mathfrak{t}^*$ spanned by all λ_i, μ_p, and define the scalar product on $\mathfrak{t}^*(\mathbb{R})$ such that λ_i, μ_p form its orthonormal basis. As usually, $\lambda \in \mathfrak{t}^*(\mathbb{R})$ is called *dominant* if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta^+$ or, equivalently, for all $\alpha \in \Pi$. Following Bott [1], we say that λ has *index* 1 if $(\lambda, \alpha) > 0$ for all $\alpha \in \Delta^+$ except of one root $\beta \in \Delta^+$, for which $(\lambda, \beta) < 0$. Now, λ is called *singular* if $(\lambda, \alpha) = 0$ for a certain $\alpha \in \Delta$. These definitions will be used with respect to $\mathfrak{g}_0$, $\mathfrak{g}_1$ as well.

Clearly, the subgroup $P = G_o$ defined above is a parabolic subgroup of G containing the Borel subgroup B^- corresponding to $-\Delta^+$. The system of simple roots of its reductive part R is $\Sigma = \Pi - \{\alpha_r, \beta_s\}$. An element $\lambda \in \mathfrak{t}^*(\mathbb{R})$ is called *R-dominant* if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Sigma$.

It is convenient to characterize an element $\lambda \in \mathfrak{t}^*(\mathbb{R})$ by the numbers $\lambda_\alpha = 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}$, $\alpha \in \Pi$, which are actually the coordinates of λ in the basis of the so-called fundamental weights. We have $\gamma_\alpha = 1$ for all $\alpha \in \Pi$. An element λ is dominant if and only if $\lambda_\alpha \geq 0$ for all $\alpha \in \Pi$.

The following proposition is well known and very easy to verify:

Proposition 2. *An element*

$$\lambda = \sum_{i=1}^{r} k_i \lambda_i,\ k_i \in \mathbb{R},$$

is dominant if and only if $k_1 \geq k_2 \geq \dots \geq |k_r|$. It is R-dominant if and only if $k_1 \geq k_2 \geq \dots \geq k_r$.

An element

$$\lambda = \sum_{j=1}^{s} l_i \mu_j,\ l_j \in \mathbb{R},$$

is dominant if and only if $l_1 \geq l_2 \geq \dots \geq l_s \geq 0$. It is R-dominant if and only if $l_1 \geq l_2 \geq \dots \geq l_s$.

We have to study the highest weights of the representations Φ_q and T_q of P defined by (9) and (10), respectively. It follows from Proposition 1 that

$$\Phi_q | R = (\rho_0 \otimes \rho_1) \bigwedge^q (\rho_0^* \otimes \rho_1^*).$$

Denote by i, i_α indices running over $1, \dots, r$, and by j, j_β those running over $1, \dots, s$. The weights of Φ_q have the form

$$\Lambda = \Lambda_0 + \Lambda_1, \tag{15}$$

where

$$\begin{aligned} \Lambda_0 &= \lambda_i - \lambda_{i_1} - \dots - \lambda_{i_q}, \\ \Lambda_1 &= \mu_j - \mu_{j_1} - \dots - \mu_{j_q}. \end{aligned} \tag{16}$$

Similarly, (3) implies that

$$\mathrm{T}_q = \mathrm{T}'_q + \mathrm{T}''_q,$$

where

$$\mathrm{T}'_q\,|R = (\bigwedge^2 \rho_0)\bigwedge^q(\rho_0^* \otimes \rho_1^*),$$
$$\mathrm{T}''_q\,|R = (\mathrm{S}^2\,\rho_1)\bigwedge^q(\rho_0^* \otimes \rho_1^*).$$

The weights of T'_q, T''_q have the form

$$\Lambda = \Lambda_0 + \Lambda_1, \tag{17}$$

where for T'_q we have

$$\begin{aligned} \Lambda_0 &= \lambda_i + \lambda_k - \lambda_{i_1} - \ldots - \lambda_{i_q},\ i < k, \\ \Lambda_1 &= -\mu_{j_1} - \ldots - \mu_{j_q}, \end{aligned} \tag{18}$$

and for T''_q

$$\begin{aligned} \Lambda_0 &= -\lambda_{i_1} - \ldots - \lambda_{i_q}, \\ \Lambda_1 &= \mu_j + \mu_l - \mu_{j_1} - \ldots - \mu_{j_q},\ j \leq l. \end{aligned} \tag{19}$$

We denote by Id_0, Id_1 the standard representations and by Ad_0, Ad_1 the adjoint representations of G_0, G_1 respectively. Remark that in the case $r = 1$ we have $G_0 = R_0 \simeq \mathrm{GL}_1(\mathbb{C})$, and $Id_0 = \rho_0 + \rho_0^*$.

Proposition 3. *Suppose that $r \geq 2$, $s \geq 1$. Then the G-module $H^0(M, \mathcal{A}_0) \simeq \mathbb{C}^{2r} \otimes \mathbb{C}^{2s}$ is irreducible with the representation $Id_0 \otimes Id_1$. For $r = 1$, $s \geq 1$, the G-module $H^0(M, \mathcal{A}_0) \simeq \mathbb{C}^{2s}$ is irreducible with the representation $\rho_0 \otimes Id_1$.*

We have

$$H^p(M, \mathcal{A}_0) = 0$$

for any $p \geq 1$.

Proof. The highest weight of $\Phi_0 = \varphi^*$ is $\lambda_1 + \mu_1$. It is dominant and is the highest weight of the representation $Id_0 \otimes Id_1$ (for $r \geq 2$) or $\rho_0 \otimes Id_1$ (for $r = 1$) of G. Our assertions follow from the theorem of Bott.

Proposition 4. *Suppose that $r \geq 1$, $r \neq 2$, $s \geq 1$. Then*

$$H^0(M, \mathcal{A}_1) \simeq \mathbb{C}$$

(the trivial G-module). In the case $r = 2$, $s \geq 1$ we have

$$H^0(M, \mathcal{A}_1) \simeq \mathbb{C} \oplus \mathfrak{sl}_2(\mathbb{C}),$$

where the first summand is the trivial G-module and the second one is the irreducible G-module with heighest weight $\lambda_1 - \lambda_2$. In both cases we have

$$H^p(M, \mathcal{A}_1) = 0,\ p \geq 1.$$

Proof. Clearly, for $r \geq 2$, $s \geq 2$ we have

$$\Phi_1|R = (\rho_0\rho_0^*) \otimes (\rho_1\rho_1^*)$$
$$= (1+\sigma_0) \otimes (1+\sigma_1) = 1 + \sigma_0 + \sigma_1 + \sigma_0 \otimes \sigma_1.$$

The trivial component gives the 1-dimensional trivial G-module. The highest weights of the non-trivial components are

$$\Lambda_0 = \lambda_1 - \lambda_r,\ \Lambda_1 = \mu_1 - \mu_s,\ \Lambda_0 + \Lambda_1.$$

The weight $\Lambda_0 + \gamma$ is singular for $r \geq 3$, since

$$(\Lambda_0 + \gamma)_{\alpha_r} = (\Lambda_0 + \gamma_0)_{\alpha_r} = -1.$$

In the case when $r = 2$ the weight $\Lambda_0 = \lambda_1 - \lambda_2$ is dominant and determines the restriction of Ad_0 onto one of the simple ideals of $\mathfrak{g}_0 \simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ (which coincides actually with $[\mathfrak{r}_0, \mathfrak{r}_0]$). Now, $\Lambda_1 + \gamma$ is singular for $s \geq 2$, since

$$(\Lambda_1 + \gamma)_{\beta_s} = (\Lambda_1 + \gamma_1)_{\beta_s} = -1.$$

Therefore, $\Lambda_0 + \Lambda_1 + \gamma$ is singular, too.

Thus, the proposition follows from the theorem of Bott. In the cases $r = 1$ or $s = 1$ the corresponding adjoint representation does not enter into the expression of Φ_1, and we get the same result.

Proposition 5. *For any $r \geq 1$, $s \geq 1$ we have*

$$H^0(M, \mathcal{A}_q) = H^1(M, \mathcal{A}_q) = 0,\ q \geq 2.$$

Proof. Let Λ be a highest weight of Φ_q. Using its expression given by (15) and (16), we easily see from Proposition 2 that Λ_0 and Λ_1 can not be dominant. Therefore the situation when Λ is dominant or $\Lambda + \gamma$ has index 1 is impossible.

Proposition 6. *For $r \geq 3$, $s \geq 1$, the G-module*

$$H^0(M, \mathcal{B}_0) \simeq \mathfrak{so}_{2r}(\mathbb{C}) \oplus \mathfrak{sp}_{2s}(\mathbb{C})$$

splits into the sum of two irreducible components with the representations Ad_0, Ad_1. *In the case $r = 2$, $s \geq 1$ the G-module*

$$H^0(M, \mathcal{B}_0) \simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sp}_{2s}(\mathbb{C})$$

splits into the sum of two irreducible components the first of which has the highest weight $\lambda_1 + \lambda_2$ while the second one is Ad_1. *In the case $r = 1$, $s \geq 1$ we have the irreducible G-module*

$$H^0(M, \mathcal{B}_0) \simeq \mathfrak{sp}_{2s}(\mathbb{C})$$

with the representation Ad_1.

We have

$$H^p(M, \mathcal{B}_0) = 0$$

for any $p \geq 1$ and all $r \geq 1$, $s \geq 1$.

Proof. By (3), the highest weights of $\mathrm{T}_0 = \tau$ are $\lambda_1 + \lambda_2$ (for $r \geq 2$) and $2\mu_1$. These are the highest weights of Ad_0 (if $r \geq 3$) and Ad_1. If $r = 2$, then $\lambda_1 + \lambda_2$ is the highest weight of the restriction of Ad_0 onto a simple ideal of $\mathfrak{g}_0$ (the complement to the ideal considered in Proposition 4).

Proposition 7. *If $r \geq 2$, $s \geq 1$, then we have*

$$H^p(M, \mathcal{B}_1) = 0$$

for any $p \geq 0$. If $r = 1$, $s \geq 1$, then

$$H^0(M, \mathcal{B}_1) \simeq \mathbb{C}^{2s}$$

is the irreducible G-module with the representation $\rho_0^* \otimes Id_1$ and

$$H^p(M, \mathcal{B}_1) = 0$$

for any $p \geq 1$.

Proof. One see easily that, for $r \geq 2$,

$$\mathrm{T}_1 \,|R = (\bigwedge^2 \rho_0 \rho_0^*) \otimes \rho_1^* + \rho_0^* \otimes (\mathrm{S}^2 \rho_1)\rho_1^*.$$

Clearly, $\lambda_r + \gamma$ and $\mu_s + \gamma$ are singular,and hence $\Lambda + \gamma$ is singular for any weight of T_1. The theorem of Bott implies our assertion.

In the case $r = 1$ we have

$$\mathrm{T}_1 \,|R = \rho_0^* \otimes (\mathrm{S}^2 \rho_1)\rho_1^*.$$

The highest weights of this representation are $-\lambda_1 + \mu_1$ and (for $s \geq 2$) $2\mu_1 - \mu_s$. The first weight is dominant and gives the representation $\rho_0^* \otimes Id_1$, while the sum of the second one with γ is singular.

Proposition 8. *Suppose that $r \geq 2$, $s \geq 1$. Then*

$$H^0(M, \mathcal{B}_2) = 0, \; H^1(M, \mathcal{B}_2) \simeq \mathbb{C}^2$$

(the trivial G-module). If $r = 1$, $s \geq 1$, then

$$H^p(M, \mathcal{B}_2) = 0, \; p = 0, 1.$$

Proof. By (3) we have

$$\begin{aligned}
\mathrm{T}_2 \,|R &= (\bigwedge^2 \rho_0 + \mathrm{S}^2 \rho_1) \bigwedge^2 (\rho_0^* \otimes \rho_1^*) \\
&= (\bigwedge^2 \rho_0 + \mathrm{S}^2 \rho_1)(\bigwedge^2 \rho_0^* \otimes \mathrm{S}^2 \rho_1^* + \mathrm{S}^2 \rho_0^* \otimes \bigwedge^2 \rho_1^*) \\
&= (\bigwedge^2 \rho_0)(\bigwedge^2 \rho_0^*) \otimes \mathrm{S}^2 \rho_1^* + (\bigwedge^2 \rho_0)(\mathrm{S}^2 \rho_0^*) \otimes \bigwedge^2 \rho_1^* \\
&+ (\bigwedge^2 \rho_0) \otimes (\mathrm{S}^2 \rho_1)(\mathrm{S}^2 \rho_1^*) + (\mathrm{S}^2 \rho_0^*) \otimes (\mathrm{S}^2 \rho_1)(\bigwedge^2 \rho_1^*).
\end{aligned}$$

The first three of these four summands exist only when $r \geq 2$. For the first one, any highest weight has the form (see (17),(18),(19))

$$\Lambda = \Lambda_0 + \Lambda_1,$$

where

$$\Lambda_0 = \lambda_i + \lambda_j - \lambda_k - \lambda_l, \ \Lambda_1 = -2\mu_s.$$

Clearly,

$$r_{\beta_s}(\Lambda_1 + \gamma_1) = r_{\beta_s}(-\beta_s + \gamma_1) = \beta_s + \gamma_1 - \beta_s = \gamma_1.$$

Hence, $\Lambda_1 + \gamma_1$ has index 1. Therefore, we have interest only in the case when Λ_0 is dominant. Using Proposition 2, one sees easily that this is possible only for $\Lambda_0 = 0$ (which is a highest weight indeed!). Then $\Lambda + \gamma$ has index 1. By the algorithm of Bott, there corresponds to Λ an irreducible component of the G-module $H^1(M, \mathcal{B}_2)$ with highest weight $r_{\beta_s}(\Lambda + \gamma) - \gamma = 0$. Quite similarly, the third summand gives (if $r \geq 2$) only the 1-dimensional trivial component of $H^1(M, \mathcal{B}_2)$.

Now let $\Lambda = \Lambda_0 + \Lambda_1$ be a highest weight of one of two remaining summands. One sees easily from Proposition 2 that neither Λ_0, nor Λ_1 is dominant ($\Lambda_0 = 0$ is not a highest weight in these cases!). Therefore Λ can not be dominant, nor can $\Lambda + \gamma$ have index 1.

Proposition 9. *Suppose that $r \geq 1$, $s \geq 1$. Then*

$$H^0(M, \mathcal{B}_q) = H^1(M, \mathcal{B}_q) = 0$$

for any $q \geq 3$.

Proof. Let Λ be a weight of T'_q. Using (18), we see, by Proposition 3, that Λ_0 can not be dominant if $q \geq 3$ and that Λ_1 can not be dominant if $q \geq 1$. Quite similarly, for any weight Λ of T''_q we see, using (19), that Λ_0 can not be dominant if $q \geq 1$ and that Λ_1 can not be dominant if $q \geq 3$. Thus, Λ can not be dominant, nor can $\Lambda + \gamma$ have index 1. The proposition follows now from the theorem of Bott.

3. The cohomology of $\tilde{\mathcal{T}}$

As in [6], we shall use here some further results of Bott's paper [1]. Let E be a holomorphic P-module. Then (see [1], Theorem I and Corollary 2 of Theorem W_2) we have an isomorphism

$$H^p(M, \mathcal{E})^G \simeq H^p(\mathfrak{n}, E)^{\mathfrak{r}}$$

between the G-invariants and the $\mathfrak{r}$-invariants of the corresponding cohomology groups. This isomorphism is compatible with the homomorphisms induced by homomorphisms of P-modules.

These considerations can be applied to calculate the cohomology of $\mathcal{A}_q$ and $\mathcal{B}_q$ by expressing explicitly the cocycles which represent the basic cohomology classes. We need such an expression for the group $H^1(M, \mathcal{B}_2)$.

We shall use the standard coordinate system on $\mathrm{IGr}_{2r|r,2s|s}$ in a neighborhood of o given by (2). As in [6], we note that the adjoint action of $\mathfrak{p}$ on $\mathfrak{n}$ coincides with τ^*; hence $\mathfrak{n}$, as a $\mathfrak{p}$-module, is isomorphic to the cotangent space $T_o(M)^*$ of M. By this isomorphism, the basis $dx_{\alpha\beta}\,(\alpha<\beta)$, $dy_{ij}\,(i\leq j)$ of $T_o(M)^*$ corresponds to the basis (12) of $\mathfrak{n}$.

The result of Bott mentioned above gives the identification

$$H^1(M,\mathcal{B}_2)=H^1(\mathfrak{n},T_o(M)\otimes\bigwedge^2 E_\phi)^{\mathfrak{r}}.$$

Since τ and ϕ are completely reducible, $\mathfrak{n}$ acts on the coefficients trivially, and hence the coboundary δ of the cochain complex $C(\mathfrak{n},T_o(M)\otimes\bigwedge^2 E_\phi)$ is zero. It follows that

$$(20)\quad H^1(\mathfrak{n},T_o(M)\otimes\bigwedge^2 E_\phi)^{\mathfrak{r}}=C^1(\mathfrak{n},T_o(M)\otimes\bigwedge^2 E_\phi)^{\mathfrak{r}}\simeq(T_o(M)\otimes T_o(M)\otimes\bigwedge^2 E_\phi)^{\mathfrak{r}}.$$

We are going to describe this vector space explicitly in terms of 1-cochains.

Proposition 10. *The following two cochains c_0, c_1 form a basis of $C^1(\mathfrak{n},T_o(M)\otimes\bigwedge^2 E_\phi)^{\mathfrak{r}}$:*

$$c_0(X_{\alpha\beta})=\sum_{i,j}\frac{\partial}{\partial y_{ij}}\otimes\xi_{\alpha i}\xi_{\beta j}+\sum_i\frac{\partial}{\partial y_{ii}}\otimes\xi_{\alpha i}\xi_{\beta i},\ c_0(Y_{ij})=0;$$

$$c_1(Y_{ij})=\sum_{\alpha,\beta}\frac{\partial}{\partial x_{\alpha\beta}}\otimes\xi_{\alpha i}\xi_{\beta j},\ c_1(X_{\alpha\beta})=0.$$

Proof. By Proposition 1, the P-module E_ϕ is identified with $(\mathbb{C}^r)^*\otimes(\mathbb{C}^s)^*$ in such a way that $\xi_{\alpha i}=x_\alpha\otimes y_i$, where x_α, y_i are the standard coordinates. Then $\bigwedge^2 E_\phi=\bigwedge^2((\mathbb{C}^r)^*\otimes(\mathbb{C}^s)^*)$ will contain an irreducible P-submodule isomorphic to $\bigwedge^2(\mathbb{C}^r)^*\otimes \mathrm{S}^2(\mathbb{C}^s)^*$ which is spanned by the elements

$$\begin{aligned}&(x_\alpha\otimes x_\beta-x_\beta\otimes x_\alpha)\otimes(y_i\otimes y_j+y_j\otimes y_i)=\\&\xi_{\alpha i}\otimes\xi_{\beta j}-\xi_{\beta i}\otimes\xi_{\alpha j}+\xi_{\alpha j}\otimes\xi_{\beta i}-\xi_{\beta i}\otimes\xi_{\alpha j}=2(\xi_{\alpha i}\xi_{\beta j}-\xi_{\beta j}\xi_{\alpha i}).\end{aligned}$$

Then, by (20), $H^1(\mathfrak{n},T_o(M)\otimes\bigwedge^2 E_\phi)^{\mathfrak{r}}$ contains the invariants of the submodule $T_o(M)\otimes T_o(M)\otimes\bigwedge^2(\mathbb{C}^r)^*\otimes\mathrm{S}^2(\mathbb{C}^s)^*$. Using (3), we see that precisely two linearly independent invariants lie there, while the complementary submodule does not contain any non-zero invariant. Since the basis $\frac{\partial}{\partial x_{\alpha\beta}}\,(\alpha<\beta)$, $\frac{\partial}{\partial y_{ij}}\,(i\leq j)$ is dual to (12), we get the basic invariants c_0, c_1 given by:

$$\begin{aligned}c_0(X_{\alpha\beta})&=\sum_{i<j}\frac{\partial}{\partial y_{ij}}\otimes(\xi_{\alpha i}\xi_{\beta j}+\xi_{\alpha j}\xi_{\beta i})+2\sum_i\frac{\partial}{\partial y_{ii}}\otimes\xi_{\alpha i}\xi_{\beta i}\\&=\sum_{i,j}\frac{\partial}{\partial y_{ij}}\otimes\xi_{\alpha i}\xi_{\beta j}+\sum_i\frac{\partial}{\partial y_{ii}}\otimes\xi_{\alpha i}\xi_{\beta i},\\c_0(Y_{ij})&=0;\\c_1(Y_{ij})&=\sum_{\alpha<\beta}\frac{\partial}{\partial x_{\alpha\beta}}\otimes(\xi_{\alpha i}\xi_{\beta j}+\xi_{\alpha j}\xi_{\beta i})=\sum_{\alpha,\beta}\frac{\partial}{\partial x_{\alpha\beta}}\otimes\xi_{\alpha i}\xi_{\beta j},\\c_1(Y_{ii})&=2\sum_{\alpha<\beta}\frac{\partial}{\partial x_{\alpha\beta}}\otimes\xi_{\alpha i}\xi_{\beta i}=\sum_{\alpha,\beta}\frac{\partial}{\partial x_{\alpha\beta}}\otimes\xi_{\alpha i}\xi_{\beta i},\\c_1(X_{\alpha\beta})&=0.\end{aligned}$$

We are now able to calculate $H^p(M,\tilde{\mathcal{T}})$, $p=0,1$.

Theorem 1. *Suppose that $r\geq 2$, $s\geq 2$ or $r\geq 3$, $s\geq 1$. Then the G-modules $H^p(M,\tilde{\mathcal{T}}_q)$, $p=0,1$; $q\geq -1$, are indicated in the following table:*

$q=$	-1	0	1	2	≥ 3
$p=0$	$\mathfrak{osp}_{2r\|2s}(\mathbb{C})_{\bar 1}$	$\mathfrak{osp}_{2r\|2s}(\mathbb{C})_{\bar 0}\oplus\mathbb{C}$	0	0	0
$p=1$	0	0	0	$\mathbb{C}$	0

Here $\mathfrak{osp}_{2r|2s}(\mathbb{C})_{\bar 0}$ and $\mathfrak{osp}_{2r|2s}(\mathbb{C})_{\bar 1}$ are endowed with the adjoint representation of G, and $\mathbb{C}$ is the trivial G-module.

If $r=2$, $s=1$, then the table has the form

$q=$	-1	0	1	2	≥ 3
$p=0$	$\mathfrak{osp}_{4\|2}(\mathbb{C})_{\bar 1}$	$\mathfrak{osp}_{4\|2}(\mathbb{C})_{\bar 0}\oplus\mathbb{C}$	0	0	0
$p=1$	0	0	0	$\mathbb{C}^2$	0

Here $\mathbb{C}^2$ is the trivial G-module.

If $r=1$, $s\geq 1$, then the corresponding table is as follows:

$q=$	-1	0	1	2	≥ 3
$p=0$	$\mathbb{C}^{2s}$	$\mathfrak{sp}_{2s}(\mathbb{C})\oplus\mathbb{C}$	$\mathbb{C}^{2s}$	0	0
$p=1$	0	0	0	0	0

Here $\mathfrak{sp}_{2s}(\mathbb{C})$ is endowed with the adjoint representation of G, $\mathbb{C}$ is the trivial G-module and $\mathbb{C}^{2s}$ for $q=-1,1$ is endowed with the representation $\rho_0\otimes Id_1$ or $\rho_0^*\otimes Id_1$ respectively.

Proof. We use the cohomology exact sequences associated with (8). Almost in all cases the mappings in these sequences are determined uniquely. The only difficulty occurs when we try to calculate $H^1(M,\tilde{\mathcal{T}}_2)$ with the help of the exact sequence

$$0\to\mathcal{A}_3\xrightarrow{\alpha}\tilde{\mathcal{T}}_2\xrightarrow{\beta}\mathcal{B}_2\to 0.$$

By Proposition 5, we have the exact sequence

$$0\to H^1(M,\tilde{\mathcal{T}}_2)\xrightarrow{\beta^*}H^1(M,\mathcal{B}_2).$$

If $r=1$ then, by Proposition 8, we have $H^1(M,\mathcal{B}_2)=0$. Hence, $H^1(M,\tilde{\mathcal{T}}_2)=0$ in this case. In what follows we assume that $r\geq 2$.

By Proposition 8, $H^1(M,\mathcal{B}_2)\simeq\mathbb{C}^2$ (the trivial G-module). The sheaves $\tilde{\mathcal{T}}_2$ and $\mathcal{B}_2$ are the sheaves of holomorphic sections of homogeneous vector bundles $\tilde{\mathbf{T}}_2$ and $\mathbf{B}_2=T(M)\otimes\bigwedge^2\mathbf{E}_\phi$, and β is induced by a homomorphism of these bundles. As we have seen in the beginning of this section, β^* is intrepreted as the homomorphism of the invariant 1-cohomology of the Lie algebra $\mathfrak{n}$:

$$H^1(\mathfrak{n},(\tilde{\mathbf{T}}_2)_o)^{\mathfrak{r}}\to H^1(\mathfrak{n},T_o(M)\otimes\bigwedge^2 E_\phi)^{\mathfrak{r}},$$

where $(\tilde{\mathbf{T}}_2)_o$ is the fibre of $\tilde{\mathbf{T}}_2$ at the point o endowed with a natural structure of the $\mathfrak{p}$-module. The group $H^1(\mathfrak{n}, (\tilde{\mathbf{T}}_2)_o)^{\mathfrak{r}}$ coincides with the 1-cohomology of the complex $C(\mathfrak{n}, (\tilde{\mathbf{T}}_2)_o)^{\mathfrak{r}}$ of $\mathfrak{r}$-invariant cochains. Since $H^1(M, \mathcal{A}_3) = 0$ by Proposition 5, the vector space $C^1(\mathfrak{n}, (\tilde{\mathbf{T}}_2)_o)^{\mathfrak{r}}$ is mapped isomorphically onto $C^1(\mathfrak{n}, T_o(M) \otimes \bigwedge^2 E_\phi)^{\mathfrak{r}}$. It follows from Proposition 10 that the cochains $c \in C^1(\mathfrak{n}, (\tilde{\mathbf{T}}_2)_o)^{\mathfrak{r}}$ have the form

$$c(X_{\alpha\beta}) = a(\sum_{i,j} \xi_{\alpha i}\xi_{\beta j}\frac{\partial}{\partial y_{ij}} + \sum_i \frac{\partial}{\partial y_{ii}}),$$
$$c(Y_{ij}) = b\sum_{\alpha,\beta} \xi_{\alpha i}\xi_{\beta j}\frac{\partial}{\partial x_{\alpha\beta}},$$

where a, $b \in \mathbb{C}$. Clearly,

$$H^1(\mathfrak{n}, (\tilde{\mathbf{T}}_2)_o)^{\mathfrak{r}} \simeq \{c \in C^1(\mathfrak{n}, (\tilde{\mathbf{T}}_2)_o)^{\mathfrak{r}} | \delta c = 0\}.$$

By the definition of δ we have

$$(\delta c)(x, y) = xc(y) - yc(x), \; x, y \in \mathfrak{n}.$$

The action of $\mathfrak{n}$ on $(\tilde{\mathbf{T}}_2)_o$ is induced by commuting the fundamental vector fields of the action of G on $\mathrm{IGr}_{2r|r,2s|s}$ with the elements of $\tilde{\mathcal{T}}_2$, followed by evaluating the commutator at $X = 0$, $Y = 0$. It follows from (13) that

$$(\delta c)(X_{\alpha\beta}, X_{\gamma\delta}) = (\delta c)(Y_{ij}, Y_{kl}) = 0$$

and that

$$(\delta c)(X_{\alpha\beta}, Y_{ij}) = (b-a)\sum_{\gamma,k}(\xi_{\alpha j}\xi_{\beta k}\xi_{\gamma i} + \xi_{\alpha k}\xi_{\beta j}\xi_{\gamma i} + \xi_{\alpha i}\xi_{\beta k}\xi_{\gamma j} + \xi_{\alpha k}\xi_{\beta j}\xi_{\gamma i})\frac{\partial}{\partial \xi_{\gamma i}}.$$

One sees easily that if $r \geq 2$, $s \geq 2$ then $\delta c = 0$ is equivalent to $a = b$. The same is true if $s = 1$, $r \geq 3$. In the remaining case $r = 2$, $s = 1$ we have $\delta c = 0$ for any invariant cochain c. Thus,

$$H^1(M, \tilde{\mathcal{T}}_2) \simeq H^1(\mathfrak{n}, (\tilde{\mathbf{T}}_2)_o)^{\mathfrak{r}} \simeq \begin{cases} \mathbb{C} & \text{if } r \geq 2,\ s \geq 2 \text{ or } r \geq 3, s = 1 \\ \mathbb{C}^2 & \text{if } r = 2,\ s = 1. \end{cases}$$

4. The cohomology of $\mathcal{T}$

In this section, we prove our main theorem about 0- and 1-cohomology of the isotropic super-Grassmannian with values in the tangent sheaf. The proof repeats that of Theorem 2 of [6]. First we state a proposition that will play the main part in it.

It is clear that on the split supermanifold $(M, \mathrm{gr}\,\mathcal{O})$ there exists a vector field $\varepsilon \in H^0(M, \tilde{\mathcal{T}}_0)$ such that $\varepsilon(f) = qf$ for any $f \in \mathrm{gr}_q\,\mathcal{O}$. This vector field commutes with any X^*, $X \in \mathfrak{g}$, and hence is a basic element of the trivial G-submodule $\mathbb{C} \subset H^0(M, \tilde{\mathcal{T}}_0)$ (see Theorem 1).

Proposition 11. *If $r \geq 2$, then ε does not lie in the image of the canonical mapping $H^0(M, \mathcal{T}_{(0)}) \to H^0(M, \tilde{\mathcal{T}}_0)$.*

Proof. We take as odd coordinates in a neighborhood of o in $(M, \operatorname{gr} \mathcal{O})$ the elements $\tilde{\xi}_{\alpha i} = \xi_{\alpha i} + \mathcal{J}^2$. Then, clearly, ε is expressed in this neighborhood as

$$\varepsilon = \sum_{\alpha,i} \tilde{\xi}_{\alpha i} \frac{\partial}{\partial \tilde{\xi}_{\alpha i}}.$$

Suppose that there exists $\hat{\varepsilon} \in H^0(M, \mathcal{T}_{(0)})$ inducing the vector field ε. One may suppose that $\hat{\varepsilon} \in (H^0(M, \mathcal{T}_{(0)})_{\bar{0}})^G$. Then $[\hat{\varepsilon}, X^*] = 0$ for any $X \in \mathfrak{g}$. Consider the action of the derivation $\hat{\varepsilon}$ in $\mathcal{O}_o$. The mapping $X \to X^*$ is a linear representation of the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, commuting with $\hat{\varepsilon}$. We see from (11) that $x_{\alpha\beta}$, y_{ij}, $\xi_{\alpha i}$ lie in the weight subspaces of this representation, corresponding to the weights $\lambda_\alpha + \lambda_\beta$, $\mu_i + \mu_j$, $\lambda_\alpha + \mu_i$ respectively. It is clear that all these weight subspaces have dimension 1. Since $\hat{\varepsilon}$ maps any weight subspace into itself, we have

$$\hat{\varepsilon} = \sum_{\alpha,i} \xi_{\alpha i} \frac{\partial}{\partial \xi_{\alpha i}} + \sum_{\alpha<\beta} a_{\alpha\beta} x_{\alpha\beta} \frac{\partial}{\partial x_{\alpha\beta}} + \sum_{i \leq j} b_{ij} y_{ij} \frac{\partial}{\partial y_{ij}},$$

where $a_{\alpha\beta}, b_{ij} \in \mathbb{C}$. Now, we have $[\hat{\varepsilon}, U^*_{\alpha\beta}] = [\hat{\varepsilon}, V^*_{ij}] = 0$ which, by (14), implies that $a_{\alpha\beta} = b_{ij} = 0$ for all $\alpha < \beta$, $i \leq j$. Thus,

$$\hat{\varepsilon} = \sum_{\alpha,i} \xi_{\alpha i} \frac{\partial}{\partial \xi_{\alpha i}}.$$

Now, by (13) we see that

$$[\hat{\varepsilon}, X^*_{\alpha\beta}](y_{ij}) = 2\xi_{\alpha i}\xi_{\beta j}.$$

This can not be 0 if $r \geq 2$, giving a contradiction.

As a corollary, we want to characterize the split isotropic super-Grassmannians.

Corollary. *The super-Grassmannian $\mathrm{I}^\circ\mathrm{Gr}_{2r|r,2s|s}$ is split if and only if $r = 1$.*

Proof. Proposition 11 shows that the super-Grassmannian is non-split if $r \geq 2$. Now, for $r = 1$ we have $H^1(M, \mathcal{B}_q) = 0$ for all $q \geq 2$, by Propositions 8 and 9. Thus, all the obstructions to the splitness are 0 (see [4], Ch.4, Sec. 2), and hence $\mathrm{I}^\circ\mathrm{Gr}_{2|1,2s|s}$ is split.

Theorem 2. *We have, for any $r \geq 1$, $s \geq 1$,*

$$H^0(M, \mathcal{T}) \simeq \mathfrak{osp}_{2r|2s}(\mathbb{C})$$

as Lie superalgebras, isomorphism being defined by the standard action of $\mathrm{OSp}_{2r|2s}(\mathbb{C})$. Also

$$H^1(M, \mathcal{T}) = \begin{cases} 0 & \text{if } (r,s) \neq (2,1) \\ \mathbb{C}^{1|0} & \text{if } r = 2,\ s = 1. \end{cases}$$

Proof. Suppose first that $(r,s) \neq (1,s)$ and $\neq (2,1)$. Then the proof goes precisely as in [6]. Using Theorem 1 and the cohomology exact sequence corresponding to (6), we see that $H^0(M,\mathcal{T}_{(q)}) = H^1(M,\mathcal{T}_{(q)}) = 0$ for $q \geq 3$. For $q = 2$ this exact sequence shows that $H^0(M,\mathcal{T}_{(2)}) = 0$ and that $H^1(M,\mathcal{T}_{(2)})$ is mapped injectively into $H^1(M,\tilde{\mathcal{T}}_2) \simeq \mathbb{C}^{1|0}$. Thus, $H^1(M,\mathcal{T}_{(2)}) \simeq \mathbb{C}^{k|0}$, $k \leq 1$. For $q = 1$ the exact sequence shows that $H^0(M,\mathcal{T}_{(1)}) = 0$ and that $H^1(M,\mathcal{T}_{(1)}) \simeq \mathbb{C}^{k|0}$. For $q = 0$ we get the exact sequence

$$\begin{aligned} 0 &\to H^0(M,\mathcal{T}_{(1)}) \to H^0(M,\mathcal{T}_{(0)}) \to H^0(M,\tilde{\mathcal{T}}_0) \\ &\to H^1(M,\mathcal{T}_{(1)}) \to H^1(M,\mathcal{T}_{(0)}) \to H^1(M,\tilde{\mathcal{T}}_0). \end{aligned} \tag{21}$$

This implies that $H^0(M,\mathcal{T}_{(0)})$ is mapped injectively into $H^0(M,\tilde{\mathcal{T}}_0)$. By Proposition 11, the trivial submodule $\mathbb{C}$ does not lie in the image. Therefore $H^1(M,\mathcal{T}_{(1)}) \neq 0$, and hence $H^1(M,\mathcal{T}_{(1)}) \simeq \mathbb{C}^{1|0}$, $H^1(M,\mathcal{T}_{(0)}) = 0$. Also, $H^0(M,\mathcal{T}_{(0)}) \simeq \mathfrak{osp}_{2r|2s}(\mathbb{C})_{\bar{0}}$. Now, for $q = -1$ we get the exact sequence

$$\begin{aligned} 0 &\to H^0(M,\mathcal{T}_{(0)}) \to H^0(M,\mathcal{T}) \to H^0(M,\tilde{\mathcal{T}}_{-1}) \\ &\to H^1(M,\mathcal{T}_{(0)}) \to H^1(M,\mathcal{T}) \to H^1(M,\tilde{\mathcal{T}}_{-1}). \end{aligned}$$

It implies that

$$\begin{aligned} &H^0(M,\mathcal{T}) \simeq H^0(M,\mathcal{T}_{(0)}) \oplus H^0(M,\tilde{\mathcal{T}}_{-1}) \simeq \mathfrak{osp}_{2r|2s}(\mathbb{C}), \\ &H^1(M,\mathcal{T}) = 0. \end{aligned}$$

For the 0-cohomology we mean here an isomorphism of G-modules. Since $\mathfrak{osp}_{2r|2s}(\mathbb{C})$ is simple [3], the homomorphism $X \rightsquigarrow X^*$ of this superalgebra into $H^0(M,\mathcal{T})$ is injective. Therefore it is an isomorphism of Lie superalgebras.

Suppose that $r = 2$, $s = 1$. Then the super-Grassmannian has dimension $2|2$. Using Theorem 1, we see that $H^1(M,\mathcal{T}_{(1)}) \simeq H^1(M,\mathcal{T}_{(2)}) \simeq \mathbb{C}^{2|0}$. Then the exact sequence (21) and Proposition 11 give that $H^1(M,\mathcal{T}_{(0)}) \simeq \mathbb{C}^{1|0}$. It follows that $H^1(M,\mathcal{T}) \simeq \mathbb{C}^{1|0}$.

The case $r = 1$ is the simplest one, and we omit the proof.

It follows from Theorem 2 that the supermanifold $\mathrm{I}^\circ\mathrm{Gr}_{2r|r,2s|s}$ is rigid if $(r,s) \neq (2,1)$ (see [8]). The remaining case $r = 2$, $s = 1$ was actually studied before. It is easy to see that $\mathrm{I}^\circ\mathrm{Gr}_{4|2,2|1}$ is precisely the supermanifold $\mathbb{G}(1,1)$ from the family $\mathbb{G}(t_1,t_2)$ constructed in [2], where the corresponding part of Theorem 2 was proved. By Theorem 4 of [2], this family is a versal deformation of $\mathrm{I}^\circ\mathrm{Gr}_{4|2,2|1}$. Thus, we get

Corollary. *The super-Grassmannian* $\mathrm{I}^\circ\mathrm{Gr}_{2r|r,2s|s}$ *is a rigid supermanifold if and only if* $(r,s) \neq (2,1)$.

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