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**SOME MATHEMATICAL ASPECTS
OF 3D X-RAY TOMOGRAPHY**

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§1. RECONSTRUCTION FORMULAE

Let \mathbf{E} be a three-dimensional Euclidean space and $\mathbf{\Lambda}$ be the variety of straight lines $L \subset \mathbf{E}$. The ray transform of a function f on \mathbf{E} with compact support is the family of integrals

$$(1) \quad Rf(L) := \int_L f dL,$$

defined on $\mathbf{\Lambda}$, where dL is the line measure in \mathbf{E} . There are several inversion methods for the operator R which could be used for construction of tomography algorithms. We discuss here some properties of presumable algorithms under the following assumptions on the inversion method:

- the data (1) is used only for a three-dimensional family Σ of straight lines or rays L (called a *pencil*);
- an exact inversion is given by a *simple* formula.

A formula is called *simple* if it is a combination of finite set of derivations, integrations and algebraic operations. Several cases are known, where there exists simple formulas:

- I Orlov's pencil $\Sigma = \Sigma(\mathbf{C}_\infty)$
- II Kirillov-Tuy's pencil $\Sigma = \Sigma(\mathbf{C}_0)$
- III pencil $\Sigma = \Sigma(\mathbf{S})$ of rays, which start with a smooth surface \mathbf{S} and are tangent to his surface.

The following *completeness* condition is assumed:

for any point $x \in \text{supp } f$ and for any plane H through this point there exists a line $L \in \Sigma$, which passes through x and belongs to H .

For each of the cases I,II,III there is a simple reconstruction formula which consists of two steps: first the two-dimensional Radon transform

$$Rf(H) \equiv Rf(p, \omega) = \int_H f dH$$

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or its derivative $\partial/\partial p Rf(p, \omega)$ is calculated. Here H means a plane with an equation $\omega \cdot x = p$. The second step is an application of the Lorentz-Radon inversion formula

$$(2) \quad f(x) = -\frac{1}{8\pi^2} \int_{\Omega} \frac{\partial^2}{\partial p^2} Rf(\omega \cdot x, \omega) d\omega.$$

We call any reconstruction of this kind *back-projection* formula, because the back-projection operation for the data $\partial^2/\partial p^2 Rf(p, \omega)$ is used on the second step.

Now we list some first step reconstruction formulae for the above cases.

I The Orlov's pencil $\Sigma(\mathbf{C}_{\infty})$ is the set of lines $L(x, e)$, where e runs over a curve \mathbf{C}_{∞} in the unit sphere $\Omega \in \mathbf{E}$ and $L(x, e)$ means the line, which contains a point x and is parallel to a unit vector e .

The completeness condition for this pencil means that any plane H is a union of parallel lines $L(x(t), e)$, $e \in \mathbf{C}_{\infty}$, $t \in \mathbb{R}$. Therefore we can find the Radon transform integrating the line data on the parameter t against the measure $dt := dH/dL$:

$$Rf(H) = \int Rf(L(x(t), e)) dt.$$

II The Kirillov-Tuy's pencil $\Sigma(\mathbf{C}_0)$ is the manifold of rays with origins on a curve $\mathbf{C}_0 \subset \mathbf{E}$. Then Grangeat-Finch formula gives a derivative of the Radon transform

$$\frac{\partial}{\partial p} Rf(H) = \int_0^{2\pi} \frac{\partial}{\partial \theta} Rf(L(x, e))|_{\theta=0} d\varphi,$$

where $L(x, e)$ means the ray in E , which starts from a point $x \in \mathbf{C} \cap H$ and is parallel to a vector $e \in \Omega$; $\varphi = \text{longitude}$ and $\theta = \text{latitude}$ are coordinates of e on Ω with the pole ω . There are other reconstruction formulae for this pencil.

III For the pencil $\Sigma(\mathbf{S})$ there is the following reconstruction formula (A.S.Denisjuk-V.P.Palamodov [1]):

$$\frac{\partial}{\partial p} Rf(H) = \frac{1}{\delta} \int_{\mathbf{C}(H)} \left[-\kappa \frac{\partial}{\partial q} + \csc^2 \psi \frac{d\psi}{ds} \right] Rf(L(x, e)) ds,$$

where s is the natural parameter on a curve $\mathbf{C}(H) \subset \mathbf{S} \cap H$, e is the tangent vector and κ is the curvature of this curve, ψ is the angle between H and \mathbf{S} at a point x ; $q = \omega \cdot e$. The curve $\mathbf{C}(H)$ is submitted to the following condition: the natural mapping $\mathbf{C}(H) \times \mathbb{R}_+ \rightarrow H$ is proper over the set $\text{supp } f \cap H$ and its degree δ does not vanish.

§2. ALGORITHMS AND ERRORS

Any simple formula may be used as a starting point for a computerized tomography algorithm. The algorithm should include the following steps:

- (d) discretization of integrals, in particular discretization of the back-projection operation (2).
- (f) filtering of derivations and

(i) interpolation of the data if necessary.

We call it a *back-projection* algorithm and claim that any back-projection algorithm produces image errors and artifacts whose geometry should have some features which do not depend on the simple formula used for the first step.

Fix a back-projection algorithm \mathcal{A} . The total error $E(f) = E(\mathcal{A}, f)$ of the reconstruction of an original f is the difference

$$E(f) := g - f,$$

where g is the result of the tomographic reconstruction. It can be written as a sum of two terms

$$E(f) = E_s(f) + E_h(f),$$

where $E_s(f)$ is the part caused by the operations (d),(f),(i) only (software part) and $E_h(f)$ the part caused by the physical origins (hardware part). Namely we define the software part as follows

$$E_s(f) := \mathcal{A}(Rf(\Sigma[d])) - f,$$

where $\mathcal{A}(Rf(\Sigma[d]))$ is the reconstruction, which is given by the algorithm \mathcal{A} with the data of integrals $Rf(L)$ for a finite sampling $\Sigma[d] \subset \Sigma$. Here $d \ll 1$ is the average distance between adjacent segments $L \cap \text{supp } f$, $L \in \Sigma[d]$. Whence, we get

$$E_h(f) = g - \mathcal{A}(Rf(\Sigma[d])).$$

This part of the total defect may be caused by several physical sources, in particular, by

(i) the effect of “partially filled volume”: for a simple model this means that

$$\int \exp(-\int f dL) c dt \neq \exp(-\iint f dL c dt),$$

where the function $c = c(t)$ is a characteristics of collimator of a detector.

- (ii) non-linearity of the detector characteristics, especially if the ratio noise/signal is not small enough;
- (iii) polychromatism of the radiation and “beam hardening”.

These effects imply together that the real data which the algorithm works up does not coincide with (and sometimes is far from) the exact values of the integral (1).

Remark Note that the software part $E_s(\cdot)$ is a *linear operator*, unlike the hardware part $E_h(f)$ which is typically a *non-linear operator*. Therefore these operators respond differently to a source of errors. The main such source is a discontinuity of the original f . This source begets an error $E_s(f)$ whose core is localized in a vicinity of singular set of f , but an essential support of $E_h(f)$ may extend far beyond of this set.

§3. CONVERGENCE OF BACK-PROJECTION ALGORITHMS

We call an algorithm \mathcal{A} *convergent* for an original f , if the software error $E_s(\mathcal{A}, f)$ tends to zero as $d \rightarrow 0$.

Claim 1. *Let \mathcal{A} be a back-projection algorithm that includes an appropriate filtering and interpolation and is applied to the integral data (1) available for a properly distributed sampling $\Sigma[d] \subset \Sigma$. Then the function $E_s(f)$ tends to zero uniformly as $d \rightarrow 0$ for any function $f \in C^q(\mathbf{E})$ with compact support, if q is big enough.*

We call a filter appropriate, if it has an effective window in the spectral domain of the size $\leq 1/2d$ like, for example, the 2D Shepp-Logan filter. Any properly distributed sampling should be d -dense in the phase space $T^*(\mathbf{E})$ in the following sense: for any point $(x, \xi) \in T^*(\mathbf{E}), x \in \text{supp } f, |\xi| = 1$ there is a line $L \in \Sigma[d]$, such that

$$\text{dist}(x, L) \leq d, \quad \text{dist}^*(\xi, L^\perp) \leq \rho \cdot \frac{d}{r}.$$

Here r is radius of the smallest ball that contains $\text{supp } f$, L^\perp means the plane in $T_x^*(\mathbf{E})$ that is orthogonal to L and dist^* means the Euclidean distance the dual space $T_x^*(\mathbf{E}) \cong \mathbf{E}^*$. This condition is akin to the Nyquist inequality; the optimal values of the factor ρ and an effective size of window of the filter should be found. See [2] for a discussion of similar problems for 2D back projection algorithms. To transform this Claim to a theorem the conditions on filters and interpolation should be specified. We shall call an algorithm that satisfies these conditions an *appropriate* Σ -algorithm.

For a function f with singularities the convergence of an appropriate algorithm is not certain even for an open set, where the function is smooth enough. Some simple examples of 2D computer simulated tomography show that the singular set of f spreads around a spot of error $E_s(f)$. The shape of this spot depends on the geometry of the singular set. To get some ideas about the shape of E_s we specify the class of originals. Suppose that the function f has the following simple form

$$(3) \quad f = a \delta(B) \quad \text{or} \quad f = a \chi(V)$$

where $a \in C_0^\infty(\mathbf{E})$, $\chi(V)$ denotes the characteristic function of an open set $V \subset \mathbf{E}$ with a smooth boundary B and δ_B is the delta-function on the boundary. In fact the delta-function is a model for the delta-like density $\frac{1}{2\varepsilon} \chi(B_\varepsilon)$, where ε is small and B_ε means ε -neighborhood of B . Apparently the value of $|E_s(f)|$ can not be small near B since $\mathcal{A}(Rf)$ is an approximation of the discontinuous original f with help of continuous functions.

We give a qualitative estimate of this value in terms of local geometry of B . In particular, we shall say that the error E_s is greater at a point $x \in B$ at one side of the surface B than at another side, if the set $y \in U : |E_s(f)| \geq l |F(x)|$, is larger at this side for a small ball U centered at x . Here $F(x)$ is an average values of the function f at this point and $l > 0$ is a small parameter. We have $F(x) \approx 1/2\varepsilon$, if $f \approx \delta(B)$ and $F(x) \approx 1/2$ if $f = \chi(V)$ for any point $x \in B$.

Claim 2. *If the set V is convex, then for an appropriate Σ -algorithm \mathcal{A} the error $E_s(\chi(V))$ tends to zero uniformly as $d \rightarrow 0$ on any compact set $K \subset E \setminus B$. In a neighbourhood of B the quantity $|E_s(f)|$ is smaller inside of V than outside.*

A theorem of this kind can be proved by means of technique like [3,4].

Let now V be an arbitrary domain with smooth boundary B . Denote by K the Gaussian curvature of the boundary. Recall that an *inflexional* tangent to a surface B is a tangent with zero normal curvature. A curve $A \subset B$ is called an *asymptotic curve*, if any tangent line L to A is inflexional for B . An inflexional tangent line L to B is called *simple*, if it is not an inflexion tangent for the corresponding asymptotic curve A . This means that

$$(4) \quad \text{dist}(x, B) \sim \text{dist}(x, B \cap L)^{k+1}$$

for $x \in L$ close to $B \cap L$, where $k = 2$. We call a straight line a *double inflexional* tangent to B , if it is a simple inflexional tangent to A . This is equivalent to the relation (4) with $k = 3$.

Claim 3. For any appropriate Σ -algorithm \mathcal{A} and any original of the form (4) such that $K(x) < 0$ the error $E_s(f)$ is greater near a point $x \in B$, where there is a double inflexional tangent $L_x \in \Sigma$ comparing with points $y \in B$, where there is no such inflexional tangent as d tends to 0.

It is plausible that $E_s(\chi(V)) \rightarrow 0$ outside the union of all inflexional tangents $L_x \subset \Sigma$. In spite of that the error E_s may be not small at least the shape of V can be recognized by means of an appropriate algorithm. This is no more the case, if the completeness condition is not fulfilled. A reconstruction will be not adequate for the open part B' of the boundary such that for any $x \in B'$ there is no line $L \in \Sigma$ such that $x \in L \subset T_x(B)$. This failure is a corollary of the lack of the data and does not depend on the kind of algorithm used. See [2,5,3] for discussion of the similar problem in 2D-tomography.

§4. GEOMETRY OF LINE SPACE AND SINGULARITY OF TANGENT LINE PENCILS

The line space Λ possesses a geometric structure of pseudo-Minkowski space: if we consider \mathbf{E} as a real twistor space, we get an exact analogy with Penrose's construction of the complex Minkowski space which starts with a complex twistor space \mathbb{C}^3 . To explain the pseudo-Minkowski geometry we choose the following charts for Λ : fix two parallel planes H_1, H_2 in \mathbf{E} and two linear functions α, β on \mathbf{E} , which are linearly independent on these plains. Taking a line L that is not parallel to these planes, we denote by α_1, β_1 and α_2, β_2 the values of α and β in the points $L \cap H_1$ and $L \cap H_2$ correspondingly. Hence we get four coordinates $\alpha_1, \beta_1, \alpha_2, \beta_2$, defined on an open part of Λ . Note some simple properties of these coordinates:

- (i) varying the functions α, β and the plane H_1 one makes a linear affine transformation of the coordinates. A change of the planes H_1, H_2 , induces a linear projective transformation of the coordinates.
- (ii) For two lines L, L' the equation

$$\det \begin{vmatrix} \alpha_1 - \alpha'_1 & \beta_1 - \beta'_1 \\ \alpha_2 - \alpha'_2 & \beta_2 - \beta'_2 \end{vmatrix} = 0, \quad \alpha'_i = \alpha_i(L'), \quad \beta'_i = \beta_i(L')$$

holds if and only if these lines have a common point or are parallel. For a fixed line L this equation defines a quadratic cone in Λ of signature $(2, 2)$, which is called *the light cone* with the vertex L .

- (iii) the set $\Lambda(x)$ of lines that contains a fixed point $x \in \mathbf{E}$ is the set of solutions of two linear affine equation on coordinates $\alpha_1, \beta_1, \alpha_2, \beta_2$. Really we may shift the plane H_2 , because of (i) hence, we may suppose that $x \in H_1$. Then the set of lines through x will be given by the equations $\alpha_1 = 0, \beta_1 = 0$. Whence, $\Lambda(x)$ is a plane in Λ .

For any plane H in \mathbf{E} the set $\Lambda(H)$ of lines $L \subset H$ is as well a plane in Λ . This implies that

- (iv) For any point $x \in \mathbf{E}$ and any plane H through x the set of lines L such that $x \in L \subset H$ is a projective line $\Lambda(x, H)$ in Λ (called *light ray*).

Now we take in the play a smooth surface $B \subset \mathbf{E}$. The pencil $\Sigma(B)$ of lines, which are tangent to B is a threefold in Λ , which may be singular. This pencil is an union of projective lines $\Lambda(x, T_x(B)), x \in B$, where $T_x(B)$ is the tangent bundle of B . It is easy to check the following

Proposition 1. *If B is strictly convex, i.e. $K > 0$, then $\Sigma(B)$ is smooth.*

Proposition 2. *If $K(x) < 0$ or $K(x) = 0, dK(x) \neq 0$ for a point $x \in B$ and L_x is a simple inflexion tangent to B , then the set $\Sigma_2(B) \subset \Sigma(B)$ of all simple inflexion tangent to B is locally diffeomorphic to the product $D_2 \times \mathbb{R}^2$, where D_2 is the plane cusp curve, given by the equation*

$$(5) \quad 4s_1^3 + 27s_0^2 = 0.$$

This means that there exists a diffeomorphism of germs

$$(s, t) : (\Lambda, L_x) \rightarrow (S \times \mathbb{R}^2, (0, 0))$$

that takes $\Sigma(B)$ onto $D_2 \times \mathbb{R}^2$ and $\Sigma_2(B)$ onto $0 \times \mathbb{R}^2$, where S is a plane with the coordinates s_0, s_1 .

The following equation holds

$$(6) \quad s_0(L) = \alpha_1 + o(\alpha_1, \alpha_2, \beta_1, \beta_2),$$

if we choose for α a linear function that vanishes on the tangent plane $T_x(B)$, for β a function that vanishes on L_x , and take for H_1 a plane that contains x .

Any line $\Lambda(x, T_x(B))$ meets $\Sigma_2(B)$ twice and is tangent to this surface.

Proposition 3. *Suppose that $K(x) < 0$ and L_x is a double inflexional tangent to B at x . Then there exists a diffeomorphism of germs*

$$(s, t) : (\Lambda, L_x) \xrightarrow{\sim} (S \times \mathbb{R}, (0, 0))$$

such that

$$\Sigma(B) \xrightarrow{\sim} D_3 \times \mathbb{R}, \quad \Sigma_3(B) \xrightarrow{\sim} \{0\} \times \mathbb{R},$$

where $\Sigma_3(B)$ is the set of all double inflexion tangents and D_3 is the discriminant surface in $S = \mathbb{R}^3$ ("swallow tail"). If we choose coordinates on Λ as in Proposition 2 and coordinates s_0, s_1, s_2 on S as below, then the equation (7) holds once more. The

light ray $\Lambda(x, T_x(B))$ is tangent to the surface given by the equations $s_0 = s_1 = 0$ in Λ .

The surface D_3 is given by the equation

$$(7) \quad 256s_0^3 - 128s_2^2s_0^2 + 16s_2^4s_0 - 4s_2^2s_1^2 + 144s_2s_1^2s_0 - 27s_1^4 = 0,$$

with the parabola $s_2 < 0, s_1 = 0, 4s_0 = s_2^2$ cut off. See [6] for details and pictures.

Proof Choose a smooth function φ such that $\varphi = 0, d\varphi \neq 0$ on B and consider the mapping $\lambda : \mathbb{R} \times \Lambda \rightarrow E$ that takes a pair (t, L) into $y = y(t, L)$, where $y(\cdot, L)$ is a smooth parameterization of lines $L \in \Lambda$. The pullback $\psi := \lambda^*(\varphi)$ is a smooth function on $\mathbb{R} \times \Lambda$ and satisfies the equation $\psi(t, L_x) \sim (t - t(x))^{k+1}$ where $y(t(x), L_x) = x$ and $k = 2, 3$ correspondingly. Applying the Malgrange's preparation theorem [7] we get a factorization $\psi = hq$, where h is a smooth function which does not vanishes at the point $(t(x), L_x)$ and

$$q(t, L) = t^{k+1} + s_k t^k + s_{k-1} t^{k-1} + \cdots + s_1 t + s_0$$

where s_0, s_1, \dots, s_k are smooth functions of L . Setting $t = r - \frac{s_k}{k+1}$, we get the equation $q(t, L) = p_k(r, s(L))$ where $p_k(r, s)$ is a similar polynomial with $s_k = 0$. A line L is tangent to B if and only if the function $\psi(\cdot, L)$ has a real multiple root. This is equivalent to the condition that $p_k(\cdot, s(L))$ has a multiple root, where $s = (s_0, s_1, \dots, s_{k-1})$. The last condition means that the point $s(L)$ belongs to the corresponding discriminant set (5),(7). This proves Propositions 2 and 3.

Corollary 4. *In a neighbourhood of the set $\Sigma_2(B)$ there exists a smooth mapping $x_2 : \Lambda \rightarrow B$ whose restriction on $\Sigma_2(B)$ coincides with the natural projection onto B .*

We define $x_2(L)$ to be the point on B , where the line $L(0, t)$ is an inflexional tangent, $t = t(L)$ is a component of the mapping given in Proposition 2 and $L = L(s, t)$ is the inverse mapping. Using Proposition 3 we define in a similar way a projection $x_3 : \Lambda \rightarrow I$. Here $I \subset B$ is a smooth curve of points y , where a double inflexion tangent L_y touche B .

§5. SINGULARITIES OF THE RAY TRANSFORM

Consider the ray transform Rf of the original like (3). This function is in C^∞ outside the threefold $\Sigma(B)$, whose singularities were described in the previous section. We give here an asymptotic representation of Rf near $\Sigma(B)$ by means some special functions.

If the body V is convex, then an asymptotic of Rf near $\Sigma(B)$ is well-known for a function like (4) as well as for more general singular functions. For the sake of completeness we write it down here. Taking a line L that is close to $\Sigma(B)$ and meets V , we find a point $x \in B$ such that the plane $P(L, x)$ through L and x contains the normal vector $n(x)$ to B . We denote $x_1(L) := x$, $\sigma(L) := \text{dist}(L, x_1)$ and define $\kappa(L)$ to be the curvature of the section $P \cap B$ at x_1 . Then we have the following asymptotic development for the ray transform of the delta-like original:

$$R(a\delta(B))(L) = [a(x_1(L)) + o(1)] |\kappa(x_1(L))|^{-\frac{1}{2}} \sigma(L)^{-\frac{1}{2}}.$$

For the original $f = a\chi(V)$ we get a similar formula where the function $\sigma^{-\frac{1}{2}}$ is changed by its primitive $2\sigma^{\frac{1}{2}}$.

Now we pass to the case of non-convex V , where the geometry of $\Sigma(B)$ was described by Propositions 2 and 3. Fix $k > 1$ and consider the following family of polynomials

$$p(r, s) = r^{k+1} + s_{k-1} r^{k-1} + s_{k-2} r^{k-2} + \cdots + s_1 r + s_0$$

with real coefficients $s = (s_0, s_1, \dots, s_{k-1})$. Let D_k be the set of points $s \in S = \mathbb{R}^k$ such that $p(r, s)$ has at least one real multiple root r . For $k = 2, 3$ this set was described in the previous section. For $s \in S \setminus D_k$ we define

$$v_k(s) := \sum \frac{1}{|p'(r_i(s), s)|},$$

where the sum is taken over the set of all simple real roots $r_i(s), i = 1, 2, \dots$ of p ; we put $v_k(s) = 0$, if there is no such a root. This function is real analytic on the complement to D_k .

Theorem 1. *Under the conditions of Proposition 2 the ray transform of the function $f = a\delta(B)$ admits the following asymptotic representation in a neighborhood of $\Sigma_2(B)$:*

$$Rf(L) = [a(x_2(L)) + o(1)] g_2(x_2(L)) v_2(s(L))$$

as $L \rightarrow \Sigma_2(B)$, where $s = s(L)$ is the submersion defined in Proposition 2, $x_2(L)$ is given in Corollary 4 and $g_2(x)$ is a smooth non-zero function on B (see below).

In a neighborhood of the set $\Sigma_3(B)$ a similar equation holds:

$$Rf(L) = [a(x_3(L)) + o(1)] g_3(x_3(L)) v_3(s(L))$$

as $L \rightarrow \Sigma_3(B)$, where g_3 is a smooth non-vanishing function on the curve I .

The quantity $o(1)$ tends uniformly to zero as L tends to a compact subset of $\Sigma_2(B)$ and of $\Sigma_3(B)$ correspondingly.

To get a similar formulae for an original like $\chi(V)$ we use the following continuous special functions:

$$w_2(s) := \int_0^{s_0} v_2(u, s_1) du, \quad w_3(s) := \int_{s_0}^{\infty} v_3(u, s_1, s_2) du.$$

Note that the function $w_3(s)$ vanishes in the component of $S \setminus D_3$, where $s_0 \geq O(|s_1|^{\frac{4}{3}} + s_2^2)$, since the function v_3 does.

Theorem 2. *Under the same hypothesis if $f = a\chi(V)$, then we have*

$$Rf(L) = [a(x_k(L)) + o(1)] g_k(x_k(L)) w_k(s(L))$$

as $L \rightarrow \Sigma'_k$, where $k = 2, 3$.

We specify the functions g_2, g_3 in the following way: take a point $x \in B$ an inflexional tangent L_x at x and the plane P that is orthogonal to B at x and

contains L_x . Then choose Euclidean coordinates α, γ in P with the origin in x , where α is as in Proposition 2; we can reach the equation $\|\alpha\| = 1$ keeping (6) and (7) and rescaling in S . Write down an equation $\alpha = \alpha(\gamma)$ of the curve $P \cap B$ and have

$$\alpha(0) = \alpha'(0) = \alpha''(0) = 0, \alpha'''(0) \neq 0, \text{ if } L_x \text{ is simple inflexional}$$

and

$$\alpha(0) = \alpha'(0) = \alpha''(0) = \alpha'''(0) = 0, \alpha''''(0) < 0, \text{ if } L_x \text{ is double inflexional.}$$

The negative sign of the fourth derivative means that the α -axis has the outward direction with respect to the (non-strictly) convex curve $P \cap B$. Then we have

$$g_2(x) = \frac{1}{2} \left| \frac{6}{|z'''(0)|} \right|^{\frac{1}{3}}; \quad g_3(x) = \left| \frac{24}{z''''(0)} \right|^{\frac{1}{4}}.$$

Remark 3. The similar statement is true as well for the case $K(x) = 0$ and any order k of contact in (4) however the mapping (s, t) may be not a diffeomorphism.

Note that the singular support of the function $v_k(s(L))$ coincides with the set $\Sigma_k(B)$ for $k = 2, 3$ since this set is a pullback of the discriminant set D_k according to Propositions 2 and 3.

For a proof of Theorems 1, 2 and Remark 3 we apply [8, Theorem 3, Ch.5, Sec.5] to the function Rf . This function belongs to the class $\Sigma_0^m(W)$, $m = 1/2$, where W is an integral subvariety in the contact manifold $K(\mathbf{\Lambda})$ (see loc.cit.Ch.5). In fact W is the set of tangent spaces to the threefold $\Sigma(B)$ or to be precise W is the image of $T(\Phi(B))$ under the natural mapping $K(\Phi) \rightarrow K(\mathbf{\Lambda})$ (see sect.6). Therefore we get an representation

$$(8) \quad Rf(L) = \sum_{j=1}^k \int v_k^j(s_0 - t, s_1, \dots, s_{k-1}) b_j(L, t) dt \quad (\text{mod } C^\infty(U)),$$

where $s_j = s_j(L)$ and v_k^j , $j = 0, 1, \dots, k-1$ are some special functions (called *versal integrals*), $b_j dt$ are some distributions which belong to the class $\Sigma_{1,0}^0$ and U is a neighbourhood of a given line $L_0 \in \Sigma_k(B)$. The equation $\text{singsupp } v_k^j = D_k$ holds for $j = 0, \dots, k-1$; the function v_k^0 is positive has the sharpest singularities and coincides with v_k as above. The inclusion $b_j dt \in \Sigma_{1,0}^0$ implies that the Fourier transform $\widehat{b}_j(L, \tau) = F_{t \rightarrow \tau}(b_j(L, t) dt)$ is bounded as $|\tau| \rightarrow \infty$. In fact the function $\widehat{b}_0(L, \cdot)$ has a non-zero limit b at infinity. It can be proven by comparing character of singularities of functions $Rf(L)$ and of function v_k on the α_1 -axis. This assertion implies that $\widehat{b}_0(\cdot, t) dt = b\delta(t) dt + b'$ where a distribution b' is smoother than delta-distribution δdt . Then we change the distribution $b_0 dt$ in (8) to $b\delta(t) dt$ and omit all the terms with $j > 0$. This gives an asymptotic for Rf and proves Theorem 1. Theorem 2 can be proved on the same lines.

An alike asymptotic representation for the ray transform $Rf(L)$, $f = \varphi_+^\lambda(x)$, $\lambda \in \mathbb{C}$ can be written under the similar conditions on geometry of the hypersurface $B = \{x \in \mathbb{R}^n, \varphi(x) = 0\}$.

§6. GEOMETRY OF NON-LINEAR ARTIFACTS

Consider a singular original that admits the following form

$$f \approx a_B \delta_B + a_C \delta_C + a_F \delta_F,$$

where a_B, a_C, a_F are in $C_0^\infty(\mathbf{E})$ and $\delta_B, \delta_C, \delta_F$ mean respectively the delta-function on a smooth surface B , smooth curve C and on a finite set F . We mean that f is in fact a sum of “delta-like” functions

$$(9) \quad f_\varepsilon = a_B \frac{1}{\omega_1 \varepsilon} \chi(B_\varepsilon) + a_C \frac{1}{\omega_2 \varepsilon^2} \chi(C_\varepsilon) + a_F \frac{1}{\omega_3 \varepsilon^3} \chi(F_\varepsilon) + a,$$

where G_ε means ε -neighborhood of a set $G \subset \mathbf{E}$ and ω_k is the volume of the k -dimensional unit ball. A reconstruction of a function of this type with small parameter d may have heavy artifacts, which are big far from the singularity set $B \cup C \cup F$ of the original. Artifacts of this kind are well-known for 2D case; they should be identified as images of the non-linear part $E_h(f)$ of the defect. We try to describe the geometry of $E_h(f)$ for an appropriate algorithm without specifying its details.

For this we need more geometry. First, introduce a flag space Φ , which consists of pairs (x, L) , where $x \in L \in \mathbf{\Lambda}$. This is a five-dimensional manifold which has two natural projections

$$\mathbf{E} \xleftarrow{\varepsilon} \Phi \xrightarrow{\lambda} \mathbf{\Lambda},$$

which send a pair (x, L) to x and L correspondingly. For any smooth surface $B \subset \mathbf{E}$ we define a threefold $\Phi(B) \subset \Phi$ that consists of pairs $(x, L), x \in B$, where L is a tangent line to B at x . Evidently $\lambda(\Phi(B)) = \Sigma(B)$. Then for any curve C in \mathbf{E} we define $\Phi(C)$ to be the threefold of pairs $(x, L), x \in C \cap L$. For any finite subset $F \subset \mathbf{E}$ a surface $\Phi(F)$ is defined in a similar way. For a function like (9) we set

$$\Phi(f) := \Phi(B) \cup \Phi(C) \cup \Phi(F).$$

This is a subvariety of Φ , which may have self-intersection. Consider the mapping

$$(10) \quad \varphi : \Phi(f) \rightarrow \mathbf{\Lambda} \supset \Sigma,$$

which is a restriction of the projection λ , Σ is a pencil. We call a line $L \in \Sigma$ a *singular* element of the pencil, if

- (i) it is a critical value of φ or
- (ii) it has several preimages under φ or
- (iii) it is a point, where the mapping φ is not transversal to Σ .

The following lines are examples of singular elements:

- any line that is tangent to B in two or more points;
- each inflexional tangent to B ;
- any line that is tangent to B and meets the set $C \cup F$;
- any line that meets $C \cup F$ two or more times.

For some cases nonlinear artifacts may appear near any singular line. To show it we choose a simple mathematical model for E_h . Set

$$g := \mathcal{A}\{Tr_\Delta(Rf(\Sigma[d]))\},$$

where Tr_Δ is the following truncation operator: $Tr_\Delta(a) = a$, if $|a| \leq \Delta$ and $Tr_\Delta(a) = \text{sign } a \cdot \Delta$, if $|a| \geq \Delta$, $a \in \mathbb{R}$. Hence we have

$$E_h(f) = \mathcal{A}\{Tr_\Delta(Rf(\Sigma[d])) - Rf(\Sigma[d])\}.$$

Claim 4. For any appropriate Σ -algorithm, any function f_ε like (8) and any singular line L of the pencil Σ the function $E_h(f_\varepsilon)$ may tend to infinity near L as $\varepsilon \sim \Delta^{-1} \sim d \rightarrow 0$.

We mean that the hardware error may tend to infinity near a given singular line for a certain relation between the small parameters ε, Δ^{-1} and d . See [9] for 2D case.

It may be worthwhile to take in account all the points of Σ , where the mapping φ has a peculiarity with respect to the quasi-Minkowski geometry. A plane $\Lambda(H)$ or $\Lambda(x)$ may have a non-generic intersection with the image of the mapping φ . We call such points x and planes H singular. The error $E_h(f_\varepsilon)$ may be relatively big near any singular point or plane.

We anticipate a complicated structure of $E_h(f)$ near singular points, lines and planes (cf. [9]). Apparently if we change the singular function $\delta(B)$ in (10) to the function $\chi(V)$, some artifacts on singular lines still may appear. But the analysis of $E_h(f)$ is more complicated in this case.

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