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Integrals in The Hida Distribution Space $(\mathcal{S})^*$

by

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Integrals in The Hida Distribution Space $(\mathcal{S})^*$.

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Abstract

We give sufficient conditions for $(\mathcal{S})^*$-integrability. The results will be applied on Skorohod integrable processes, where we show the equality between the Skorohod integral and an $(\mathcal{S})^*$-integral involving white noise and Wick product, only using the Skorohod integrability requirement.

1 Introduction.

This article considers integrals in $(\mathcal{S})^*$, and the connection between an $(\mathcal{S})^*$-integral and the Skorohod integral. Moreover we will show that

$$\int_0^t Y_s \diamond W_s ds = \int_0^t Y_s \delta B_s \tag{1}$$

without any significant restrictions on the process Y_s, except the natural requirement of Skorohod integrability. The left hand side of this equation is to be understood as a Lebesgue integral in the $(\mathcal{S})^*$-sense. The right hand side is the familiar Skorohod integral. The symbol $\diamond$ denotes the Wick-product and W_s the white noise process. All these notions are discussed in detail.

In [LØU], Th.(3.3) there is an elegant proof of (1). However, the authors have put severe restrictions on the process Y_s. The restrictions are given in the following way: Define λ as the measure on the product-σ-algebra on $\mathbb{R}^n$ such that

$$\int f(y)d\lambda(y) = \int_{\mathbb{R}} \cdots (\int_{\mathbb{R}} (\int_{\mathbb{R}} f(y_1,\ldots,y_n)e^{-1/2y_1^2}\frac{dy_1}{\sqrt{2\pi}})e^{-1/2y_2^2}\frac{dy_2}{\sqrt{2\pi}})\ldots e^{-1/2y_n^2}\frac{dy_n}{\sqrt{2\pi}}$$

Let $z^\alpha = z_1^{\alpha_1}\ldots z_m^{\alpha_m}$ where $z_j \in \mathbb{C}$. If Y_t has the Wiener-Ito chaos expansion (*to be defined later, see (13)*)

$$Y_t = \sum_\alpha c_\alpha(t)H_\alpha(\omega)$$

then we define the Hermite transform of Y_t to be

$$\tilde{Y}_t = \sum_\alpha c_\alpha(t)z^\alpha$$

To obtain (1), [LØU] requires that

$$\int_0^t (\int\int |\tilde{Y}_s\tilde{W}_s|^2 d\lambda(x)d\lambda(y))ds < \infty$$

This restriction ensures the existence of the inverse Hermite transform (*for more information about the Hermite transform, see [LØU]*). In this article however, we prove that Skorohod integrability of Y_t is sufficient to ensure the existence of the left hand side of (1). The proof of the equality goes by a direct calculation, without using any Hermite transforms.

2 The Spaces $(\mathcal{S})$ and $(\mathcal{S})^*$.

We start by recalling some of the basic definitions and features of *the white noise probability space.* This brief introduction is mostly taken from [GHLØUZ]. For a more complete account, see [HKPS].

As usual, let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$ denote the space of tempered distributions on $\mathbb{R}^d$, which is the dual of the well-known Schwartz space $\mathcal{S}(\mathbb{R}^d)$. By the Bochner-Minlos theorem there exists a measure μ on $\mathcal{S}'$ such that

$$\int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2}, \phi \in \mathcal{S}(\mathbb{R}^d) \tag{2}$$

where $||.||$ is the $L^2(\mathbb{R}^d)$-norm. This measure corresponds to the bilinear form

$$\mathcal{E}(\phi,\psi) = \int_{\mathbb{R}^d} \phi\psi dx; \phi, \psi \in \mathcal{S}(\mathbb{R}^d)$$

Let $\mathcal{B}$ denote the Borel sets on $\mathcal{S}'$ (equipped with the weak star topology). Then the triple $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}, \mu)$ is called *the white noise probability space.*

Definition 1 *The white noise process is a map*

$$W : \mathcal{S} \times \mathcal{S}' \to \mathbb{R}$$

given by

$$W(\phi,\omega) = W_\phi(\omega) = \langle\omega,\phi\rangle, \omega \in \mathcal{S}', \phi \in \mathcal{S} \tag{3}$$

■

Since $\mathcal{S}$ is dense in L^2, we can define $\langle\omega,\phi\rangle$ for $\phi \in L^2$ by

$$\langle\omega,\phi\rangle = \lim_{n\to\infty} \langle\omega,\phi_n\rangle$$

where $\phi_n \in \mathcal{S}'$ is a sequence converging to $\phi \in L^2$. In particular, if we define

$$\tilde{B}_x(\omega) := \tilde{B}_{x_1,\ldots,x_d}(\omega) := \langle\omega, \mathcal{X}_{[0,x_1]\times\ldots\times[0,x_d]}(\cdot)\rangle \tag{4}$$

then $\tilde{B}_x$ has an x-continuous version B_x which then becomes a *d-parameter Brownian motion.*

The *d-parameter Wiener-Ito integral* of $\phi \in L^2$ is defined by

$$\int_{\mathbb{R}^d} \phi(y) dB_y(\omega) = \langle\omega,\phi\rangle \tag{5}$$

The left hand side coincides with the Ito integral if supp$(\phi) \subset [0,\infty)$. (*See [LØU], p.4*). Of special interest now will be the space $L^2(\mathcal{S}'(\mathbb{R}^d),\mu)$ or $L^2(\mu)$ for short. *The Wiener-Ito chaos expansion theorem* says that every $F \in L^2(\mu)$ has the form

$$F(\omega) = \sum_{n=0}^{\infty} \int_{(\mathbb{R}^d)^n} f_n(u) dB_u^{\otimes n}(\omega) \tag{6}$$

where $f_n \in L^2(\mathbb{R}^{nd})$ and f_n is symmetric in all its nd variables (in the sense that $f_n(u_{\sigma_1},\ldots,u_{\sigma_{nd}}) = f_n(u_1,\ldots,u_{nd})$ for all permutations σ.) The right hand side of (6) are the *multiple Ito integrals.*

With F, f_n as in (6) we have

$$||F||^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n!||f_n||^2_{L^2(\mathbb{R}^{nd})} \tag{7}$$

There is an equivalent expansion of $F \in L^2(\mu)$ in terms of the Hermite polynomials

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}); n = 0, 1, 2, \ldots$$

We now explain this more closely. Define the *Hermite function* of order n as $\xi_n(x)$

$$\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x) \tag{8}$$

where $x \in \mathbb{R}, n = 0, 1, 2, \ldots$. $\{\xi_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Therefore the family $\{e_\alpha\}$ of tensor products

$$e_\alpha := e_{\alpha_1,\ldots,\alpha_m} := \xi_{\alpha_1} \otimes \ldots \otimes \xi_{\alpha_d} \tag{9}$$

(where α denotes the multi-index $(\alpha_1, \ldots, \alpha_d)$) forms an orthonormal basis for $L^2(\mathbb{R}^d)$. Assume that the family of all multi-indices $\beta = (\beta_1, \ldots, \beta_d)$ is given a fixed ordering

$$(\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(n)}, \ldots)$$

where $\beta^{(k)} = (\beta_1^{(k)}, \ldots, \beta_d^{(k)})$. Put

$$e_n = e_{\beta^{(n)}}; n = 1, 2, \ldots$$

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a multi-index. It was shown by Ito that

$$\int_{(\mathbb{R}^d)^n} e_1^{\hat{\otimes}\alpha_1} \hat{\otimes} \ldots \hat{\otimes} e_m^{\hat{\otimes}\alpha_m} dB^{\hat{\otimes}n} = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) \tag{10}$$

where $\theta_j(\omega) = \int_{\mathbb{R}^d} e_j(x) dB_x(\omega), n = |\alpha|$ and $\hat{\otimes}$ denotes the *symmetrized tensor product*, so that, e.g., $f\hat{\otimes}g(x,y) = \frac{1}{2}[f(x)g(y) + f(y)g(x)]$ if $x, y \in \mathbb{R}$ and similarly for more than two variables.

If we define, for each multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$,

$$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) \tag{11}$$

then we see that (10) can be written

$$\int_{(\mathbb{R}^d)^n} e^{\hat{\otimes}\alpha} dB^{\otimes|\alpha|} = H_\alpha(\omega) \tag{12}$$

using multi-index notation: $e^{\hat{\otimes}\alpha} = e_1^{\hat{\otimes}\alpha_1} \hat{\otimes} \ldots \hat{\otimes} e_m^{\hat{\otimes}\alpha_m}$ if $e = (e_1, e_2, \ldots)$. Since the family $\{e^{\hat{\otimes}\alpha}; |\alpha| = n\}$ forms an orthonormal basis for the symmetric functions in $L^2((\mathbb{R}^d)^n)$, we see by combining (6) and (12) that we have the representation

$$F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \tag{13}$$

(the sum being taken over all multi-indeces α of nonnegative integers). Morover, it can be proved that

$$||F||^2_{L^2(\mu)} = \sum_\alpha \alpha! c_\alpha^2 \tag{14}$$

where $\alpha! = \alpha_1! \ldots \alpha_m!$.

There is a subspace of $L^2(\mu)$ which in some sense corresponds to the Schwartz subspace $\mathcal{S}(\mathbb{R}^d)$ of $L^2(\mathbb{R}^d)$. This space is called the *Hida test function space* and is denoted $(\mathcal{S})$. Using the characterization due to Zhang in [Z], a simple description of $(\mathcal{S})$ can be given as follows:

Definition 2 *Let $F \in L^2(\mu)$ have the chaos expansion*

$$F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

Then F is a Hida test function, *i.e. $F \in (\mathcal{S})$, if*

$$\sup_\alpha c_\alpha^2 \alpha!(2\mathbb{N})^{\alpha k} < \infty \ \forall \textit{ natural numbers } k < \infty \tag{15}$$

where

$$(2\mathbb{N})^\alpha := \prod_{j=1}^m (2^d \beta_1^{(j)} \dots \beta_d^{(j)})^{\alpha_j} \ \textit{if } \alpha = (\alpha_1, \dots, \alpha_m) \tag{16}$$

■

In this article, the dual of $(\mathcal{S})$, denoted $(\mathcal{S})^*$, will be studied. It is therefore of great importance to have a nice characterization of this space, which is called the *Hida distribution space.* Another theorem in [Z] states the following:

Theorem 3 *A* Hida distibution *G is a formal series*

$$G = \sum_\alpha b_\alpha H_\alpha \tag{17}$$

where

$$\sup_\alpha b_\alpha^2 \alpha!((2\mathbb{N})^{-\alpha})^q < \infty \textit{ for some } q > 0 \tag{18}$$

■

If $G \in (\mathcal{S})^*$ is given by (17) and $F \in (\mathcal{S})$ is given by (13), the action of G on F is given by

$$\langle G, F\rangle = \sum_\alpha \alpha! b_\alpha c_\alpha \tag{19}$$

Note that no assumptions are made regarding the convergence of the formal series in (17).

We can in a natural way regard $L^2(\mu)$ as a subspace of $(\mathcal{S})^*$. In particular, if $X \in L^2(\mu)$ then by (19) the action of X on $F \in (\mathcal{S})$ is given by

$$\langle X, F\rangle = E[X \cdot F]$$

Before we look at an example, we define the important *Wick product* of two Hida distributions F, G:

Definition 4 *Let $F = \sum_\alpha a_\alpha H_\alpha, G = \sum_\alpha b_\beta H_\beta$ be two elements of $(\mathcal{S})^*$. Then the* Wick product *of F and G is the element $F \diamond G$ in $(\mathcal{S})^*$ given by*

$$F \diamond G = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta} \tag{20}$$

■

We will in the rest of the article only consider $d = 1$, i.e. $\mathcal{S} = \mathcal{S}(\mathbb{R})$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$. Now, we turn the attention to an important element in $(\mathcal{S})^*$, namely *the white noise*, $W_t(\omega)$. This element is defined as

$$W_t(\omega) = \sum_{k=1}^\infty \xi_k(t) H_{\epsilon_k}(\omega) = \sum_{k=1}^\infty \xi_k(t) h_1(\theta_k) \tag{21}$$

where $\epsilon_k = (0, \dots, 0, 1)$ with 1 on the k'th place, $k = 1, 2, \dots$.We show that the white noise is an $(\mathcal{S})^*$-element: By (16)

$$(2\mathbb{N})^{\epsilon_k} = 2k$$

and (18) becomes

$$\sup_\alpha b_\alpha^2 \alpha!(2\mathbb{N})^{-\alpha q} = \sup_k \xi_k(t)^2 1 (2k)^{-q} < \infty$$

for some $q > 0$, since $\sup_t |\xi_k(t)| = O(k^{-1/12})$. (*See [HP], p.571*).

After this rather brief introduction to the white noise theory, we will have a look at integrals in $(\mathcal{S})^*$.

3 Integrals in $(\mathcal{S})^*$.

The two concepts of integrability of stochastic processes we are going to study, are the following:

Definition 5 *A process $Y_s \in (\mathcal{S})^*$ for $s \in [0,t]$ is called $(\mathcal{S})^*$-integrable if*

$$\langle Y_s, \psi \rangle \in L^1([0,t]), \forall \psi \in (\mathcal{S}) \tag{22}$$

The $(\mathcal{S})^*$-integral is then defined as the unique $(\mathcal{S})^*$-element

$$\langle \int_0^t Y_s ds, \psi \rangle = \int_0^t \langle Y_s, \psi \rangle ds \tag{23}$$

where $\psi \in (\mathcal{S})$. (See prop (6.1) in [HKPS]).

■

Definition 6 *A process $Y_s \in L^2(\mu)$ for $s \in [0,t]$ is* Skorohod *integrable if*

$$\int_0^t E[Y_s^2] ds + \sum_{m=1}^{\infty} (m+1)! ||\tilde{f}_m||^2 < \infty \tag{24}$$

where $\tilde{f}_m$ is the symmetrization of $f_m(.)\mathcal{X}_{(0,t)}(.)$ in the chaos expansion, (6). Moreover

$$\int_0^t Y_s \delta B_s = \int_0^t f_o(s) dB_s + \sum_{m=1}^{\infty} \int_{\mathbb{R}^{m+1}} \tilde{f}_m dB^{\otimes m+1} \tag{25}$$

■

If we have chaos expanded an $(\mathcal{S})^*$-integrable process, what is its $(\mathcal{S})^*$-integral ? The answer to this question is:

Proposition 7 *Assume $Y_s \in (\mathcal{S})^*, s \in [0,t]$, has the chaos expansion*

$$Y_s = \sum_{\alpha} c_\alpha(s) H_\alpha$$

where

$$\sum_{\alpha} \alpha! |a_\alpha| \int_0^t |c_\alpha(s)| ds < \infty$$

$\forall \psi = \sum a_\alpha H_\alpha \in (\mathcal{S})$. *Then Y_s is $(\mathcal{S})^*$-integrable, and*

$$\int_0^t Y_s ds = \sum_{\alpha} (\int_0^t c_\alpha(s) ds) H_\alpha \tag{26}$$

■

Proof: Since

$$\int_0^t |\langle Y_s, \psi \rangle| ds \leq \sum_{\alpha} \alpha! |a_\alpha| \int_0^t c_\alpha(s)| ds < \infty$$

by assumption, the $(\mathcal{S})^*$-integrability follows from (22). By Th.(2.25) in [F] we can change sums and integrals. Now invoking the definition of $(\mathcal{S})^*$-integrals, we get

$$\langle \int_0^t Y_s ds, \psi \rangle = \int_0^t \langle Y_s, \psi \rangle ds = \int_0^t (\sum_{\alpha} \alpha! a_\alpha c_\alpha(s)) ds$$

$$= \sum_\alpha \alpha! a_\alpha \int_0^t c_\alpha ds = \langle \sum_\alpha \int_o^t c_\alpha(s) ds H_\alpha, \psi \rangle$$

The proposition follows. ■

Example: Assume

$$f \in L^2((0,t)) \text{ a.e. } s \in [0,t]$$

Then for $\psi = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})$ we have

$$\sum_k |a_{\epsilon_k}| \int_0^t |f(s)||\xi_k(s)| ds < \infty$$

This is so because we can define the element

$$Z = \sum_k (\int_0^t |f(s)||\xi_k(s)| ds) H_{\epsilon_k}$$

which is in $(\mathcal{S})^*$, since

$$\sup_k (\int_0^t |f(s)||\xi_k(s)| ds)^2 (2k)^{-q} \leq \sup_k (\int_0^t f^2 ds)(\int_0^t \xi_k^2 ds)(2k)^{-q}$$

$$\leq ||f||^2 1 \sup_k (2k)^{-q} < \infty, \forall q > 0$$

This implies

$$\infty > \langle Z, \bar{\psi} \rangle = \sum_k |a_{\epsilon_k}| \int_0^t |f(s) \xi_k(s)| ds$$

Here $\bar{\psi} = \sum_\alpha |a_\alpha| H_\alpha$. By prop.(7) it follows that $f(s)W_s$ is $(\mathcal{S})^*$-integrable on $[0,t]$ and

$$\int_0^t f(s) W_s ds = \sum_k \int_0^t f(s) \xi_k(s) ds H_{\epsilon_k}$$

Note the following equality:

$$\int_0^t f(s) dB_s = \sum_k (f \mathcal{X}_{(0,t)}, \xi_k) \int_{\mathbb{R}} \xi_k(s) dB_s = \sum_k \int_0^t f(s) \xi_k(s) ds H_{\epsilon_k} = \int_0^t f(s) W_s ds \qquad (27)$$

To proceed, we need a useful lemma:

Lemma 8 *Assume*

$$\sup_\alpha \alpha! \int_0^t |c_\alpha(s)|^2 ds < \infty$$

Then

$$X = \sum_\alpha (\int_0^t |c_\alpha(s)| ds) H_\alpha$$

will be an element of $(\mathcal{S})^*$. ■

Proof: We must show that

$$\sup_\alpha (\int_0^t |c_\alpha(s)| ds)^2 \alpha! (2\mathbb{N})^{-\alpha q} < \infty$$

for a $q > 0$. By the Hölder inequality, we get

$$\sup_\alpha (\int_0^t |c_\alpha(s)| ds)^2 \alpha! (2\mathbb{N})^{-\alpha q} \leq t \sup_\alpha (\int_0^t c_\alpha^2(s) ds) \alpha! (2\mathbb{N})^{-\alpha q}$$

$$\leq t \sup_\alpha \left(\int_0^t c_\alpha^2(s) ds \right) < \infty$$

since

$$(2\mathbb{N})^\alpha = 2^{|\alpha|}(1^{\alpha_1} 2^{\alpha_2} \ldots m^{\alpha_m}) \geq 1 \ , \ \forall \alpha$$

■

To prove (1), we must classify the processes Y_s which make $Y_s \diamond W_s$ $(\mathcal{S})^*$-integrable. The next proposition deals with this:

Proposition 9 *Assume* $Y_s \in (\mathcal{S})^*$ *for* $s \in [0,t]$ *with chaos expansion*

$$Y_s = \sum_\alpha c_\alpha(s) H_\alpha$$

such that

$$\sup_\alpha \alpha! \int_0^t |c_\alpha(s)|^2 ds < \infty$$

Then $Y_s \diamond W_s$ *is* $(\mathcal{S})^*$*-integrable on* $[0,t]$ *and*

$$\int_0^t Y_s \diamond W_s ds = \sum_{\alpha,k} \left(\int_0^t c_\alpha(s) \xi_k(s) ds \right) H_{\alpha+\epsilon_k} \tag{28}$$

■

Proof: Let $\psi = \sum_\alpha a_\alpha H_\alpha$. According to (22), the proposition is proved if

$$\sum_{\alpha,k} (\alpha + \epsilon_k)! \int_0^t |c_\alpha(s)||\xi_k(s)| ds |a_{\alpha+\epsilon_k}| < \infty$$

By the estimate $\sup_{s\in\mathbb{R}} |\xi_k(s)| = O(k^{-1/12})$ in [HP], p.571, we have

$$\sum_{\alpha,k} (\alpha + \epsilon_k)! \int_0^t |c_\alpha(s)||\xi_k(s)| ds |a_{\alpha+\epsilon_k}| \leq \sum_{\alpha,k} (\alpha + \epsilon_k)! C k^{-1/12} \left(\int_0^t |c_\alpha(s)| ds \right) |a_{\alpha+\epsilon_k}|$$

$$\leq C \sum_{\alpha,k} (\alpha + \epsilon_k)! \left(\int_0^t |c_\alpha(s)| ds \right) |a_{\alpha+\epsilon_k}|$$

Now put

$$X = \sum_\alpha \int_0^t |c_\alpha(s)| ds H_\alpha$$

$$Z = \sum_k 1 \cdot H_{\epsilon_k}$$

By lemma(8),$X \in (\mathcal{S})^*$.Since

$$\sup_k (2k)^{-q} < \infty, \forall q > 0$$

we have that $Z \in (\mathcal{S})^*$. Hence $X \diamond Z \in (\mathcal{S})^*$ and

$$X \diamond Z = \sum_{\alpha,k} \int_0^t |c_\alpha(s)| ds H_{\alpha+\epsilon_k}$$

which implies

$$\langle X \diamond Z, \bar{\psi} \rangle = \sum_{\alpha,k} (\alpha + \epsilon_k)! \int_0^t |c_\alpha(s)| ds |a_{\alpha+\epsilon_k}| < \infty$$

where $\bar{\psi} = \sum_\alpha |a_\alpha| H_\alpha$. Hence the proposition follows. ■

An important consequence of this proposition is

Corollary 10 *Assume Y_s is Skorohod integrable on $[0,t]$. Then $Y_s \diamond W_s$ is $(\mathcal{S})^*$-integrable, and*

$$\int_0^t Y_s \diamond W_s ds = \sum_{\alpha,k} \left(\int_0^t c_\alpha(s) \xi_k(s) ds \right) H_{\alpha+\epsilon_k} \tag{29}$$

■

Proof: By the Skorohod integrability, (24), we have

$$\int_0^t E[Y_s^2] ds = \int_0^t \left(\sum_\alpha \alpha! c_\alpha(s)^2 \right) ds = \sum_\alpha \alpha! \int_0^t c_\alpha^2(s) ds < \infty$$

and hence

$$\sup_\alpha \alpha! \int_0^t |c_\alpha(s)|^2 ds \leq \sum_\alpha \alpha! \int_0^t c_\alpha^2(s) ds < \infty$$

■

We are now ready to prove the main result of this article:

Theorem 11 *Assume Y_s Skorohod integrable on $[0,t]$. Then*

$$\int_0^t Y_s \delta B_s = \int_0^t Y_s \diamond W_s ds \tag{30}$$

■

Proof: In the proof we use the definitions of the Wick product and the Skorohod integral. Direct calculation will then show (30).

The Wiener-Ito chaos expansion gives

$$Y_s = f_0(s) + \sum_{m=1}^\infty \int_{\mathbb{R}^m} f_m(s;u) dB_u^{\otimes m}$$

$$= f_0(s) + \sum_{m=1}^\infty \sum_{\substack{\alpha \\ |\alpha|=m}} (f_m(s;.), \xi^{\hat{\otimes}\alpha}) \int_{\mathbb{R}^m} \xi^{\hat{\otimes}\alpha} dB^{\otimes m}$$

$$= f_0(s) + \sum_{\substack{\alpha \\ |\alpha|\geq 1}} (f_{|\alpha|}(s;.), \xi^{\hat{\otimes}\alpha}) H_\alpha$$

Hence, taking the Wick product with W_s, we get

$$Y_s \diamond W_s = f_0(s) W_s + \sum_{\alpha,k} (f_{|\alpha|}(s;.), \xi^{\hat{\otimes}\alpha}) \xi_k(s) H_{\alpha+\epsilon_k}$$

By corollary (10):

$$\int_0^t Y_s \diamond W_s ds = \int_0^t f_0(s) W_s ds + \sum_{\alpha,k} \left(\int_0^t (f_{|\alpha|}(s;.), \xi^{\hat{\otimes}\alpha}) \xi_k(s) ds \right) H_{\alpha+\epsilon_k}$$

By the definition of the Skorohod integral, we have

$$\int_0^t Y_s \delta B_s = \int_0^t f_0(s) dB_s + \sum_{m=1}^\infty \int_{\mathbb{R}^{m+1}} \tilde{f}_m dB^{\otimes m+1}$$

$$= \int_0^t f_0(s)dB_s + \sum_{m=1}^{\infty} \sum_{\substack{\alpha \\ |\alpha|=m+1}} (\tilde{f}_m, \xi^{\hat{\otimes}\alpha}) \int_{\mathbb{R}^{m+1}} \xi^{\hat{\otimes}\alpha} dB^{\otimes m+1}$$

$$= \int_0^t f_0(s)dB_s + \sum_{\substack{\alpha \\ |\alpha|\geq 2}} (\tilde{f}_{|\alpha|-1}, \xi^{\hat{\otimes}\alpha}) H_\alpha$$

From (27)

$$\int_0^t f_0(s)dB_s = \int_0^t f_0(s)W_s ds$$

Hence, we must show that

$$\sum_{\substack{\alpha,k \\ |\alpha|\geq 1}} \int_0^t (f_{|\alpha|}(s;.), \xi^{\hat{\otimes}\alpha})\xi_k(s)ds H_{\alpha+\epsilon_k} = \sum_{\substack{\alpha \\ |\alpha|\geq 2}} (\tilde{f}_{|\alpha|-1}, \xi^{\hat{\otimes}\alpha}) H_\alpha$$

Considering the right hand side, we find that

$$\sum_{\substack{\alpha \\ |\alpha|\geq 2}} (\tilde{f}_{|\alpha|-1}, \xi^{\hat{\otimes}\alpha}) H_\alpha = \sum_{\substack{\alpha,k \\ |\alpha|\geq 1}} (\tilde{f}_{|\alpha|}, \xi^{\hat{\otimes}(\alpha+\epsilon_k)}) H_{\alpha+\epsilon_k}$$

Hence, it is sufficient to show that

$$\sum_{\substack{\alpha,k \\ |\alpha|=n}} (\tilde{f}_{|\alpha|}, \xi^{\hat{\otimes}(\alpha+\epsilon_k)}) H_{\alpha+\epsilon_k} = \sum_{\substack{\alpha,k \\ |\alpha|=n}} (\int_0^t (f_{|\alpha|}(s;.), \xi^{\hat{\otimes}\alpha})\xi_k(s)ds) H_{\alpha+\epsilon_k} \tag{31}$$

Let $|\alpha| = n$. Then we might write α as

$$\alpha = \epsilon_{i_1 i_2 \ldots i_n}$$

where $\epsilon_{i_1 i_2 \ldots i_n}$ has ones on the coordinats $i_1, \ldots, i_n$, and zeros everywhere else. If $i_j = i_k$, then the multi-index has 2 on the coordinate i_j, and so on. In addition we have

$$\epsilon_{i_1 i_2 \ldots i_n} + \epsilon_k = \epsilon_{i_1 i_2 \ldots i_n k}$$

If $u \in \mathbb{R}^{n+1}$, we get

$$\xi^{\hat{\otimes}(\epsilon_{i_1 \ldots i_n}+\epsilon_k)}(u) = \frac{1}{(n+1)!} \sum_\sigma \xi_{i_1}(u_{\sigma_1}) \ldots \xi_{i_n}(u_{\sigma_n}) \xi_k(u_{\sigma_{n+1}})$$

where the sum is taken over all permutations σ of the set $\{1, \ldots, n+1\}$. In addition

$$\tilde{f}_n(u) = \frac{1}{n+1} \sum_{j=1}^{n+1} \mathcal{X}_{(0,t)}(u_j) f_n(u_j; u_1, \ldots, \hat{u}_j, \ldots, u_{n+1})$$

Therefore

$$(\tilde{f}_n, \xi^{\hat{\otimes}(\epsilon_{i_1 \ldots i_n}+\epsilon_k)}) = (1/((n+1)!(n+1))) \sum_{j=1}^{n+1} \sum_\sigma \int_{\mathbb{R}^{n+1}} \mathcal{X}_{(0,t)}(u_j) f_n(u_j; u_1, \ldots, \hat{u}_j, \ldots, u_{n+1})$$

$$\times \xi_{i_1}(u_{\sigma_1}) \ldots \xi_k(u_{\sigma_{n+1}}) du$$

$$= (1/((n+1)!(n+1))) \sum_{j=1}^{n+1} n! \{ \int_0^t (f_n(u_j;.), \xi^{\hat{\otimes}(\epsilon_{i_2 \ldots i_n}+\epsilon_k)}) \xi_{i_1}(u_j) du_j + \ldots$$

$$+\int_0^t (f_n(u_j;.), \xi^{\hat{\otimes}(\epsilon_{i_1}\dots_{i_{n-1}}+\epsilon_k)})\xi_{i_n}(u_j)du_j + \int_0^t (f_n(u_j;.), \xi^{\hat{\otimes}(\epsilon_{i_1}\dots_{i_n})})\xi_k(u_j)du_j\}$$

$$= (1/((n+1)!(n+1)))n!(n+1)\{\int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_2}\dots_{i_n k}})\xi_{i_1}(s)ds + \dots + \int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_1}\dots_{i_n}})\xi_k(s)ds\}$$

$$= (1/(n+1))\{\int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_2}\dots_{i_n k}})\xi_{i_1}(s)ds + \dots + \int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_1}\dots_{i_n}})\xi_k(s)ds\}$$

This inserted in the left hand side of (31), gives

$$\sum_{\substack{\alpha,k \\ |\alpha|=n}} (\tilde{f}_{|\alpha|}, \xi^{\hat{\otimes}\alpha+\epsilon_k})H_{\alpha+\epsilon_k} = \sum_{i_1,\dots,i_n,k} (\tilde{f}_n, \xi^{\hat{\otimes}\epsilon_{i_1}\dots_{i_n}+\epsilon_k})H_{\epsilon_{i_1}\dots_{i_n}+\epsilon_k}$$

$$= (1/(n+1)) \sum_{i_1,\dots,i_n,k} \{\int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_2}\dots_{i_n k}})\xi_{i_1}(s)ds + \dots + \int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_1}\dots_{i_n}})\xi_k(s)ds\}H_{\epsilon_{i_1}\dots_{i_n k}}$$

$$= (1/(n+1))\{\sum_{i_2,\dots,i_n,k} (\sum_{i_1} \int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_2}\dots_{i_n k}})\xi_{i_1}(s)dsH_{\epsilon_{i_2}\dots_{i_n k}+\epsilon_{i_1}}) + \dots$$

$$+ \sum_{i_1,\dots i_n} (\sum_k \int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_1}\dots_{i_n}})\xi_k(s)dsH_{\epsilon_{i_1}\dots_{i_n}+\epsilon_k})\}$$

$$= \sum_{i_1,\dots,i_n,k} (\int_0^t (f_n(s;.), \xi^{\hat{\otimes}\epsilon_{i_1}\dots_{i_n}})\xi_k(s)ds)H_{\epsilon_{i_1}\dots_{i_n}+\epsilon_k}$$

$$= \sum_{\substack{\alpha,k \\ |\alpha|=n}} (\int_0^t (f_n(s;.), \xi^{\hat{\otimes}\alpha})\xi_k(s)ds)H_{\alpha+\epsilon_k}$$

which shows (31), and hence the theorem.

■

Corollary 12 *Assume $Y_s \in L^2(\mu)$, $s \in [0,t]$ is Ito integrable then*

$$\int_0^t Y_s dB_s = \int_0^t Y_s \diamond W_s ds \tag{32}$$

■

Proof: Ito integrability implies Skorohod integrability. See [NZ] for more information about this. ■

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