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Abstract

In this paper we will show how to construct what we would like to call inverse Wick powers of white noise. It is easily seen that no element W^{-1} in the space $(S)^*$ of Hida distributions can have the property that the Wick product $W \diamond W^{-1} = 1$. We can, however, find objects $W^{-\diamond n}$ s.t. $W^{\diamond n} \diamond W^{-\diamond n} = 1$ except from delta function singularities at the origin.

§1 Introduction

The simplest way to describe a white noise W is through a mapping $W : \mathcal{D} \times \Omega \to \mathbb{R}$:

$$W(\phi, \omega) = \int \phi(s) dB_s(\omega) \tag{1.1}$$

where $\mathcal{D} = \mathcal{D}(\mathbb{R})$ is the space of Schwartz test functions, B_t is a Brownian motion defined on a probability space $(\Omega, P, \mathcal{F})$ and $\int \phi dB$ denotes an Ito integral, see e.g. [Ø]. It is then easy to see that W is the derivative of B_t in the sense of Schwartz distributions. Since a Brownian motion is a.s. nowhere differentiable, W is a pure distribution (i.e. not a function). In a stochastic calculus one wants to study products of such objects. This, however, is certainly nontrivial and several spaces and constructions have been introduced to deal with this problem. One such space is the space $(S)^*$ of Hida distributions, see [HKPS] or [K]. A very convenient property of $(S)^*$ is that this space is closed under Wick multiplication, see [HKPS] or [GHLØUZ].

Wick multiplication is closely related to Hermite polynomials. In particular one can show that if $\phi \in \mathcal{D}$ is such that $||\phi||_{L^2(\mathbb{R})} = 1$, then:

$$W^{\diamond n}(\phi, \omega) = W \diamond W \diamond \ldots \diamond W(\phi, \omega) = H_n(W(\phi, \omega)) \tag{1.2}$$

where the $\diamond$ indicates that we perform Wick products between these distributions. As a consequence, we get:

$$W^{\diamond m} \diamond W^{\diamond n}(\phi, \omega) = H_{m+n}(W(\phi, \omega)) \tag{1.3}$$

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§2 Properties of Hermite polynomials

Hermite polynomials can be defined in several different ways. According to the context, different scalings and normalizations have been preferred. The Hermite polynomials we use in this paper, are defined through the relation:

$$(2.1) \qquad H_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x+it)^n e^{-\frac{t^2}{2}} dt \qquad n = 0, 1, 2, \dots$$

These Hermite polynomials are solutions of the differential equation:

$$(2.2) \qquad y'' - xy' + ny = 0$$

and satisfies the classical relations:

$$(2.3) \qquad H_{n+1}(x) = x H_n(x) - n H_{n-1}(x)$$

$$(2.4) \qquad \frac{d}{dx} H_n(x) = n H_{n-1}(x)$$

$$(2.5) \qquad H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}})$$

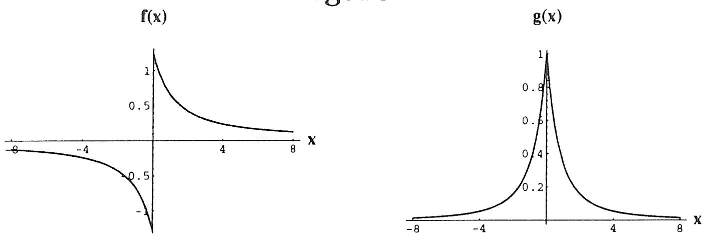
$$(2.6) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-\frac{x^2}{2}} dx = \begin{cases} n! & n = m \\ 0 & n \neq m \end{cases}$$

If $x \neq 0$, the definition (2.1) also makes sense when n is negative integer. The functions that appear by this construction, are not polynomials. It turns out, however, that these functions have several nice properties with natural connections to the ordinary Hermite polynomials. By a slight abuse of notation, we will call them Hermite polynomials of negative index, and we define:

$$(2.7) \qquad H_{-n}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{(x+it)^n} e^{-\frac{t^2}{2}} dt \qquad n = 0, 1, 2, \dots, \ x \neq 0$$

We can, with some minor modifications, extend the fundamental properties (2.2)-(2.6) of the Hermite polynomials to the case with negative indices. To integrate the functions correctly across the origin, however, a correction term must sometimes be introduced in the formulas. We start out to examine the case $H_{-1}(x)$.

Figure 1



In figure 1 we have shown plots of $H_{-1}(x) = f(x)$ and $H_{-2}(x) = g(x)$. The behaviour we see is typical. When the index is odd we always get odd functions, and they are even when the index is even. The following proposition gives detailed information. To simplify the notation we introduce the constants:

$$C_n = \begin{cases} \frac{\sqrt{2\pi}}{2^k k!} & \text{if } n = 2k+1, k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

$$n!! = \begin{cases} 1 \cdot 3 \cdot 5 \cdots n & \text{if } n = 2k+1,\ k = 0, 1, \ldots \\ 2 \cdot 4 \cdot 6 \cdots n & \text{if } n = 2k,\ k = 1, 2, \ldots \\ 1 & \text{if } n = 0 \end{cases} \tag{2.16}$$

Proposition 2.2
The Hermite polynomials of negative index have the following properties:

$$H_{-(n+1)}(x) = -\frac{x}{n} H_{-n}(x) + \frac{1}{n} H_{-(n-1)}(x) \qquad \text{for all } x \neq 0, n = 1, 2, \ldots \tag{2.17}$$

$$H_{-n}(x) = \begin{cases} \text{odd} & \text{if } n \text{ is odd} \\ \text{even} & \text{if } n \text{ is even} \end{cases} \tag{2.18}$$

$$\lim_{x \to 0^+} H_{-n}(x) = \begin{cases} \frac{\sqrt{\frac{\pi}{2}}}{(n-1)!!} & \text{if } n = 2k+1, k = 0, 1, \ldots \\ \frac{1}{(n-1)!!} & \text{if } n = 2k, k = 1, 2, \ldots \end{cases} \tag{2.19}$$

(2.20)
$$\frac{d}{dx} H_{-n}(x) = -n H_{-(n+1)}(x) + C_n \delta(x) \qquad \text{in } \mathcal{D}', \delta(x) = \text{Dirac's delta function}$$

$$H''_{-n} - x H'_{-n} - n H_{-n} = C_n \delta' - (n C_{n+1} + x C_n)\, \delta \qquad \text{in } \mathcal{D}', n = 1, 2, \ldots \tag{2.21}$$

We want to prove that these objects converge to well defined objects in $\mathcal{D}'$, and start out with the following lemma.

Lemma 2.3

$$H_1 \diamond H_{-n}(x) = H_{-(n-1)}(x) - C_n\delta(x) \tag{2.29}$$

Proof

Choose $\phi \in \mathcal{D}(\mathbb{R})$. To prove the lemma we have to show that:

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}\sum_{k=0}^{N} a_{nk}H_{k+1}(x)\phi(x)dx = \int_{-\infty}^{\infty} H_{-(n-1)}(x)\phi(x)dx - C_n\phi(0) \tag{2.30}$$

We first use the relations (2.3) and (2.4) to rewrite the expression to the following form:

$$\begin{aligned}
&\lim_{N\to\infty}\int_{-\infty}^{\infty}\sum_{k=0}^{N} a_{nk}H_{k+1}(x)\phi(x)dx \\
&= \lim_{N\to\infty}\int_{-\infty}^{\infty}\sum_{k=0}^{N} a_{nk}\left(xH_k(x) - kH_{k-1}(x)\right)\phi(x)dx \\
&= \lim_{N\to\infty}\int_{-\infty}^{\infty} x\sum_{k=0}^{N} a_{nk}H_k(x)\phi(x)dx - \lim_{N\to\infty}\int_{-\infty}^{\infty}\sum_{k=0}^{N} a_{nk}\frac{d}{dx}H_k(x)\phi(x)dx \\
&= \lim_{N\to\infty}\int_{-\infty}^{\infty} x\sum_{k=0}^{N} a_{nk}H_k(x)\phi(x)dx + \lim_{N\to\infty}\int_{-\infty}^{\infty}\sum_{k=0}^{N} a_{nk}H_k(x)\phi'(x)dx \\
&= \int_{-\infty}^{\infty} xH_{-n}(x)\phi(x)dx + \int_{-\infty}^{\infty} H_{-n}(x)\phi'(x)dx
\end{aligned} \tag{2.31}$$

To proceed further we use (2.17) and (2.20):

$$\begin{aligned}
&\int_{-\infty}^{\infty} xH_{-n}(x)\phi(x)dx + \int_{-\infty}^{\infty} H_{-n}(x)\phi'(x)dx \\
&= \int_{-\infty}^{\infty}\left(H_{-(n-1)}(x) - nH_{-(n+1)}(x)\right)\phi(x)dx \\
&+ \int_{-\infty}^{\infty} nH_{-(n+1)}(x)\phi(x)dx - C_n\phi(0) \\
&= \int_{-\infty}^{\infty} H_{-(n-1)}(x)\phi(x)dx - C_n\phi(0)
\end{aligned} \tag{2.32}$$

□

Proposition 2.4
For $k = 0, 1, \ldots$ the Wick product $H_k \diamond \delta(x) \in \mathcal{D}'$ and there exist constants D_{nk} s.t.:

$$H_k \diamond \delta(x) = \sum_{n=0}^{k} D_{nk}\delta^{(n)}(x) \tag{2.39}$$

Proposition 2.5
If either $m \geq 0$ or $n \geq 0$, then:

$$H_m \diamond H_n = H_{m+n} - \sum_{k=1}^{m\vee n} C_{k-(m+n)} H_{k-1} \diamond \delta \tag{2.40}$$

Proof
Trivial if both $m, n \geq 0$ (all the C-s are 0). Assume $m \geq 0$ and $p \geq 0$, then:

$$\begin{aligned} H_m \diamond H_{-p} &= H_{m-1} \diamond \left(H_1 \diamond H_{-p}\right) \\ &= H_{m-1} \diamond \left(H_{-(p-1)} - C_p\delta\right) \\ &= H_{m-1} \diamond H_{-(p-1)} - C_p H_{m-1} \diamond \delta \\ &= H_{m-2} \diamond H_{-(p-2)} - C_{p-1}H_{m-2} \diamond \delta - C_p H_{m-1} \diamond \delta \\ &\ \vdots \\ &= H_{m-p} - \sum_{k=0}^{m-1} C_{p-k} H_{m-k-1} \diamond \delta \\ &= H_{m-p} - \sum_{k=1}^{m} C_{k-m+p} H_{k-1} \diamond \delta \end{aligned} \tag{2.41}$$

□

Corollary 2.6
If $n \geq 0$, then:

$$H_n \diamond H_{-n} = 1 - \sum_{k=1}^{n} C_k H_{k-1} \diamond \delta \tag{2.42}$$

and there exist constants K_{nk} s.t.:

$$H_n \diamond H_{-n} = 1 - \sum_{k=0}^{n-1} K_{nk}\delta^{(k)} \tag{2.43}$$

Remark: It is also interesting to note that $H_n \diamond H_{-n}$ acts as the zero functional on the chaos of order $\leq n$. We will return to these questions in §3.

Proof

By (2.21) we see that for $x \neq 0$, then:

$$nH_{-n}(x) = H''_{-n}(x) - xH'_{-n}(x) \tag{2.52}$$

We use this result together with integration by parts to carry out the following calculation:

$$\begin{aligned}
& n\int_0^\infty H_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= \int_0^\infty H''_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} - \int_0^\infty xH'_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= \frac{1}{\sqrt{2\pi}}H'_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\Big|_0^\infty - \int_0^\infty H'_{-n}(x)\left(H_{-m}(x)e^{-\frac{x^2}{2}}\right)'\frac{dx}{\sqrt{2\pi}} \\
&- \int_0^\infty xH'_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= -\frac{1}{\sqrt{2\pi}}H'_{-n}(0^+)H_{-m}(0^+) - \int_0^\infty H'_{-n}(x)H'_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&+ \int_0^\infty xH'_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} - \int_0^\infty xH'_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= -\frac{1}{\sqrt{2\pi}}H'_{-n}(0^+)H_{-m}(0^+) - \int_0^\infty H'_{-n}(x)H'_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}
\end{aligned} \tag{2.53}$$

By the same calculation we see that:

$$\begin{aligned}
& -m\int_0^\infty H_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= \frac{1}{\sqrt{2\pi}}H'_{-m}(0^+)H_{-n}(0^+) + \int_0^\infty H'_{-n}(x)H'_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}
\end{aligned} \tag{2.54}$$

We add the results of (2.53) and (2.54) together, and divide by $n-m$ to get:

$$\begin{aligned}
& \int_0^\infty H_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= \frac{1}{\sqrt{2\pi}(n-m)}\left(H'_{-m}(0^+)H_{-n}(0^+) - H'_{-n}(0^+)H_{-m}(0^+)\right)
\end{aligned} \tag{2.55}$$

Using (2.20) we can rewrite this:

$$\begin{aligned}
& \int_0^\infty H_{-n}(x)H_{-m}(x)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \\
&= \frac{1}{\sqrt{2\pi}(n-m)}\left(nH_{-m}(0^+)H_{-(n+1)}(0^+) - mH_{-(m+1)}(0^+)H_{-n}(0^+)\right)
\end{aligned} \tag{2.56}$$

Proposition 2.9

Let $n \geq 1, m \geq 0$. If one of the pair m, n is even and the other is odd, then:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_{-n}(x) H_m(x) e^{-\frac{x^2}{2}} dx = 0 \tag{2.61}$$

If m and n are both odd, then with $m = 2k + 1$:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_{-n}(x) H_m(x) e^{-\frac{x^2}{2}} dx = \frac{(-1)^k m!}{(n+m)(m-1)!!(n-1)!!} \tag{2.62}$$

If m and n are both even, then with $m = 2k$:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_{-n}(x) H_m(x) e^{-\frac{x^2}{2}} dx = \frac{(-1)^k n\, m!}{(n+m) m!! n!!} \tag{2.63}$$

If n is odd, then:

$$H_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+2k+1)(2k)!!(n-1)!!} H_{2k+1}(x) \tag{2.64}$$

If n is even, then:

$$H_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k n}{(n+2k)(2k)!! n!!} H_{2k}(x) \tag{2.65}$$

Proof

(2.61) is trivial. Use exactly the same calculation as in (2.52) - (2.56) to get:

$$\begin{aligned} &\int_0^{\infty} H_{-n}(x) H_m(x) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{2\pi}(n+m)} \left(n H_m(0^+) H_{-(n+1)}(0^+) + m H_{m-1}(0^+) H_{-n}(0^+) \right) \end{aligned} \tag{2.66}$$

We then apply the formulas (2.33) and (2.19) to get (2.62) and (2.63). The last two results follows easily from (2.62) and (2.63).

□

§3 Inverse Wick powers of white noise

We will now consider the Schwartz space $S(\mathbb{R})$ together with its dual $S'(\mathbb{R})$ equipped with the white noise measure μ which is uniquely characterized by the characteristic function:

$$\int_{S'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}||\phi||^2_{L^2(\mathbb{R})}}; \qquad \text{for all } \phi \in S \tag{3.1}$$

For $\lambda \in \mathbb{C}$ and $\phi \in S(\mathbb{R})$ we may extend the definition above to $X \in (S)^*$ by the dual pairing:

(3.9) $$SX(\lambda\phi) =< X, Exp[\lambda W_\phi] >$$

It is well known that the elements $X \in (S)^*$ are uniquely characterized by their S-transforms; see [PS]. S-transforms always extend to ray entire functionals $\Psi = SX$ satisfying an estimate of the form:

(3.10) $$|\Psi(\lambda\,\phi)| \leq C\,e^{K|\lambda|^2|\phi|^2_{2,p}}$$

where C and K are constants, p a non-negative integer and $|\phi|_{2,p} := |A^p\phi|_{L^2(\mathbb{R})}$. Conversely every ray entire functional Ψ satisfying an estimate as above is the S-transform of some element $X \in (S)^*$. The S-transform can be used to prove the convergence in $(S)^*$. In this case we have the following result, see [PS]:

Proposition 3.1 [PS]
Let $X_n, X \in (S)^*$. Then $X_n \to X$ if and only if the following conditions are satisfied:

(3.11) $$SX_n(\phi) \to SX(\phi)$$

(3.12) $$|SX_n(\lambda\,\phi)| \leq C\,e^{K|\lambda|^2|\phi|^2_{2,p}}$$

where the constants C, K and p do not depend on n.

□

This result is very convenient when we try to make sense out of various functionals of white noise. It is easily seen that:

(3.13) $$S(W_\phi)(\xi) = \int \phi(s)\xi(s)ds$$

In this formula, we let ϕ approach a delta function concentrated at a point t. We then consider:

(3.14) $$S(W_t)(\xi) = \lim_{\phi\to\delta_t} \int \phi(s)\xi(s)ds = \xi(t)$$

Although this calculation is only formal, we see using proposition 3.1, that (3.14) uniquely characterizes an element $W_t \in (S)^*$. Since $(S)^*$ is closed under Wick multiplication and we have the convenient property (1.8), it is easy to see that:

Once we have proved lemma 3.2, it is clear that the formulas corresponding to proposition 2.5 and corollary 2.6 follow in exactly the same way as before.

Theorem 3.3
If either $m \geq 0$ or $n \geq 0$, then:

$$W_\phi^{\diamond m} \diamond W_\phi^{\diamond n} = W_\phi^{\diamond (m+n)} - \sum_{k=1}^{m \vee n} C_{k-(m+n)} W_\phi^{\diamond (k-1)} \diamond \delta(W_\phi) \tag{3.21}$$

□

Theorem 3.4
If $n \geq 0$, then:

$$W_\phi^{\diamond n} \diamond W_\phi^{-\diamond n} = 1 - \sum_{k=1}^{n} C_k W_\phi^{\diamond (k-1)} \diamond \delta(W_\phi) \tag{3.22}$$

□

In the two previous theorems we also want to be able to pass to the limit, i.e. replace W_ϕ by W_t. Since we know how to calculate the S-transforms of $W_\phi^{\diamond n}$ and $\delta(W_\phi)$, we can use (3.22) to calculate the S-transform of $W_\phi^{-\diamond n}$. The result is the following:

Theorem 3.5
If $n \geq 0$, then:

$$S(W_\phi^{-\diamond n})(\xi) = e^{-\frac{1}{2}<\phi,\xi>^2} \sum_{k=[\frac{n}{2}+1]}^{\infty} \frac{1}{2^k k!} < \phi, \xi >^{2k-n} \tag{3.23}$$

where $\left[\frac{n}{2}+1\right]$ denotes the integer part of $\frac{n}{2}+1$.

Proof
We take S-transforms on both sides in (3.22) and use (2.15), (3.15) and (3.17) to get:

$$< \phi, \xi >^n \cdot S(W_\phi^{-\diamond n})(\xi) = 1 - \sum_{k=0}^{[\frac{n}{2}]} \frac{\sqrt{2\pi}}{2^k k!} < \phi, \xi >^{2k} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}<\phi,\xi>^2} \tag{3.24}$$

The formula:

$$e^{\frac{1}{2}<\phi,\xi>^2} = \sum_{k=0}^{\infty} \frac{1}{2^k k!} < \phi, \xi >^{2k} \tag{3.25}$$

is then used to rewrite (3.24) to the form (3.23).

□

§4 Some properties of the Hermite transform

The $\mathcal{H}$-transform was introduced in [LØU 1] and [LØU 2]. This transform has turned out to be a useful tool when we address problems related to stochastic partial differential equations. One of the main advantages is the existence of an explicit inverse, see [GHLØUZ] for more details.

$$\mathcal{H} : (L^2) \times \mathbb{C}_0^{\mathbb{N}} \to \mathbb{C} \tag{4.1}$$

where $\mathbb{C}_0^{\mathbb{N}}$ denotes all finite sequences of complex numbers, can be defined as follows:

If $X \in (L^2) = \sum_\alpha a_\alpha H_\alpha(\omega)$, see (1.4), then:

$$\mathcal{H}X(z) = \sum_\alpha a_\alpha z^\alpha \tag{4.2}$$

where $z = (z_1, z_2, \cdots)$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots$.
In particular we have:

$$\mathcal{H}W_\phi(z) = ||\phi||_{L^2(\mathbb{R})} z_1 \tag{4.3}$$

In this case we end up with an entire function of one complex variable. This is typical when we consider power series in the white noise. If the $\mathcal{H}$-transform is a function $f(z)$ of one complex variable only, the inverse transform takes on the form:

$$\mathcal{H}^{-1}(f(z)) = \int_{-\infty}^{\infty} f(x + iy) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \Big|_{x=\int \phi dB} \tag{4.4}$$

When we apply the $\mathcal{H}$-transform to solve some stochastic differential equations, see [HLØUZ 1] and [HLØUZ 2], it turns out that we sometimes encounter transforms that are not entire functions. It is therefore of considerable interest to be able to apply inverse transforms to meromorphic functions or even functions that are only defined locally. In this connection we observe the following:

Theorem 4.1

If $f(z)$ is a meromorphic function with a finite number of poles, all located on the real axes, and there exist constants C, λ and r s.t.:

$$|f(z)| \le Ce^{\lambda|z|} \qquad |z| \ge r \tag{4.5}$$

then $\mathcal{H}^{-1}(f(z)) \in (L^2)$.

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