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Abstract

In this paper we consider the transformation of measure induced by a not-necessarily-invertible perturbation of the identity. The Radon-Nikodym density for the image of the Wiener measure and the associated Girsanov-type density are derived. An application of these results yields an extension of the degree theorem.

1 Introduction

Let (W, H, μ) denote an abstract Wiener space and let $F(\omega)$ be an H-valued random variable. The transformation $T\omega = \omega + F(\omega)$, induces the measure $\mu \circ T^{-1}$ on W (i.e. $(\mu \circ T^{-1})(A) = \mu(T^{-1}A))$ and in many cases there also exists a Girsanov-type measure ν defined by $\nu(T^{-1}A) = \mu(A)$ for all measurable subsets A of W, (*cf* e.g. lemma 2.1 of [14]). We derive conditions for absolute continuity between these measures and the related Radon-Nikodym derivatives. These problems were considered by many authors (*cf* e.g. [2, 3, 5, 7, 8, 9, 10, 13, 14] and the references therein). The case where W is the standard Wiener space and the shift F is adapted to the canonical Brownian filtration is covered by the well known Cameron-Martin-Maruyama-Girsanov theorem. Cameron and Martin were the first to consider in 1949 invertible nonadapted shifts, their work was extended by Gross, Kuo, Skorohod and others. Their results represented the density of the image measure in terms of the Fredholm determinant of $(I_H + \nabla F)$, where ∇F denotes the gradient and the exponent of a Stratonovitch-type integral of F. This representation imposed the strong assumptions such as ∇F is of trace class and F is Stratonovitch integrable. It was Ramer who in 1975 extended considerably the previous results by representing the density as the product of a Carleman-Fredholm determinant (which exists under the condition that ∇F is Hilbert-Schmidt) and the exponent of what he called "an abstract version of the Ito integral"(this integral was called the Ito-Ramer integral in [5] and appeared, independently, in the Malliavin calculus literature as the divergence or Skorohod integral). The results of Ramer [10] were shown by Kusuoka [5] to hold under considerably weaker conditions.

In this paper we extend the results of Ramer [10] and Kusuoka [5] (*cf* also [8]). In particular, T is not required to be invertible and the requirement that F be $H-C^1$ is replaced by a weaker condition. Moreover we give the explicit expressions of the corresponding Radon-Nikodym densities.

The main results of this paper are presented in sections 3 and 4 and can be summarized as follows: Let $F(\omega)$ be a random variable with values in the Cameron-Martin space H and in some Sobolev space of positive index. F will be said to belong to $H - C^1_{loc}$ if there exists a random variable $Q(\omega) > 0$ almost surely and for almost all ω, $h \to F(\omega + h)$ is continuously differentiable on the set $\{h \in H : \|h\|_H \leq Q(\omega)\}$. Let $N(\omega, M)$ denote the cardinality of the set $T^{-1}\{\omega\} \cap M$, where M denotes the set on which the Carleman-Fredholm determinant of $I_H + \nabla F$ is non-zero. If we denote by Λ_F the random variable defined by the equation (2.2), then

Theorem 1.1 *For* $T\omega = \omega + F(\omega)$

- *The cardinality of the set* $T^{-1}\{\omega\} \cap M$ *is almost surely at most countably infinite and the restriction of the measure* $\mu \circ T^{-1}$ *to* M *is absolutely continuous with respect to* μ *furthermore the Radon-Nikodym derivative is given by*

$$\frac{d\mu \circ T^{-1}}{d\mu}|_M(\omega) = \sum_{\theta \in T^{-1}\{\omega\} \cap M} |\Lambda_F(\theta)|^{-1}.$$

- *For every positive, bounded, measurable function* f *on* W

$$E[f(T\omega) \cdot |\Lambda_F|] = E[f(\omega) N(\omega, M)].$$

More generally, for any such f *and* g*, we have*

$$E\left[f(T\omega) \cdot g(\omega) \cdot |\Lambda_F(\omega)|\right] = E\left[f(\omega) \sum_{\theta \in T^{-1}\{\omega\} \cap M} g(\theta)\right].$$

In the next section we first summarize the notation of the Malliavin calculus and recall the definition of the $\rho_A(\omega)$ random variable of Kusuoka [5]. Then we introduce the $H - C^1_{loc}$ class which generalizes the $H - C^1$ class [10, 5] and show that the elements of the $H - C^1_{loc}$ are Skorohod integrable. In section 3 we consider the case where $T\omega = \omega + F(\omega)$ with $\| \nabla F \| \leq c < 1$ generalizing previously known results (theorem 6.1 of [5]). In section 4 it is shown that under the $H - C^1_{loc}$ assumption on F, W can be decomposed into the ω set on which $(I + \nabla F)$ is not invertible and a countable union of disjoint sets M_i such that $T\omega = \omega + F(\omega)$ is injective on M_i, moreover, the restrictions of $\mu \circ T^{-1}$ to M_i, $\mu \circ T^{-1}|_{M_i}$ is absolutely continuous with respect to $\mu|_{TM_i}$ (note that TM_i is a measurable subset of W

since T is injective on M_i) and the density of the related Girsanov-type measure is evaluated. This result is applied in section 5 to give an extension of previously known results for the degree theorem ([4, 6, 14]) on Wiener space. Also an extension of the results of fourth section, by removing the hypothesis of strict positivity of the control random variable $Q(\omega)$, is given in the same section.

2 Preliminaries

Let (W, H, μ) be an abstract Wiener space. We start with a short summary of the notation of the Malliavin calculus. For $h \in H^* = H$, the Wiener integral $W(h)$ will also be denoted $\langle h, \omega \rangle$, $\omega \in W$. Let X be a real separable Hilbert space; smooth, X-valued functionals on (W, H, μ) are functionals of the form

$$a(\omega) = \sum_1^N \eta_i(\langle h_1, \omega \rangle, \cdots, \langle h_m, \omega \rangle)\, x_i$$

with $x_i \in X$ and $\eta_i \in C_b^\infty(IR^m)$, $h_i \in W^* \subset H$. For smooth X-valued functionals, define

$$\nabla a(\omega) = \sum_{i=1}^N \sum_{j=1}^m \partial_j \eta_i \Big(\langle h_1, \omega \rangle, \cdots, \langle h_j, \omega \rangle \Big) \cdot x_j \otimes h_i \,,$$

and ∇^k, $k = 2, 3, \cdots$ are defined recursively. The Sobolev spaces $ID^{p,k}(X)$ $p > 1$, $k \in IN$ are the completion of X-valued smooth functionals with respect to the norm

$$\| a \|_{p,k} = \sum_{i=0}^k \| \nabla^i a \|_{L^p(\mu, X \otimes H^{\otimes i})} \,. \tag{2.1}$$

The gradient $\nabla : ID^{p,k}(X) \to ID^{p,k-1}(X \otimes H)$ denotes the closure of ∇ as defined for smooth functionals under the norm of (2.1). The gradient ∇a is considered as a mapping from H to X and $(\nabla a)^*$ will denote the adjoint of ∇a and is a mapping from X to H. The adjoint of ∇ under the Wiener measure μ is denoted by δ and called the divergence or the Skorohod integral. (for the memory, it is defined by the" integration by parts formula" $E(G\delta u) = E\langle \nabla G, u \rangle_H$ for smooth real-valued G and H-valued u). Recall that if F is in

$ID^{p,1}(H)$, for some $p > 1$, then for a.e. ω, $\nabla F(\omega)$ is an Hilbert-Schmidt operator from H to H and for any smooth H-valued F and any complete orthonormal basis of H, say $\{e_i,\ i = 1, 2, \cdots\}$ we have

$$\delta F = \sum_{i=0}^{\infty} \langle F, e_i \rangle_H \langle e_i, \omega \rangle - \Big\langle \nabla(\langle F, e_i \rangle_H), e_i \Big\rangle_H .$$

An X-valued random variable F is said to be in $ID^{p,k}_{loc}(X)$ if there exists a sequence (A_n, F_n) where A_n are measurable subsets of $W, \cup_n A_n = W$ almost surely,$F_n \in ID^{p,k}(X)$ and for every n, $F_n = F$ almost surely on A_n.

Let K be a linear operator from H to H and let λ_i, $i = 1, 2, \cdots$ be the sequence of eigen values of K repeated according to their multiplicity. The Carleman-Fredholm determinant is defined as:

$$\det{}_2(I + K) = \prod_{i=1}^{\infty} (1 + \lambda_i) e^{-\lambda_i}$$

and the product is known to converge for Hilbert-Schmidt operators. For $F \in ID^{2,1}_{loc}(H)$, ∇F is Hilbert-Schmidt and define

$$\Lambda_F(\omega) = \det{}_2(I + \nabla F) \exp(-\delta F - \frac{1}{2} \parallel F \parallel_H^2) . \tag{2.2}$$

The following lemma will be needed in section 4. The proof is straightforward (*cf* lemma 6.1 of [5] or lemma 1.5 of [8]).

Lemma 2.1 *Let F_1, F_2, F_3 belong to $ID^{2,1}_{loc}(H)$ and let $T_i \omega = \omega + F_i(\omega)$, $i = 1, 2, 3$. Assume that: (i) $\mu \circ T_2^{-1} \ll \mu$ and (ii) $T_3 = T_2 \circ T_1$ (i.e. $F_3 = F_2 + F_1 \circ T_2$). Then*

(a) $I + \nabla F_3 = [I + (\nabla F_1)(T_2)](I + \nabla F_2)$

(b) $\Lambda_{F_3} = (\Lambda_{F_1} \circ T_2 \omega) \cdot \Lambda_{F_2}(\omega)$.

Remark: Recall that for any measurable set A on W there exists a *σ-compact modification of* A, i.e. there exists a σ-compact set G such that $G \subset A$ and $\mu(G) = \mu(A)$.

Following Kusuoka [5] we associate with every measurable subset A of W the following random variable $\rho_A(\omega)$ which plays an important role in the construction of a class of mollifiers:

Definition 2.0 *Let A be a measurable subset of W, set*

$$\rho_A(\omega) = \inf_{h \in H}\{\| h \|_H \colon \ \omega + h \in A\} \tag{2.3}$$

and $\rho_A(\omega) = \infty$ if $\omega \notin A + H$.

Clearly, $\rho_A(\omega) = 0$ if $\omega \in A$, moreover (*cf* [5] or [8]):

(i) If $A \subset B$ then $\rho_A(\omega) \geq \rho_B(\omega)$.

(ii) $|\rho_A(\omega) - \rho_A(\omega + h)| \leq \| h \|_H$.

(iii) $A_n \nearrow A$ implies $\rho_{A_n}(\omega) \searrow \rho_A(\omega)$.

(iv) If G is σ-compact and $\varphi \in C_0^\infty(IR)$ (compact support) then $\varphi(\rho_G(\omega)) \in ID^{p,1}$ for all p and

$$\begin{aligned} \| \nabla\varphi(\rho_G(\omega)) \|_H &\leq \| \varphi' \|_\infty \cdot \mathbf{1}_{\{\varphi'(\rho_G) \neq 0\}} \\ &\leq \| \varphi' \|_\infty \ . \end{aligned} \tag{2.4}$$

(v) It follows also from the $0-1$ law on the Wiener space that, if $\mu(A) > 0$, then $\rho_A < +\infty$ almost surely.

The following lemma is implicit in [5] (*cf* also [8]), it is formulated here explicitly since it demonstrates the applicability of ρ_A to the construction of certain mollifiers and thus clarifies the role of ρ_A in section 4.

Lemma 2.2 *Let $F \in ID^{2.1}_{loc}(X)$, consider the H-parameterized random fields on $\Omega \times \{\| h \|_H \leq 1\}$*

$$\begin{aligned} V(\omega, h) &= F(\omega + h) \\ U(\omega, h) &= \nabla F(\omega + h)\,. \end{aligned}$$

Assume that the random fields $V(\omega, h), U(\omega, h)$ are separable and consequently $\sup_{\|h\|\leq 1} \|F(\omega + h)\|$ and $\sup_{\|h\|\leq 1} \|\nabla F(\omega + h)\|$ are well defined random variables. Set

$$A = \left\{ \omega : \sup_{\|h\|_H \leq 1} \| F(\omega + h) \|_X \leq K \quad and \quad \sup_{\|h\|_H \leq 1} \| \nabla F(\omega + h) \|_{X \otimes H} \leq K \right\}. \tag{2.5}$$

Let $\varphi \in C^\infty_o(IR)$ satisfy $|\varphi(x)| \leq 1$, $\varphi(x) = 1$ for $|x| \leq \frac{1}{3}$, $\varphi(x) = 0$ for $|x| \geq \frac{2}{3}$ and $\|\varphi'\|_\infty \leq 4$.

Let G be the σ-compact modification of A. Set

$$\varphi(\rho_G(\omega))F(\omega) = F_G(\omega)\,. \tag{2.6}$$

Then

(i) for every $\omega \in G, F_G(\omega) = F(\omega)$ and $\nabla F_G(\omega) = \nabla F(\omega)$,

(ii) $\| F_G \|_X \leq K$ and $\| \nabla F_G \|_{X \otimes H} \leq 5K$, almost surely.

Remark: As defined above, G may be empty and further assumptions are needed in order to apply this result.

Proof: Obviously $F(\omega) = F_G(\omega)$ on G, therefore by the locality property of ∇,

$$\mathbf{1}_{\{F - F_G = 0\}} \nabla(F - F_G) = 0$$

therefore $\nabla F_G = \nabla F$ on G. Turning to (ii), by definition

$$\| F_G(\omega) \|_X \leq \mathbf{1}_{\{\rho_G \leq \frac{2}{3}\}}(\omega) \cdot \| F(\omega) \|_X \;.$$

However, if $\rho_G(\omega) \leq \frac{2}{3}$, then there exists an $h_o \in H$ such that $\| h_o \| \leq \frac{2}{3}$ and $\omega + h_o \in G$. Therefore

$$\sup_{\|h\|_H \leq 1} \| F(\omega + h + h_o) \|_X \leq K$$

and, in particular, $\| F(\omega) \|_X \leq K$ for ω for which $\rho_G(\omega) \leq \frac{2}{3}$. Now

$$\nabla F_G(\omega) = \varphi(\rho_G(\omega))\nabla F(\omega) + F(\omega)\nabla\varphi(\rho_G(\omega))\cdot$$
$$\| \nabla F_G(\omega) \|_{X\otimes G} \leq \mathbf{1}_{\{\rho_G<\frac{2}{3}\}}(\omega)\cdot \| \nabla F(\omega) \| +4 \| F(\omega) \| \cdot \mathbf{1}_{\{\frac{1}{3}\leq\rho_G\leq\frac{2}{3}\}}(\omega)$$

and by the same arguments as above it follows that $\| \nabla F_G \|_{X\otimes H} \leq 5K$, which completes the proof.

Definition 2.1 *A random variable F will be said to be in $H - C^1$ if, for almost all ω, the mapping $h \to F(\omega + h)$ is continuously differentiable in H.*

Definition 2.2 *F will be said to be in $H - C^1_{loc}$ if there exists a random variable $Q(\omega)$ such that $Q(\omega) > 0$ a.s. and the mapping $h \to F(\omega + h)$ is continuously differentiable in the region $\{h \in H \| h \|_H < Q(\omega)\}$.*

As an example of a r.v. in $H - C^1_{loc}$ but not in $H - C^1$, consider $\psi(W(h_o))$ where h_o is a fixed element of H and $\psi(X), X \in IR$ is the periodic continuous function defined by $\psi(0) = 0$ and $\psi(X) = sign\,(\cos \pi X)$, i.e. $|\psi'(X)| = 1$ except at the points of discontinuity $X = (\frac{1}{2} + n)$, $n = 0, \pm 1, \pm 2, \cdots$. Consequently, $\psi(W(h_o))$ is in $H - C^1_{loc}$ but not in $H - C^1$.

Proposition 2.1 *$H - C^1_{loc} \subset ID^{\infty,1}_{loc}$ where $ID^{\infty,1}_{loc}$ denotes the set of Wiener functionals which are locally in L^∞ as well as their first order Sobolev derivatives.*

Remark: This extends the result $H - C^1 \subset ID^{1,2}_{loc}$ ([5]).

Proof: Let $F \in H - C^1_{loc}$. Set

$$A_n = \{\omega \in W : \begin{array}{ll} \text{(a)} & Q(\omega) \geq 4/n \\ \text{(b)} & \sup_{\|h\|_H < \frac{2}{n}} |F(\omega + h)| \leq n \\ \text{(c)} & \sup_{\|h\|_H < \frac{2}{n}} \| \nabla F(\omega + h) \|_H \leq n\} \end{array}$$

Then $(A_n) \nearrow W$ almost surely. Let G_n denote the σ-compact modification of A_n and

$$F_n(\omega) = \varphi(n\rho_{G_n}(\omega))F(\omega)$$

where φ is in $C_0^\infty(IR)$ and ρ_G is as was defined earlier. Choose φ such that $\|\varphi\|_\infty < 1$, $\| \varphi' \|_\infty < 4$, $\varphi(t) = 1$ on $|t| \leq \frac{1}{3}$ and zero on $|t| \geq \frac{2}{3}$. Then:

1. On $\{\omega : n\rho_{G_n}(\omega) < \frac{1}{3}\}$, $F_n(\omega) = F(\omega)$ and these sets increase to W since $A_n \nearrow W$ almost surely.

2. In order to show that F_n is bounded, note that

$$|F_n(\omega)| \leq \mathbf{1}_{\{n\rho_{G_n} < \frac{2}{3}\}} \cdot |F(\omega)| \, .$$

Now, for a given ω, $n\rho_{G_n}(\omega) < \frac{2}{3}$ implies that there exists an h_o with $\omega + h_o \in G_n$ and $\| h_0 \| < \frac{2}{3}n$. Hence

$$\sup_{|\eta| \leq \frac{2}{n}} F(\omega + h_o + \eta) \leq n$$

and $|F(\omega)| \leq n$ on $\{\omega : n\rho_{G_n}(\omega) < \frac{2}{3}\}$.

3. Similarly, in order to show that ∇F_n is bounded:

$$\nabla F_n = [\nabla\varphi(n\rho_{G_n}(\omega))]F + \varphi(n\rho_{G_n}(\omega)) \cdot \nabla F$$

and since $|\nabla\varphi(n\rho_{G_n}(\omega)| \leq \| \varphi' \|_\infty \cdot n \cdot \mathbf{1}_{\{\omega: \varphi'(n\rho_{G_n}) \neq 0\}}$ it follows that

$$\begin{aligned} &\| \nabla[\varphi(n\rho_{G_n}(\omega)) \cdot F] \|_H \\ &\quad \leq |F| \cdot \| \varphi' \|_\infty \cdot n \mathbf{1}_{\{n\rho_{G_n} \leq \frac{2}{3}\}} + \mathbf{1}_{\{n\rho_{G_n} \leq \frac{2}{3}\}} \cdot \| \nabla F \|_H \, . \end{aligned}$$

Again as in (2) above, if $n\rho_{G_n}(\omega) \leq \frac{2}{3}$, then there exists an h, $\|h\|_H < \frac{2}{3}n$ such that $\omega + h \in G_n$, hence $|F(\omega)| \leq n$, $\|\nabla F(\omega)\|_H \leq 5n$, on $\{\omega : n\rho_{G_n}(\omega) < \frac{2}{3}\}$.

3 The case $\| \nabla F \| \leq c < 1$

Theorem 3.1 *Let $F : W \to H$ be a measurable map belonging to $I\!D^{p,1}(H)$ for some $p > 1$. Assume that there exists constants c, d (with $c < 1$) such that for almost every ω*

$$\| \nabla F(\omega) \| \;\leq\; c < 1 \tag{3.1}$$

and

$$\| \nabla F(\omega) \|_{H-S} \;\leq\; d < \infty \tag{3.2}$$

where $\| \cdot \|$ denotes the operator norm and $\| \cdot \|_{H-S} = \| \cdot \|_{H \otimes H}$ denotes the Hilbert-Schmidt (or $H \otimes H$) norm (otherwise stated, for a.c. ω, $\| F(\omega + h) - F(\omega) \|_H < c \| h \|_H$ for all $h \in H$ where c is a constant, $c < 1$ and $\nabla F \in L^\infty(\mu, H \otimes H)$). Then:

(a) Almost surely $\omega \mapsto T\omega = \omega + F(\omega)$ is bijective, cf [5].

(b) The measures μ and $\mu \circ T^{-1}$ are mutually absolutely continuous.

(c)
$$Ef(\omega) = E\{f(T\omega) \cdot |\Lambda_F(\omega)|\} \tag{3.3}$$

for all bounded and measurable $f(\omega)$ and in particular $E|\Lambda_F| = 1$.

Remarks: (a) In theorem 6.1 of [5] it was shown that without the assumption $\| \nabla F \|_{H-S} \leq d < \infty$, that ($\alpha$) T is bijective (β) $\mu \circ T^{-1} \ll \mu$ and (γ) $Ef(\omega) \geq Ef(T\omega)|\Lambda_F(\omega)|$. (*cf* also [2] and [3]).

(b) Note that by the assumptions above, $E[\exp \lambda |F|^2] < \infty$ for all $\lambda < \frac{1}{2d^2}$ (*cf* [11, 12]).

(c) From the conclusions of the theorem, we have immediately

$$\begin{aligned} \frac{d(\mu \circ T^{-1})}{d\mu} &= \frac{1}{|\Lambda_F \circ T^{-1}|}, \\ \frac{d(\mu \circ (T^{-1})^{-1})}{d\mu} &= |\Lambda_F|. \end{aligned}$$

Proof: Let us choose a complete, orthonormal basis (e_n) of H from $W^\star$. Let V_n be the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$ and denote by π_n the orthogonal projection from H onto $span\{e_1, \ldots, e_n\}$. Define F_n, $n = 1, 2, \ldots$, as

$$F_n = E[P_{1/n}\pi_n F | V_n],$$

where $P_{1/n}$ represents the Ornstein-Uhlenbeck semigroup on W. It is evident that the boundedness properties of ∇F are inherited by ∇F_n, i. e.,

$$\|\nabla F_n\| \leq c \text{ and } \|\nabla F_n\|_{H-S} \leq d$$

almost surely, for any n. Furthemore F_n's are now smooth functionals. Let $T_n\omega = \omega + F_n(\omega)$, then as in [5], one can construct via iteration the inverse of T_n, noted $T_n^{-1}\omega = \omega + G_n(\omega)$. G_n satisfies the relation $F_n \circ T_n^{-1} = -G_n$. We have

$$\begin{aligned}
\|G_n(\omega + h) - G_n(\omega)\|_H &= \|F_n \circ T_n^{-1}(\omega + h) - F_n \circ T_n^{-1}(\omega)\|_H \\
&= \|F_n(\omega + h + G_n(\omega + h)) - F_n(\omega + G_n(\omega))\|_H \\
&\leq c\|h\|_H + c\|G_n(\omega + h) - G_n(\omega)\|_H,
\end{aligned}$$

therefore

$$\|G_n(\omega + h) - G_n(\omega)\|_H \leq \frac{c}{1-c}\|h\|_H.$$

We also have

$$\nabla G_n = -(I_H + \nabla G_n)^\star \nabla F_n \circ T_n^{-1},$$

hence

$$\begin{aligned}
\|\nabla G_n\|_{H-S} &\leq \|I_H + \nabla G_n\| \|\nabla F_n \circ T_n^{-1}\|_{H-S} \\
&\leq (1 + \frac{c}{1-c}) \|\nabla F_n \circ T_n^{-1}\|_{H-S} \\
&\leq \frac{d}{1-c}
\end{aligned}$$

almost surely. From the finite dimensional Jacobi theorem (or from [10]) we know already that the image of μ under T_n or its inverse are equivalent to μ, with $E[f \circ T_n |\Lambda_n|] = E[f]$ for any $f \in C_b(W)$, where Λ_n denotes Λ_{F_n}. As n goes to infinity, (F_n) converges to F in $I\!D^{p,1}$ for any $p > 1$, taking a subsequence, if necessary, $\Lambda_n \to \Lambda$ almost surely. To prove the relation (3.3), it is then sufficient to prove the uniform integrability of the sequence $(\Lambda_n; n = 1, 2, \ldots)$. From the de la Vallé Poussin lemma, the uniform integrability is implied if we can show that

$$\sup_n E[|\Lambda_n| |\log |\Lambda_n||] < +\infty.$$

Hence it suffices to show

$$\sup_n E[|(\delta F_n) \circ T_n^{-1}|] < +\infty,$$

in fact $|det_2(I + \nabla F_n)| \leq \exp d^2$ from a well-known inequality on the Carleman-Fredholm determinants. From the formula (cf for example [13])

$$(\delta F_n) \circ T_n^{-1} = -\delta G_n + ||G_n||_H^2 + trace[(\nabla F_n \circ T_n^{-1}).\nabla G_n],$$

and since $||\delta\xi||_p \leq c_p ||\xi||_{p,1}$ (with $c_2 = 1$), using the bound for the Hilbert-Schmidt norm of ∇G_n, we obtain

$$\begin{aligned} E[((\delta F_n) \circ T_n^{-1})^2]^{1/2} &\leq E[(\delta(F_n \circ T_n^{-1}))^2]^{1/2} + E[||G_n||_H^2]^{1/2} \\ &+ E[||\nabla F_n \circ T_n^{-1}||_{H-S}^2 ||\nabla G_n||_{H-S}^2]^{1/2} \\ &\leq ||G_n||_2 + ||\nabla G_n||_{L^2(\mu, H\otimes H)} + \frac{d^2}{1-c} \\ &\leq ||G_n||_2 + \frac{d}{1-c} + \frac{d^2}{1-c}. \end{aligned}$$

Since $||\nabla G_n||_{H-S} \leq \frac{d}{1-c}$, it follows from [11, 12] that

$$\sup_n E[\exp \alpha ||G_n||_H^2] < +\infty,$$

for any $\alpha < \frac{1}{2}(\frac{1-c}{d})^2$ and this completes the proof.

4 Decomposing $\{\det_2(I_H + \nabla F) \neq 0\}$ into sets on which T is injective

Theorem 4.1 *Let $F : W \to H$ be a $H - C^1_{loc}$ map, $T\omega = \omega + F(\omega)$. Let M denote the set*

$$M = \{\omega : \det_2(I_H + \nabla F(\omega)) \neq 0\}$$

or, what is the same, M is the set on which $I_H + \nabla F$ is invertible. Then there exists a measurable partition of $(M_n; n = 1, 2, \ldots)$ of M and a sequence of shifts $(T_n; n = 1, 2, \ldots)$ with $T_n\omega = \omega + F_n(\omega)$, $F_n \in I\!D^{p,1}_{loc}$ for some $p > 1$ such that, for each n, $T_n = T$ almost surely on M_n and $T_n : W \to W$ is bijective. Moreover

$$E[f \circ T_n |\Lambda_n|] = E[f],$$

for any $f \in C_b(W)$. Consequently

(i) For almost all ω, the cardinal of the set $T^{-1}\{\omega\} \cap M$, denoted by $N(\omega, M)$ is at most countably infinite.

(ii) For any $f \in C_b^+(W)$, we have

$$E[f \circ T |\Lambda|] = E[f N(\omega, M)]\,,$$

and for any $n \geq 1$,

$$E[\mathbf{1}_{TM_n}(\omega) f(\omega)] = E[\mathbf{1}_{M_n}(\omega) f(T\omega) |\Lambda_F(\omega)|]. \tag{4.1}$$

(iii) $\mu \circ T^{-1}|_M \ll \mu$ with

$$\frac{d(\mu \circ T^{-1})|_M}{d\mu}(\omega) = \sum_{\theta \in T^{-1}\{\omega\} \cap M} \frac{1}{|\Lambda_F(\theta)|}.$$

Remark: The first part of (iii) restricted to F in $H - C^1$ is the first half of theorem 6.2 of [5]. In the particular case where the restriction of T to a measurable set D is injective, (4.1) yields (by replacing M_n in (4.1) with $D \cap M_n$ and summing over n):

$$E[\mathbf{1}_{T(D \cap M)}(\omega) f(\omega)] = E[\mathbf{1}_D(\omega) f(T\omega) |\Lambda_F(\omega)|]\,. \tag{4.2}$$

This improves upon part (ii) of theorem 6.2 of [5] where it is shown that for $M^c = \phi$, $E\mathbf{1}_{TD}(\omega)f(\omega) \geq E\mathbf{1}_D(\omega)f(T\omega)|\Lambda_F|$.

Proof: The proof follows the approach of Kusuoka [5] (*cf* also Nualart [8]). Due to its length, we explain the main idea in the first part, then the rigourous proof is given .

Let e_i, $i = 1, 2, \cdots$ be a complete, orthonormal basis of H. Let $\boldsymbol{\lambda} = \{\lambda_{i,j}, 1 \leq i, j \leq n\}$ be a real valued $n \times n$ matrix such that $I_{n\times n} + \boldsymbol{\lambda}$ is invertible. Let $T_{\boldsymbol{\lambda}}(\omega) = \omega + F_{\boldsymbol{\lambda}}(\omega)$ where

$$F_{\boldsymbol{\lambda}}(\omega) = \sum_{i,j=1}^{n} \lambda_{i,j}\langle\omega, e_j\rangle e_i \tag{4.3}$$

and note that $\nabla F_{\boldsymbol{\lambda}} = \sum \lambda_{i,j} e_j \otimes e_i$ is deterministic and $\mu \circ T_{\boldsymbol{\lambda}}^{-1} \sim \mu$ and $T_{\boldsymbol{\lambda}}$ is bijective. With $T = I + F$, consider the following factorization of T

$$T = (T \circ T_{\boldsymbol{\lambda}}^{-1}) \circ T_{\boldsymbol{\lambda}} = T_c \circ T_{\boldsymbol{\lambda}}\,. \tag{4.4}$$

Therefore,

$$T_c = T \circ T_{\boldsymbol{\lambda}}^{-1};\ T_c\omega = T_\lambda^{-1}\omega + F(T_{\boldsymbol{\lambda}}^{-1}\omega)\,.$$

Since $T_{\boldsymbol{\lambda}}^{-1}\omega = \omega - F_{\boldsymbol{\lambda}}(T_\lambda^{-1}\omega)$, it follows that

$$T_c\omega = \omega - F_{\boldsymbol{\lambda}}(T_{\boldsymbol{\lambda}}^{-1}\omega) + F(T_{\boldsymbol{\lambda}}^{-1}\omega)\,.$$

Setting $T_c\omega = \omega + F_c(\omega)$, yields

$$F_c(\omega) = F(T_{\boldsymbol{\lambda}}^{-1}\omega) - F_{\boldsymbol{\lambda}}(T_{\boldsymbol{\lambda}}^{-1}\omega)\,, \tag{4.5}$$

and since $T\omega = T_c(T_{\boldsymbol{\lambda}}\omega) = \omega + F_{\boldsymbol{\lambda}}(\omega) + F_c(T_{\boldsymbol{\lambda}}\omega)$

$$\nabla F = \nabla F_{\boldsymbol{\lambda}}(\omega) + ((\nabla F_c) \circ T_{\boldsymbol{\lambda}}\omega) \cdot \nabla T_{\boldsymbol{\lambda}}\,. \tag{4.6}$$

A rough outline of the proof is as follows:

Let ω_o be a particular ω in M, choose n and $\boldsymbol{\lambda}$ such that $\| \nabla F(\omega_o) - \nabla F_{\boldsymbol{\lambda}}(\omega_0) \|_{H-S} \leq \epsilon$. Assume that there exists a "neighborhood of ω_o", say $V = V(\omega_o) \subset M$ on which $\| \nabla F(\omega) - \nabla F_{\boldsymbol{\lambda}}(\omega) \|_{H-S} \leq 2\epsilon$, we can use this relation to define $V(\omega_o)$. Therefore, by

(4.5) it must hold that on $V(\omega_o)$, $\| (\nabla F_c) \circ T_{\boldsymbol{\lambda}}\omega \|_{H-S}$ is small. Consequently, $T = T_c \circ T_\lambda$, $T_c\omega = \omega + F_c(\omega)$ where T_λ is very well behaved for all ω and $\| \nabla F_c \|_{H-S}$ is small on the set $T_\lambda^{-1}V(\omega_o)$ and in order to apply the results of the previous section to T_c in the vicinity of $T_\lambda^{-1}V(\omega_o)$ we have to mollify F_c outside this region. Namely, we have constructed a mollifier, say $\psi(\omega; T_{\boldsymbol{\lambda}}^{-1}V)$ such that for $\omega \in T_{\boldsymbol{\lambda}}^{-1}V(\omega_0)$

$$\psi(\omega, T_{\boldsymbol{\lambda}}^{-1}V)F_c(\omega) = F_c(\omega) \tag{4.7}$$

and $\psi \cdot F_c$ is in $ID^{\infty,1}(H)$, and

$$\| \nabla(\psi(\omega, T_{\boldsymbol{\lambda}}^{-1}V)F_c(\omega)) \|_{H-S} \leq c < 1 \, .$$

This will enable us to apply part (b) of theorem 3.1 to yield the equivalence of the measures $\mu \circ T_c^{-1}\big|_{T_{\boldsymbol{\lambda}}^{-1}V}$ and $\mu\big|_{T_{\boldsymbol{\lambda}}^{-1}V}$, by part (a) of theorem 3.1. The restriction of T to V is injective and by lemma 2.2 and part (c) of theorem 3.1 yields equation (4.1) with M_n replaced by V. Assuming that a countable sequence of V's, say V_n, covers M, setting $M_n = V_n \cap \left(\bigcup_{i=1}^{n-1} V_i\right)^c$ yields the decomposition of M. Note that instead of the relation

$$T_c\omega = \omega + \psi(\omega)F_c(\omega) \qquad \text{on } w \in T_{\boldsymbol{\lambda}}^{-1}V$$

we can also use

$$T_c\omega = \omega + v + \psi(\omega)(F_c(\omega) - v)$$

for some fixed $v \in H$.

Returning to the proof, let $\boldsymbol{\lambda}$ denote an $n \times n$ matrix and assume that $I_{n\times n} + \boldsymbol{\lambda}$ is invertible. Denote by $\| (I_{n\times n} + \boldsymbol{\lambda})^{-1} \|$ the operator norm of $(I_{n\times n} + \boldsymbol{\lambda})^{-1}$ and set

$$(\| (I_{n\times n} + \boldsymbol{\lambda})^{-1} \|)^{-1} = \gamma(\boldsymbol{\lambda}) \, .$$

Let $A(n, v, \boldsymbol{\lambda})$ denote the set

$$A(n, v, \boldsymbol{\lambda}) \quad = \quad \left\{ \omega : \; Q(\omega) > \frac{4}{n}, \sup_{\|h\|_H < \frac{1}{n}} \| F(\omega + h) - F_{\boldsymbol{\lambda}}(\omega + h) - v \|_H \leq a \frac{\gamma(\boldsymbol{\lambda})}{n} \right.$$

$$\text{and} \quad \sup_{\|h\|_H<\frac{1}{n}} \| (\nabla F)(\omega+h) - \nabla F_{\boldsymbol\lambda} \|_{H-S} \leq a\gamma(\boldsymbol\lambda) \Big\}$$

where F_λ is as defined by (4.3) and a is a constant to be determined later. Let A denote any of the sets $A(n,v,\boldsymbol\lambda)$ and let G be a σ-compact modification of $A\cap M$, i.e. $G\subset A\cap M$ and $\mu(G)=\mu(A\cap M)$. Let ψ and ρ be as defined in section 2. Consider

$$\tilde F_c(\omega) = \varphi(n\rho_G(T_\lambda^{-1}\omega))F_c(\omega)$$

then obviously $\tilde F_c(\omega)=F_c(\omega)$ on $\{\omega : T_{\boldsymbol\lambda}^{-1}\omega\in G\}$. We want to show now that $\| \nabla\tilde F_c(\omega)\|_{H-S}\leq 5a$, for this purpose it suffices to show that

$$\| \nabla(\tilde F_c(T_{\boldsymbol\lambda}\omega))\|_{H-S}\leq 5a\gamma(\boldsymbol\lambda)\,. \tag{4.8}$$

By (4.4)

$$\nabla(\tilde F_c(T_\lambda\omega)) = \nabla\left\{\varphi(n\rho_G(\omega))[F(\omega)-F_{\boldsymbol\lambda}(\omega)-v]\right\}.$$

Hence

$$\begin{aligned}\| \nabla(\tilde F_c(T_{\boldsymbol\lambda}\omega))\|_{H-S} &\leq \mathbf{1}_{\{\rho_G(\omega)\leq\frac{2}{(3n)}\}}\| \nabla F(\omega)-\nabla F_{\boldsymbol\lambda}\|_{H-S} \\ &+4\cdot n\cdot\mathbf{1}_{\{\rho_G(\omega)\leq\frac{2}{(3n)}\}}\| F(\omega)-F_{\boldsymbol\lambda}(\omega)-v\|_H\ .\end{aligned} \tag{4.9}$$

If $\rho_G(\omega)\leq\frac{2}{3n}$ then there exists an $|h_o|\leq\frac{2}{3n}$ such that $\omega+h_o\in G$, hence $\|\nabla F(\omega)-\nabla F_{\boldsymbol\lambda}\|_{H-S}\leq a\gamma(\boldsymbol\lambda)$. Similarly, if $\rho_G(\omega)\leq\frac{2}{3n}$, then $\| F(\omega)-F_{\boldsymbol\lambda}(\omega)-v\|_H\leq a\gamma(\boldsymbol\lambda)/n$. Substituting in (4.8) yields (4.7) with $a<1/6$.

Set $\tilde T_c\omega=\omega+\tilde F_c(\omega)$, then $T=\tilde T_c\circ T_{\boldsymbol\lambda}$ on G, recall now that $G=G(n,v,\boldsymbol\lambda)$, assume that the elements of $\boldsymbol\lambda$ are rational and v belongs to a countable dense set in H, the totality of $G(n,v,\boldsymbol\lambda)$ is countable, say $\{G_\nu,\nu=1,2,\cdots\}$ and $\cup_\nu G_\nu=M$. Setting $M_n=G_n\cap\left(\bigcup_{i=1}^{n-1}G_k\right)^c$ yields the first the existence of the partition. Let us denote by $T_n(\omega)=\omega+F_n(\omega)$ the shift $\tilde T_c\circ T_{\boldsymbol\lambda}$ which corresponds to the set $G(n,v,\boldsymbol\lambda)$, then with the help of the Theorem 3.1, we have

$$E[f\circ T_n|\Lambda_n|]=E[f],$$

for any $f \in C_b(W)$, which completes the proof of the first part of the theorem. We have

$$\begin{aligned} T^{-1}\{\omega\} \cap M &= \{\theta \in M : T(\theta) = \omega\} \\ &= \bigcup_{n=1}^{\infty} \{\theta \in M_n : T_n(\theta) = \omega\}, \end{aligned}$$

since the shifts T_n are bijective, the cardinal of the above set is at most countably infinite, and this proves (i).

For (ii), we have

$$\begin{aligned} E[f \circ T|\Lambda|] &= \sum_{n=1}^{\infty} E[\mathbf{1}_{M_n} f \circ T_n |\Lambda_n|] \\ &= \sum_n E[\mathbf{1}_{T_n(M_n)} \circ T_n \cdot f \circ T_n |\Lambda_n|] \\ &= \sum_n E[\mathbf{1}_{T_n(M_n)} f] \\ &= E[f N(\omega, M)], \end{aligned}$$

for any $f \in C_b^+(W)$, with the convention that if one side of the equality is $+\infty$ so does the other side. For the second part of (ii) we have

$$\begin{aligned} E[\mathbf{1}_{T(M_n)} f] &= E[\mathbf{1}_{T_n(M_n)} f] \\ &= E[\mathbf{1}_{M_n} f \circ T_n |\Lambda_n|] \\ &= E[\mathbf{1}_{M_n} f \circ T |\Lambda|]. \end{aligned}$$

To prove (iii), we have, for any $f \in C_b(W)$,

$$\begin{aligned} E[\mathbf{1}_M \, f \circ T] &= \sum_n E[\mathbf{1}_{M_n} f \circ T_n] \\ &= \sum_n E[\mathbf{1}_{M_n} f \circ T_n \frac{|\Lambda_n|}{|\Lambda| \circ T_n^{-1} \circ T_n}] \\ &= \sum_n E[\mathbf{1}_{T(M_n)} f \, \frac{1}{|\Lambda_n \circ T_n^{-1}|}] \\ &= E[\sum_{\theta \in T^{-1}\{\omega\} \cap M} f \, \frac{1}{|\Lambda(\theta)|}], \end{aligned}$$

which completes the proof.

The following corollary, whose proof is immediate, gives a more symmetric version of the identity of (ii) of the theorem:

Corollary 4.1 *Let F be as in the theorem 4.1, then, for any positive, bounded, Borel measurable, real valued functions f and g, we have*

$$E\left[f(T\omega)\,g(\omega)\,|\Lambda|\right] = E\left[f(\omega)\cdot \sum_{\theta\in T^{-1}\{\omega\}\cap M} g(\theta)\right].$$

In particular, for $g = \mathbf{1}_B$, $B \in \mathcal{B}(W)$ we have

$$E[f\circ T\cdot \mathbf{1}_B\,|\Lambda|] = E[f\cdot N(\omega, B\cap M)],$$

where $N(\omega, B\cap M)$ denotes the cardinal of the set $T^{-1}\{\omega\}\cap B\cap M$.

5 A generalization of the degree theorem and extensions

Let $T\omega = \omega + F(\omega)$, $F \in I\!D^{p,1}_{loc}(H)$ for some $p > 1$ and $E[N(\omega, M)] < \infty$. Let

$$\begin{aligned} N^+(\omega) &= \text{card}\{\theta : T\theta = \omega, \Lambda_F > 0\} \\ N^-(\omega) &= \text{card}\{\theta : T\theta = \omega, \Lambda_F < 0\} \end{aligned}$$

Consider the following three statements:

(A) $E[f(T\omega)\Lambda_F(\omega)] = E\{f(\omega)(N^+(\omega) - N^-(\omega))\}$ for every bounded and measurable f.

(B) $N^+(\omega) - N^-(\omega) = E(N^+(\omega) - N^-(\omega))$.

(C) $E[f(T\omega)\Lambda_F(\omega)] = E[\Lambda_F(\omega)]E[f(\omega)]$ for every measurable and bounded F.

Note that (A), if F is $H - C^1_{loc}$, follows from the theorem 4.1 and (B) and (C) form the degree theorem (*cf* [4, 6, 14]). Moreover statements (A) and (B) imply (C) and statements (A) and (C) imply (B). This yields the degree theorem under weaker assumption as follows.

Theorem 5.1 *Let $F(\omega) \in H - C^1_{loc}$, suppose that for some $\gamma > 0$ and $r > (1+\gamma)/\gamma$, $F(\omega) \in I\!D^{r,2}(H)$, $\Lambda_F \in L^{1+\gamma}(\mu)$ and $\Lambda_F \cdot (I_H + \nabla v)^{-1} \cdot v \in L^{1+\gamma}(\mu, H)$ for all $v \in W^*$. Then, a.s.*

$$\begin{aligned} E\Lambda_F &= \sum_{\theta \in T^{-1}\{\omega\}} sign\, \Lambda_F(\theta) \\ &= N^+(\omega) - N^-(\omega) . \end{aligned}$$

Proof: Statement (C) holds under the assumed integrability conditions by theorem 3.1 of [14] and statement (A) holds under the assumptions of theorem 4.1 and since $E[N(\omega, M)]$ is finite. (B) follows directly from (A) and (C).

In the fourth section we have used a strictly positive random variable Q to define the region of Fréchet differentiability of the drift F in H. In fact, it is not necessary to suppose Q strictly positive as one can see below:

Theorem 5.2 *Let $Q : W \to I\!R_+$ be a random variable with $\mu\{\omega : Q(\omega) > 0\} > 0$ and denote this set with Q_+. Let also $F : W \to H$ be in $I\!D^{p,1}_{loc}$ for some $p > 1$. Suppose that $h \to F(\omega + h)$ is continuously Fréchet differentiable on the set $\{h \in H : \|h\|_H < Q(\omega)\}$ for almost all $\omega \in Q_+$. Define $T\omega$ as $\omega + F(\omega)$ and denote by M the set*

$$M = \{\omega : \det_2(I_H + \nabla F(\omega)) \neq 0\}$$

or, what is the same, M is the set on which $I_H + \nabla F$ is invertible. Then there exists a measurable partition of $(M_n; n = 1, 2, \ldots)$ of $M \cap Q_+$ and a sequence of shifts $(T_n; n = 1, 2, \ldots)$ with $T_n\omega = \omega + F_n(\omega)$ and with $F_n \in I\!D^{p,1}_{loc}$ for some $p > 1$. Furthemore, for each n, $T_n = T$ almost surely on M_n and $T_n : W \to W$ is bijective with

$$E[f \circ T_n |\Lambda_n|] = E[f],$$

for any $f \in C_b(W)$. Consequently

- *(i) For almost all ω, the cardinal of the set $T^{-1}\{\omega\} \cap M \cap Q_+$, denoted by $N(\omega, M \cap Q_+)$ is at most countably infinite.*

(ii) For any $f \in C_b^+(W)$, we have

$$E[f \circ T \cdot \mathbf{1}_{Q_+} | \Lambda|] = E[f(\omega) N(\omega, M \cap Q_+)]\,,$$

and for any $n \geq 1$,

$$E[\mathbf{1}_{TM_n}(\omega) \cdot f(\omega)] = E[\mathbf{1}_{M_n}(\omega) f(T\omega) |\Lambda_F(\omega)|]\cdot \tag{4.10}$$

(iii) $\mu \circ T^{-1}|_{M \cap Q_+} \ll \mu$ with the Radon-Nikodym density

$$\frac{d(\mu \circ T^{-1})|_{M \cap Q_+}}{d\mu}(\omega) = \sum_{\theta \in T^{-1}\{\omega\} \cap M \cap Q_+} \frac{1}{|\Lambda_F(\theta)|}\cdot$$

Proof: The proof goes exactly as in the proof of the theorem 4.1, once we realize that the sets $\{A(n, v, \boldsymbol{\lambda})\}$form a covering of the set $M \cap Q_+$.

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