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# Projective algebraic geometry in positive characteristic

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### Abstract

The geometry of projective algebraic varieties defined over a field of positive characteristic may be quite different from the usual characteristic 0 geometry. Here we have a look at some of the phenomena that may occur — in particular we consider the geometry of plane curves; the theory of tangent and osculating spaces and the Gauss map for arbitrary varieties; the special behavior of the Fermat hypersurfaces. We also discuss a geometric approach to the number theoretic problem of finding a bound for the number of integer (or rational) solutions to a polynomial equation.

This paper, which is based on the talk given at the 21st Nordic Congress of Mathematicians, is meant as a brief introduction to the subject. The only new material included is a corrected proof of Proposition 11 in the paper "On the inseparability of the Gauss map" by Kleiman and the author [**15**, p.120] — as Kaji has pointed out, there was an error in the previous proof.

## 1. Introduction

Let $k$ be an algebraically closed field, denote by $\mathbf{P}_k^n$ projective $n$-space over $k$, and suppose $X \subset \mathbf{P}_k^n$ is a variety. In the classical case, i.e., if the characteristic of $k$ is 0, there is a well known geometric theory of $X$, including the tangent spaces and osculating spaces, the Gauss map, the conormal variety of $X$, the dual variety, the points of hyperosculation (or generalized Weierstrass points), and so forth. Moreover, one can give enumerative formulas — like the generalized Plücker formulas that relate the characters of $X$ to those of its dual variety, or in some cases formulas counting the number of hyperosculating points. One of the classical results, *biduality* or *reflexivity*, says that the double dual variety of $X$

is equal to $X$, or that the conormal variety of $X$ is equal to the conormal variety of its dual variety.

In the case that the characteristic of $k$ is positive, however, the "geometry" of $X$ may be quite different. Some of the phenomena that are new to positive characteristic are: $X$ may have concurrent tangent spaces, all tangents to $X$ may be multitangents, the Gauss map may be inseparable, the gap sequence may be non-classical.

We first consider the case of plane curves, where these phenomena are fairly well understood.

It was shown by Lluis **[18]** that the only nonsingular curves with concurrent tangents are conics in characteristic 2. This was generalized in **[15]**, where we showed that the only nonsingular hypersurfaces with concurrent tangent spaces are odd-dimensional quadrics in characteristic 2.

The Gauss map of a given variety is the map that sends a nonsingular point to the tangent space at that point, considered as a point in the Grassmann variety. To say that the Gauss map is purely inseparable, is to say that a general tangent space is tangent at only one point, i.e., that most tangent spaces are not "multitangents". For curves there is a sufficient condition for the Gauss map to be purely inseparable ([**12**], [**15**]). In higher dimensions, not much is known. It was shown in **[15]** that complete intersection surfaces have purely inseparable Gauss maps — however, Kaji pointed out an error in the proof of Proposition 11 of [**15**], which is used to prove the above mentioned result. A corrected proof of this proposition is presented below. A recent result by Kaji **[13]** gives another sufficient condition for arbitrary dimensional varieties $X$ to have purely inseparable Gauss map, namely that their cotangent bundle be generically ample.

Even in characteristic 0, it is well known that the geometry of Fermat hypersurfaces is a bit special. Here we consider Fermat hypersurfaces of degree $q+1$, where $q = p^e$ and $p$ is the characteristic of $k$, i.e., hypersurfaces projectively equivalent to the one given by $\Sigma T_i^{q+1} = 0$. For example, Beauville **[2]** has given several characterizations of such Fermat hypersurfaces, related to hyperplane sections, polar divisors, and dual varieties. In [**15**] it was conjectured — and proved in the case of surfaces — that these characterizations are equivalent to the condition that the dual variety is nonsingular.

For curves in positive characteristic, Laksov **[16]** defined, and gave a formula for the number of hyperosculating points. Xu **[26]** gave a formula for the number of hyperosculating points in the case that $X$ is a Fermat surface as above, and he obtained a bound for this number in the case of higher dimensional Fermat hypersurfaces **[27]**.

Finally we shall give an example of some work of the "Brazilian school" of how the geometry of curves in positive characteristic can be used to deduce bounds for the number of rational points of a curve defined over a finite field.

## 2. Geometry of plane curves

Let $k$ be an algebraically closed field and let $\mathbf{P}_k^2$ denote the projective plane over $k$. The zero set of a homogeneous polynomial $F(x,y,z)$ defines a plane curve $X \subset \mathbf{P}_k^2$, of degree equal to the degree of $F$. Suppose $F$ has no multiple factors. Then the *dual map* of $X$ is the rational map

$$\gamma : X \to (\mathbf{P}_k^2)^*,$$

which sends a nonsingular point $x \in X$ to the tangent line to $X$ at $x$, considered as a point in the dual projective plane. In other words, $\gamma$ is the map given by the partial derivatives of $F$: $\gamma = (F_x, F_y, F_z)$. The (closure of the) image $X^*$ of $\gamma$ is called the *dual curve* of $X$.

If the base field $k$ has characteristic 0, it is well known that the map $\gamma$ is birational onto the dual curve $X^*$. The birational inverse is the dual map of $X^*$ and $(X^*)^* = X$ holds (when we identify $\mathbf{P}^2_k$ with the dual space of $(\mathbf{P}^2_k)^*$). In particular the curve $X$ and the dual curve $X^*$ have the same geometric genus. It also follows that a general tangent line to the curve $X$ is tangent at only one point. Moreover, the inflection points of $X$ are the (nonsingular) points of ramification of the map $\gamma$ and are finite in number, or, most tangent lines have order of contact 2 with the curve at the point of tangency. In fact, the inflection points are the nonsingular points of the intersection of $X$ with its *Hessian curve*, the curve defined by the determinant of the Hessian matrix of $F$.

Let $d$ denote the degree of $X$ and $d^*$ the degree of $X^*$ — also called the *class* of $X$. Then $d^*$ is equal to the number of tangent lines to $X$ that pass through a given (general) point of $\mathbf{P}^2_k$, or to the number of lines through a general point that are tangent to $X$.

The enumerative characters of $X$ and $X^*$ are related by the *Plücker formulas*. One of them is the following:

$$d^* = d(d-1) - \sum_{P \in \operatorname{Sing} X} e_P, \tag{$I$}$$

where $e_P$ denotes the multiplicity of the Jacobian ideal in the local ring of $X$ at a point $P$. Moreover, if $g$ denotes the geometric genus of $X$ and $X^*$, and if $X$ has only $\delta$ nodes and $\kappa$ ordinary cusps as singularities, then

$$d^* = d(d-1) - 2\delta - 3\kappa = 2d + 2g - 2 - \kappa,$$

and there are corresponding dual formulas

$$d = d^*(d^* - 1) - 2\tau - 3\iota = 2d^* + 2g - 2 - \iota,$$

where $\tau$ is the number of bitangents and $\iota$ the number of inflection points (flexes).

Suppose now that the field $k$ has characteristic $p > 0$. Then the "geometry" of $X$ may be quite different:

(1) $X$ may have concurrent tangent lines (i.e., all tangents to $X$ at nonsingular points pass through a common point).

(2) $X$ may have infinitely many multitangents (i.e., almost all tangents are tangent at several points of $X$).

(3) All points of $X$ may be inflection points (i.e., every tangent has order of contact $> 2$ with $X$).

In fact, it turns out that the dual map $\gamma$ is not necessarily birational — it may have inseparable degree $> 1$, or both inseparable and separable degrees $> 1$ — and $X$ and $X^*$ need not have the same geometric genus. This phenomenon was first studied seriously by Wallace [**25**] (see also [**14**] and references therein). Curves with concurrent tangents are often called *strange* curves — they have recently been studied in [**1**] and [**9**].

The Plücker formula (I) still holds, provided the class $d^*$ is replaced by $\deg\gamma \cdot d^*$.

**Example 1.** Let $X$ be the curve given by the equation

$$F(x,y,z) = xy^{p-1} - z^p = 0.$$

Then the partial derivatives of $F$ are $F_x = y^{p-1}$, $F_y = -xy^{p-2}$, and $F_z = 0$. The tangent line at the point $x = (a,b,c)$ is the line $bx - ay = 0$ — hence all tangents pass through the point $(0,0,1)$. Therefore, the dual curve $X^*$ is the line in $(\mathbf{P}^2)^*$ dual to the point $(0,0,1)$, and the dual map is just the projection of $X$ from the point $(0,0,1)$.

In this case, we have $d^* = 1$, $\deg\gamma = p$, and $e_P = p(p-2)$, which checks with the Plücker formula (here $P = (1,0,0)$ is the only singular point of $X$, if $p > 2$).

**Example 2.** ([14]) Let $X$ be the curve given by the equation

$$F(x,y,z) = xy^{q-1}z^{q^2-q} + x^q z^{q^2-q} + y^{q^2} = 0,$$

where $q = p^e$, with $p^e > 2$. One sees that all tangents pass through $(0,0,1)$, but one also checks that every tangent line is tangent at $q-1$ distinct points, and has order of contact $q$ at each of these points. In this case the dual map has separable degree $q-1$ and inseparable degree $q$. Note that the singular points of $X$ are a high order cusp at $(1,0,0)$ and an ordinary multiple point at $(0,0,1)$.

We have $\deg\gamma = q(q-1)$. Since the multiplicity of the Jacobian ideal at an ordinary $q$-uple point is $q(q-1)$, we conclude from the Plücker formula that the multiplicity of the Jacobian ideal at the cusp $(1,0,0)$ is equal to $q(q-1)^2(q+2)$.

The curves of Examples 1 and 2 have singular points. Indeed, it was shown by Lluis ([**18**], see also Samuel [**21**]) that the only *nonsingular* curves with concurrent tangents are conics in characteristic 2. A similar result holds for hypersurfaces (of arbitrary dimension):

**Theorem.** (Kleiman–Piene [**15**, Th. 7, p.118]) *The only nonsingular hypersurfaces $X \subset \mathbf{P}^n$, of degree $d \geq 2$, with concurrent tangent hyperplanes, are odd-dimensional quadrics in characteristic* 2.

The idea of the proof is the following: if $X$ has concurrent tangent hyperplanes, then the dual variety $X^*$ is a hyperplane in $\mathbf{P}^{n*}$, in particular it is nonsingular. Now if the dual variety is nonsingular, then one can show that it has the same top Chern class as $X$ — since the dual variety has degree 1, this is only possible if $X$ has degree 1 or 2.

Let us consider the case $p = 2$, i.e., $X \subset \mathbf{P}^2_k$ is a plane curve over an algebraically closed field of characteristic 2 ([4], [**23**], [**3**]). In this case the dual map is never separable (the Hessian determinant is 0). If $X$ is a conic, $X$ has concurrent tangents. If $X$ is a nonsingular cubic, then so is $X^*$, and $X^{**} \neq X$ unless $X$ is projectively equivalent to a Fermat cubic, given by $x^3 + y^3 + z^3 = 0$. This last condition is equivalent to the $j$-invariant of $X$ being 0, or to the Hasse invariant of $X$ being 0. Moreover if $X$ is not projectively equivalent to a Fermat curve, then $X \cap X^{**}$ is equal to the set of inflection points, so that in this case the double dual curve $X^{**}$ plays the role of the Hessian curve in the characteristic 0 case. If $X$ is projectively equivalent to a Fermat curve, then $X^{**} = X$ holds, and $X$ "behaves" like a conic in characteristic 0. If $X$ is a nodal cubic, then $X^{**}$ is the line through the three inflection points of $X$!

The typical question in the theory of *enumerative geometry* of plane curves is the following: find the number $N_{\alpha,\beta}$ of curves in a given family that go through $\alpha$ given points and are tangent to $\beta$ given lines, where $\alpha+\beta$ is equal to the dimension of the family. The numbers $N_{\alpha,\beta}$ are called the *characteristic numbers* of the family. Even in characteristic 0 these numbers are known only for families of curves of degree $d \leq 4$ — of these, only the characteristic numbers for families of curves of degree 2 and 3, originally determined by Chasles, Zeuthen, and Maillard, have been rigorously verified according to modern standards.

For families of curves in positive characteristic $p$, where $p$ is small with respect to the degree $d$ of the curves, not much is known. The case of conics in characteristic 2 was done by Vainsencher [**23**]. The case of cubics in characteristic 2 was treated by Berg [**3**], who obtained the following characteristic numbers for families of cubics with $j$-invariant 0 (i.e., the Fermat cubics):

$$\begin{aligned} N_{8,0} &= N_{0,8} = 1 \\ N_{7,1} &= N_{1,7} = 2 \\ N_{6,2} &= N_{2,6} = 4 \\ N_{5,3} &= N_{3,5} = 8 \\ N_{4,4} &= 10 \end{aligned}$$

The symmetry of the numbers reflects the fact that the dual of a cubic curve with $j$-invariant 0 is again a cubic curve with $j$-invariant 0.

## 3. Gauss maps

Let $X \subset \mathbf{P}^n$ be a variety of dimension $r$. The *Gauss map* of $X$ is the rational map

$$\gamma : X \to \mathrm{Grass}(r, n)$$

sending a nonsingular point $x$ of $X$ to the (projective) tangent space to $X$ at $x$, considered as a point in the Grassmann variety of $r$-dimensional linear subspaces of $\mathbf{P}^n$. If $X$ is a hypersurface, i.e., if $r = n-1$, then the Gauss map is the same as the dual map. Set $\mathcal{L} = \mathcal{O}_{\mathbf{P}^n}(1)|_X$, let $\mathcal{P}^1_X(\mathcal{L})$ denote the 1st order sheaf of principal parts of $\mathcal{L}$, and let

$$a^1 : \mathcal{O}_X^{n+1} \to \mathcal{P}^1_X(\mathcal{L})$$

denote the Taylor map, taking a coordinate function to its 1st jet [**20**]. On the open subvariety where $X$ is nonsingular, $a^1$ determines an $(r+1)$-dimensional quotient, and the Gauss map is the map defined by this quotient. In characteristic 0 it is known that the Gauss map is birational, and finite if $X$ is nonsingular. In positive characteristic the Gauss map need not be birational — it may have inseparable degree strictly greater than 1, and possibly also separable degree strictly greater than 1.

For curves we have the following result:

**Theorem.** (Kaji [**12**, Th.0.2], Kleiman–Piene [**15**, Th.8, p.118]) *Let $X \subset \mathbf{P}^n$ be a (reduced and irreducible) curve, of geometric genus $g$. Let $X' \to X$ denote the normalization map, and $\kappa$ the degree of the ramification divisor of this map. If $2g - 2 > g$ holds, then the Gauss map of $X$ is purely inseparable.*

It is reasonable to believe, as was conjectured in [**15**, Conj.1, p.107], that many nonsingular projective varieties should have purely inseparable Gauss map. The two known results are the following.

**Theorem.** (Kleiman–Piene [**15**, Th.13, p.124]) *If $X$ is a nonsingular complete intersection surface of degree $\geq 2$, then its Gauss map is purely inseparable.*

**Theorem.** (Kaji [**13**]) *Every nonsingular projective variety with generically ample cotangent bundle has purely inseparable Gauss map.*

Kaji observed that there is an error in the proof of Proposition 11 of [**15**, p.120]. There we claimed that for the extension of function fields $K(X)/K(X^\vee)$ defined by the Gauss map, if $X'$ denotes the normalization of $X^\vee$ in the purely inseparable closure of $K(X^\vee)$ in $K(X)$, then the induced map $X \to X'$ is separable. This claim only holds for function fields of dimension 1. Kaji shows that for a finite extension $L/K$ of fields, if $K_s$ and $K_i$ denote the separable and purely inseparable closures of $K$ in $L$, then $L$ is not necessarily separable over $K_i$ though $L$ is purely inseparable over $K_s$. Indeed, he gives an example of 2-dimensional function fields where this statement fails, namely: $K = k(x,y)$, $L = K(z)$, where $z$ is a root of the equation $Z^{2p} + xZ^p + y = 0$, and $k$ is a field of characteristic $p > 2$. (Note that this field extension does not come from a Gauss map.)

Here is the statement of the above mentioned proposition, and a new proof:

**Proposition.** [**15**, Prop.11, p.120] *Let $X$ be a smooth projective surface of degree $d \geq 2$, with very ample invertible sheaf $\mathcal{L}$. Set $h := c_1(\mathcal{L})$. Assume that $c_1(\mathcal{P}^1_X(\mathcal{L}))$ is numerically equivalent to a rational multiple of $h$ and that $c_1(X) \leq 4h$ and $c_2(X) > 4d$. Then the Gauss map of $X$ is purely inseparable.*

**Proof.** Let $X^\vee$ denote the image of the Gauss map, and $X''$ the normalization of $X^\vee$ in the separable closure of its function field in the function field of $X$. Then the Gauss map factors through a map $\epsilon : X \to X''$ which is finite, purely inseparable, and of degree $i$; moreover, $\epsilon$ is flat off the finite number of singular points of $X''$.

As in [**15**, Setup 1, p.112], let $Y = \mathbf{P}(\mathcal{P}^1_X(\mathcal{L}))$ denote the (abstract) tangent developable of $X$, so that $Y$ is the pullback to $X$ of the incidence variety, under the Gauss map. Let $Y''$ and $Y^\vee$ denote the pullbacks of the incidence variety to $X''$ and $X^\vee$, and $\eta : Y \to Y''$ the induced map. The tangent developable $Y \to X$ has an obvious section — let $S \subset Y$ denote its image. Let $D'$ denote the image of $S$ in $Y^\vee$ and $D$ the pullback of $D'$

to $Y$; hence, $D$ is equal to the pullback under $\eta$ of $D''$, where $D''$ is the pullback of $D'$ to $Y''$. Therefore,

$$\eta_*[D] = i[D''].\tag{$*$}$$

Moreover, $D''$ is reduced, because $X'' \to X^\vee$, and hence $Y'' \to Y^\vee$, is separable. Hence, if $S''$ denotes the image of $S$ in $Y''$, then the cycle $[D'']$ contains $[S'']$ with multiplicity 1. Hence $[D]$ contains $\eta^*[S'']$ with multiplicity 1; that is, if, say, $\eta^*[S''] = j[S]$, then $[D]$ contains $[S]$ with multiplicity $j$. Finally, $j = i$; indeed, $(*)$ implies that $\eta_* j[S] = i[S'']$, and $\eta_*[S] = [S'']$ because $S \to S''$ is birational (in fact, an isomorphism). Thus we have shown that $[D]$ contains $[S]$ with multiplicity $i$.

Let $\tilde{\pi} : \tilde{Y} \to Y$ be the blowup along $S$, and let $\tilde{S}$ be the strict transform of $D$. It now follows that the direct image $\tilde{\pi}_*[\tilde{S}]$ is equal to the difference $[D] - i[S]$. By [**15**, Lemma 2, p.114], [D] is numerically equivalent to $is[S]$, where $s$ denotes the separable degree of $X \to X^\vee$. Hence $\tilde{\pi}_*[\tilde{S}]$ is numerically equivalent to $is[S] - i[S] = i(s-1)[S]$. The rest of the proof [**15**, pp.120–121] now goes through as given, after replacing each occurrence of $(s-1)$ by $i(s-1)$.

## 4. Fermat varieties

Suppose $X \subset \mathbf{P}^2_k$ is a Fermat curve given by the equation

$$x^{q+1} + y^{q+1} + z^{q+1} = 0,$$

where $q = p^e$ is a power of the characteristic $p$ of $k$. Then the Gauss map is just the iterated Frobenius: $\gamma(x, y, z) = (x^q, y^q, z^q)$, hence it has separable degree 1 and inseparable degree $q$. In this case, the dual curve $X^*$ is a Fermat curve of the same type, and $X^{**} = X$ holds.

It has been shown that the only nonsingular plane curves of degree $d \geq 4$ such that the dual curve is also nonsingular are the ones projectively equivalent to Fermat curves of degree $q + 1$ as above ([**19**], [**10**], [**11**], [**6**]). It was conjectured in [**15**, Conj.2, p.107] that the same holds for hypersurfaces in any dimension. Beauville gave several charactersizations of hypersurfaces projectively equivalent to Fermat ones; using this, and the purely inseparability of the Gauss map, it was shown in [**15**, Th.14, p.124] that the conjecture holds for surfaces.

Given a variety $X \subset \mathbf{P}^n$ of dimension $r$, set $\mathcal{L} = \mathcal{O}_{\mathbf{P}^n}(1)|_X$, let $\mathcal{P}^m_X(\mathcal{L})$ denote the sheaf of principal parts of order $m$ of $\mathcal{L}$, and let

$$a^m : \mathcal{O}_X^{n+1} \to \mathcal{P}^m_X(\mathcal{L})$$

denote the Taylor map, taking a coordinate function to its $m$th jet [**20**]. The rank of each $a^m$ is constant, equal to $s(m)+1$, say, on some open subvariety of $X$. This gives a sequence

$$0 = s(0) < r = s(1) \leq s(2) \leq \ldots \leq s(\bar{m}) = n,$$

where $\bar{m}$ is the smallest integer $m$ such that $a^m$ has generic rank $n+1$. Discarding those $s(j)$ such that $s(j) = s(j-1)$, we obtain a strictly increasing sequence

$$0 = s(0) < r = s(1) < s(b_2) < \ldots < s(b_l) = n.$$

The sequence

$$b_0 = 0 < b_1 = 1 < b_2 < \ldots < b_t$$

is called the *gap sequence* of $X \subset \mathbf{P}^n$. It was introduced for curves by Laksov [**16**, p.49] and used for hypersurfaces by Xu ([**26**], [**27**]). For varieties in characteristic 0, not contained in a linear subspace, the gap sequence is always $0 < 1 < 2 < \ldots$ — such a sequence is called *classical.*

A *point of hyperosculation*, also called a *generalized Weierstrass point*, of $X$, is a point where the rank of $a^m$ is smaller than $s(m)$, for some $m$.

Laksov [**16**] gave generalized Plücker formulas for curves in $\mathbf{P}^n$ in positive characteristic, in particular he found formulas for the (weighted) number of hyperosculating points. For example, if $X \subset \mathbf{P}^2_k$ is a Fermat curve of degree $q+1$, $q = p^e$, then the gap sequence is $0 < 1 < q$, and the total number of points of hyperosculation is

$$(q+1)(q^2 - q + 1).$$

There is no reason to expect a variety $X \subset \mathbf{P}^n$ of dimension $\geq 2$ to have only finitely many points of hyperosculation, but if it does, one would like a formula for their number as in the case of curves. For example, a *general* surface in $\mathbf{P}^3$ has *no* points of hyperosculation, since no plane intersects such a surface in a point of multiplicity $\geq 3$. On the other hand, one can give a formula for the number of hyperosculating points of a *ruled surface*, *assuming* that it only has a finite number.

Suppose next that $X \subset \mathbf{P}^n$ is a Fermat hypersurface of degree $q+1$, with $q = p^e > 2$ equal to a power of the characteristic $p > 0$. Then almost every tangent hyperplane to $X$ has order of contact $q$ at the point of tangency. The gap sequence is $0 < 1 < q$, and the points of hyperosculation of $X$ are the points such that the tangent hyperplane to $X$ at that point has order of contact strictly greater than $q$ (hence equal to $q+1$). For $n = 3$, Xu [**26**, Th.2.2, Th.3.3] showed that a general surface of degree $d = tq + 1$, where $q = p^e$, and with gap sequence $0 < 1 < q$, has finitely many points of hyperosculation, their (weighted) number being equal to

$$((q^2 + q + 1)d^2 - 4q(q+1)d + 6q^2)d.$$

If $X$ is a Fermat surface, then the number of hyperosculating points is finite and equal to

$$(q+1)(q^4 - q^3 + 2q^2 - q + 1).$$

In the case of arbitrary dimensional hypersurfaces, Xu gave a bound for the number of hyperosculating points [**27**], provided the number is finite.

## 5. An application to number theory

The geometry of curves in positive characteristic can be used to answer number theoretical questions (see e.g. [**22**], [**7**], [**24**], [**17**], [**5**], [**8**]) — as an example, let us look at the following situation [**24**].

Let $X$ be a nonsingular curve defined over a finite field $\mathbf{F}_q$, $q = p^e$, and let

$$N := \#X(\mathbf{F}_q)$$

denote the number of rational points of $X$. Then Weil proved the Riemann Hypothesis for curves over finite fields, namely

$$|N - q - 1| \leq 2g\sqrt{q}.$$

One can show this by considering an embedding $X \subset \mathbf{P}^n_k$, where $k$ is an algebraic closure of $\mathbf{F}_q$. Let $F : X \to X$ denote the iterated Frobenius

$$F(x_0, ..., x_n) = (x_0^q, ..., x_n^q).$$

Then the rational points of $X$ are just the fixed points of $F$, and their number can be computed using $l$-adic cohomology and the Lefschetz Trace Formula.

One can also give a proof along the following lines ([**22**], [**24**]): We may assume that the embedding $X \subset \mathbf{P}^n_k$ is such that $X$ has a classical gap sequence; moreover, we may assume $X$ is nonspecial and linearly normal, so that $d = g + n$ holds, where $d$ is the degree and $g$ is the genus of $X$. The fixed points of $F$ can be bounded by the number of points $x \in X$ such that $F(x)$ is contained in an osculating hyperplane to $X$ at $x$. This last number can be computed as follows. Set $\mathcal{L} = \mathcal{O}_{\mathbf{P}^n}(1)|_X$, and consider the Taylor map

$$a^{n-1} : \mathcal{O}_X^{n+1} \to \mathcal{P}_X^{n-1}(\mathcal{L}).$$

Since the gap sequence is classical, the map $a^{n-1}$ is generically surjective. Its kernel $K$ is locally free (since $X$ is a nonsingular curve) with rank 1. We have $F^*\mathcal{L} \cong \mathcal{L}^{\otimes q}$, and the composition

$$K \to \mathcal{O}_X^{n+1} \to F^*\mathcal{L}$$

gives a section

$$\mathcal{O}_X \to K^{-1} \otimes \mathcal{L}^{\otimes q}.$$

This section is not identically zero, and obviously its zeros are the points considered above, each ocurring with multiplicity $\geq n$. Therefore, we get

$$\begin{aligned} N &\leq \frac{1}{n} \deg K^{-1} \otimes \mathcal{L}^{\otimes q} \\ &= \frac{1}{n}(q(g+n) + n(g+n) + (n-1)(g-1)) \\ &= (n-1)(g-1) + \frac{1}{n}(q+n)(n+g). \end{aligned}$$

We may assume $q$ is a square and take $n = \sqrt{q}$; this gives

$$N \leq 1 + n^2 + 2gn = 1 + q + 2g\sqrt{q}.$$

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