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Introduction

Enumerative geometry seeks to determine how many algebro-geometric objects of a given type satisfy certain given conditions. Questions of this type have been posed by mathematicians since Apollonius. More than 2000 years ago he considered problems like the following: find all circles that are tangent to 3 given circles. He showed that there are 8 such circles, and that they are constructible.

In the 17th century Fermat and Descartes represented geometric objects, namely plane curves, as solution sets of polynomial equations. Later Möbius introduced homogeneous coordinates.

The geometric theory of plane curves was further developed and consolidated in the 19th century — by Monge, Poncelet, Plücker, and others.*

In terms of complex projective geometry Apollonius' question translates into: How many conics (in the complex projective plane) are tangent to 5 given conics? The answer 3264 was found by Chasles in 1864 — with this achievement, "modern" enumerative geometry really starts.

In the late 1800's the subject flourished, prompting Hilbert's 15th problem: "To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him."

Much of contemporary enumerative geometry has dealt with giving proofs, acceptable by modern standards, of formulas obtained by 19th century mathematicians like Schubert and Zeuthen. A common method has been to find a parameter space for the objects and realize the conditions as cycles on this space. The results are then obtained from the intersection theory on the parameter space. One of the main goals in the enumerative

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* The great Norwegian mathematician Sophus Lie (1842–1899) was decisively influenced by Poncelet and Plücker — he decided to pursue mathematics seriously only after having read their work on geometry in 1868.

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theory of plane curves is to determine the *characteristic numbers* of a given family, i.e., the numbers $N_{\alpha,\beta}$ of curves of the family that pass through α points and touch β lines. For example, much work has recently been done in the case of plane cubic curves.

It turns out that the problem of enumerating curves, especially rational curves, pops up in various contexts. In 1986, Clemens asked if there are only finitely many rational curves of a given degree on a generic *quintic threefold*, i.e., a hypersurface of degree 5 in $\mathbf{P}^4$ — and, if so, determine their number ([**25**], [**27**]).

This problem, and the more general question of studying rational curves on certain projective varieties, is related to at least three interesting problems: one concerns intermediate Jacobians and the Abel–Jacobi map, another is the classification problem for higher dimensional varieties, and a third has to do with string theory in theoretical physics.

The number 2875 of lines on a generic quintic threefold was first found by Schubert ([**83**], [**46**]), using his calculus on the Grassmann variety of lines in $\mathbf{P}^4$. The number $609,250$ of conics was computed by Katz [**53**]. The method was to realize the variety of conics on the threefold as the zeros of a section of a bundle on the variety of conics in $\mathbf{P}^4$, and then compute the top Chern class of this bundle. Similarly, but more involved, was the computation of the number $317,206,375$ of twisted cubics (rational curves of degree 3), done by Ellingsrud and Strømme [**38**]. They used the knowledge of the Hilbert scheme (and of its intersection ring) of twisted cubics in $\mathbf{P}^3$ to construct a parameter variety for twisted cubics in $\mathbf{P}^4$; the computations were performed in the intersection ring of this parameter space.

A striking new approach to this type of enumerative problems has recently come from string theory in theoretical physics. Rational curves on three-dimensional Calabi–Yau varieties are interpreted as *instantons.* One uses topological quantum field theory to find a polynomial whose coefficients determine the number of rational curves of given type (e.g., degree) on a given Calabi–Yau variety, provided this number is finite. In the case of a generic quintic threefold, Candelas et. al. were thus able to predict the number of rational curves of any given degree [**19**]. Their approach has been carefully studied by Aspinwall and Morrison ([**13**], [**69**], [**70**]). It is also worth remarking that an interesting, unexpected phenomenon in algebraic geometry, called "mirror symmetry", has popped up in this context [**10**].

In addition to the classical open problems, like determining the characteristic numbers of plane curves of degree d, for $d \geq 4$, there are other open problems related to enumerative geometry. One such is to give a description of certain Hilbert schemes and their intersection rings (e.g., for rational normal curves of degree 4 or more). Another, mentioned above, is the question of existence of rational curves on (special) Calabi–Yau threefolds. A third, maybe more peripheral, consists in investigating Fermat hypersurfaces (or intersections of such) in this connection. These varieties play a special role both in characteristic 0 and in positive characteristic — among other things they enter in the construction of special Calabi–Yau threefolds via group actions, and they have special properties with respect to existence of lines.

1. Plane curves

Let $\mathbf{P}^2 = \mathbf{P}^2_k$ denote the projective plane over an algebraically closed base field k of characteristic 0. A *plane curve of degree d*, $C \subset \mathbf{P}^2$, is defined (uniquely up to multiplication by a non-zero element of k) by a homogeneous polynomial in 3 variables, of degree d. Hence the set of all plane curves of degree d can be identified with

$$\mathcal{C}_d := \mathbf{P}(H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d))) \cong \mathbf{P}^{\frac{1}{2}d(d+3)}.$$

Moreover, the "universal polynomial of degree d" defines a universal family

$$\begin{array}{ccc} \mathcal{F}_d & \subset & \mathcal{C}_d \times \mathbf{P}^2 \\ & \overset{\phi}{\searrow} & \downarrow \\ & & \mathcal{C}_d \end{array}$$

such that $\phi^{-1}(c) = C$ is the curve corresponding to the point $c \in \mathcal{C}_d$.

By a *family* of plane curves of degree d we shall mean a subvariety $Z \subset \mathcal{C}_d$. The *characteristic numbers* of a given family Z are defined as

$$N_{\alpha,\beta} = \#\{C \in Z | C \quad \text{passes through } \alpha \text{ points and touches } \beta \text{ lines}\}$$

for $\alpha + \beta = \dim Z$.

Many enumerative problems for plane curves, like the ones about conics, can be reduced to finding the characteristic numbers (cf. Contact Theorem [**41**]). Therefore, a main goal in the enumerative theory of plane curves is to determine the characteristic numbers for "all" families of curves of any given degree d.

The characteristic numbers for the 5-dimensional family of all conics were determined by Chasles.

β	$\mathbf{N}_{5-\beta,\beta}$
0	1
1	2
2	4
3	4
4	2
5	1

Plane conics

(For a historical survey, and a review of the modern work on conics, see [**40**], [**57**].)

For plane cubics, the numbers were determined independently by Maillard [**65**] and Zeuthen [**106**]. Here are some of them:

β	$\mathbf{N}_{9-\beta,\beta}$	$\mathbf{N}_{8-\beta,\beta}$	$\mathbf{N}_{7-\beta,\beta}$
0	1	12	24
1	4	36	60
2	16	100	114
3	64	240	168
4	256	480	168
5	976	712	114
6	3,424	756	60
7	9,766	600	24
8	21,004	400	
9	33,616		
	smooth	*nodal*	*cuspidal*

Plane cubics

Zeuthen also determined the characteristic numbers for the various families of plane quartic curves — for the 14-dimensional family of smooth quartics he found the following numbers [**107**]:

β	$\mathbf{N}_{\alpha,\beta}$
0	1
1	6
2	36
3	216
4	1,296
5	7,776
6	46,656
7	279,600
8	1,668,096
9	9,840,040
10	56,481,396
11	308,389,896
12	1,530,345,504
13	6,533,946,576
14	23,011,191,144

Non-singular quartics
$(\alpha + \beta = 14)$

In order to solve an enumerative problem one also has to prove that the found numbers determine true solutions to the problem and that one does not count "multiple" solutions. This can be considered as part of Hilbert's 15th problem cited in the introduction.

One of the great achievements in mathematics in our century has been to lay down the foundations of algebraic geometry. Of particular importance for enumerative geometry is the general intersection theory that has been developed (see [**57**] for a historical survey and bibliography).

Over the past twenty years there has been a renewed interest in the classical problems of enumerative geometry, due in part to the availability of the modern intersection theory (see [**40**]). For example, the results of Chasles, Maillard, and Zeuthen on conics and cubics have been rigorously established through the efforts of many people — for conics Fulton–MacPherson [**40**], Kleiman [**56**], Casas–Xambó [**21**]; for cubics Aluffi ([**4**], [**5**], [**6**], [**9**]), Kleiman–Speiser ([**58**], [**59**], [**60**]), Sacchiero ([**77**], [**78**]), Sterz ([**86**], [**87**]), Miret–Xambó ([**66**], [**67**], [**68**]). However, only a few of Zeuthen's numbers for quartics have been verified ([**7**], [**8**], [**43**]).

The methods used in the above cited works differ slightly — here we shall only review the approach that uses a parameter space of "complete curves" ([**5**], [**35**]).

Let $\mathbf{P}^{2*}$ denote the *dual projective plane*, i.e., the projective plane whose points corresponds to the lines of $\mathbf{P}^2$. If $C \subset \mathbf{P}^2$ is a nonsingular curve of degree d, its *dual curve* $C^* \subset \mathbf{P}^{2*}$ is defined. It is a curve of degree

$d^* = d(d-1)$ and its points correspond to the tangent lines of C. Thus we obtain a rational map

$$\Gamma : \mathcal{C}_d \cong \mathbf{P}^{\frac{1}{2}d(d+3)} - - \to \mathcal{C}_{d^*} \cong \mathbf{P}^{\frac{1}{2}d^*(d^*+3)}$$

which sends a (nonsingular) curve to its dual curve. The map Γ has a unique extension to the subset $\mathcal{U}_d$ whose points correspond to reduced curves.

For $P \in \mathbf{P}^2$ let

$$\Pi_P = \{C \in \mathcal{C}_d | P \in C\}.$$

The "point condition" Π_P is a hyperplane in $\mathcal{C}_d$.

For $L \in \mathbf{P}^{2*}$, i.e., for L a line in $\mathbf{P}^2$, let

$$\Lambda_L = \{C \in \mathcal{C}_d | \ L \quad \text{is tangent to } C\}.$$

The "line condition" Λ_L is a hypersurface in $\mathcal{C}_d$ of degree $2(d-1)$ and is equal to the (closure of) the pullback of the hyperplane

$$\Pi_L = \{D \in \mathcal{C}_{d^*} | L \in D\}.$$

The map Γ is defined by the linear system generated by the Λ_L's.

In order to compute the numbers $N_{\alpha,\beta}$ we want to intersect α point conditions $\Pi_{P_1}, ..., \Pi_{P_\alpha}$ and β line conditions $\Lambda_{L_1}, ..., \Lambda_{L_\beta}$. One of the main problems of the theory is that the intersection of the line conditions need not be proper, even for general choices of lines.

Define a *variety of complete curves of degree d* to be a variety B together with a surjective morphism $B \to \mathcal{C}_d$, which is an isomorphism above $\mathcal{U}_d$ and such that Γ extends to a morphism $\tilde{\Gamma} : B \to \mathcal{C}_{d^*}$. A "point condition" $\tilde{\Pi}_P$ on B is the strict transform of Π_P. A "line condition" $\tilde{\Gamma}^{-1}(\Pi_L)$ on B is the strict transform $\tilde{\Lambda}_L$ of $\Lambda_L \subset \mathcal{C}_d$, and the fact that Γ extends to $\tilde{\Gamma}$ on B means that the intersection of the strict transforms $\tilde{\Lambda}_L$ is empty. On a variety of complete curves, the characteristic numbers $N_{\alpha,\beta}$ are given as the intersection numbers $\tilde{\Pi}^\alpha \tilde{\Lambda}^\beta$ of α general point conditions and β general line conditions on B, with $\alpha + \beta = \frac{1}{2}d(d+3)$ ([**5**, Cor. I, p.505]).

Example. For $d = 2$, the blowup $B \to \mathcal{C}_2$ of the Veronese surface

$$V = \{\text{double lines}\} \subset \mathcal{C}_2 \cong \mathbf{P}^5$$

is a variety of complete conics.

In the case of cubic curves, one has the following result:

Theorem 1. (Aluffi [**5**, Theorem III, p.513], Sterz [**86**]) *A nonsingular variety $B = B_5$ of complete cubics is obtained by a sequence of 5 blowups $B_i \to B_{i-1}$, where $B_0 = \mathcal{C}_3 \cong \mathbf{P}^9$, with nonsingular centers. Moreover, the intersection ring of B can be computed.*

Aluffi was able to deduce the characteristic numbers for cubics from this. He also tried to determine the characteristic numbers for plane curves of higher degree, using a similar approach. By using one blowup, he obtained the following numbers for curves of degree d:

β	$\mathbf{N}_{\alpha,\beta}$
$0 \le s \le 2d-2$	$(2d-2)^s$
$2d-1$	$(2d-2)^{2d-1} - 2^{d-3}d(d-1)(d^2-d+2)$
$2d$	$(2d-2)^{2d} - 2^{d-4}d(d-1)(8d^4-21d^3+19d^2-20d+32)$

Non-singular curves of degree d
$(\alpha+\beta = \frac{1}{2}d(d+3))$

Note that the numbers for $\beta \le 2d-2$ are trivial, in the sense that $2d-2$ (general) line conditions intersect properly already on $\mathcal{C}_d$. The numbers for $\beta = 2d-1$ and $\beta = 2d$ agree with Zeuthen's for $d = 4$ and are new for $d \ge 5$.

However, it seems rather out of question to use the above method to determine all the other characteristic numbers for curves of degree d. Van Gastel ([**43**], [**42**]) has suggested another method, by which he too was able to obtain the above numbers. His method is based on the intersection algorithm due to Stückrad and Vogel, together with a study of limits of the conormal varieties of the curves (see also [**55**]) — this amounts to a study of 1-parameter families of *complete* curves. Unfortunately, his method does not so far give any more numbers than Aluffi's method.

Positive characteristic. Suppose the base field k has characteristic $p > 0$. Then the intersection numbers may count the characteristic numbers with a multiplicity (equal to a power of p) [**41**, Contact Theorem].

If $p \ne 2$ then the characteristic numbers for conics are the same as in the characteristic 0 case. The case $p = 2$ was treated by Vainsencher [**90**]. The only non-zero characteristic numbers are $N_{5,0} = 1$, $N_{4,1} = 1$, and $N_{3,2} = 1$. From this, he deduced that for $p = 2$ there are 51 conics tangent to 5 given conics.

In the case of cubics, the numbers are the same as in the characteristic 0 case as long as $p \ne 2$ and $p \ne 3$. In the case $p = 2$ most of the numbers have been determined by Berg [**16**], in particular he found those for the 8-dimensional family of cubics with j-invariant 0. (Note that a plane cubic curve has j-invariant 0 if and only if it has Hasse invariant 0, or again, if and only if it is projectively equivalent to a cubic defined by the "Fermat equation" $\sum X_i^3 = 0$.)

β	$\mathbf{N}_{\mathbf{8}-\beta,\beta}$
0	1
1	2
2	4
3	8
4	10
5	8
6	4
7	2
8	1

Non-singular cubics in characteristic 2 with j-invariant 0

The symmetry of this table reflects the fact that in characteristic 2 the dual of a cubic with j-invariant 0 is also a cubic with j-invariant 0. This is similar to the case of smooth conics and of cuspidal cubics in characteristic 0.

2. Curves in higher dimensional space

We have seen that in the case of plane curves, there are satisfactory results only for curves of genus 0 or 1. Hence one would not expect much enumerative geometry to be known for more general curves, i.e., curves in $\mathbf{P}^n$, for arbitrary n, of genus greater than 1.

Let $C \subset \mathbf{P}^n = \mathbf{P}^n_k$ be a curve in projective n-space, $n \geq 2$. Just as in the case of plane curves, one can consider various conditions for curves in a given "family": to pass through a given point, to meet a given line, to touch a given plane, to osculate a given linear space, etc. — and ask to enumerate the curves in the family that satisfy certain conditions.

As in the case of plane curves, the natural strategy would be to find a good parameter space for the family (or a modification of it), represent the conditions as cycles on this space, and do intersection theory — taking care that one counts the exact number of solutions to the original problem in this way.

The first obstacle one meets is how to find a parameter space. There are two natural candidates to look at, the *Chow variety* and the *Hilbert scheme.*

The Chow varieties parametrize curves of given degree d:

$$\mathrm{Chow}^d(\mathbf{P}^n) = \{C | C = \sum m_i C_i, m_i \geq 0, C_i \quad \text{reduced and}$$
$$\text{irreducible curve in } \mathbf{P}^n, \sum m_i \mathrm{deg} C_i = d\}.$$

In general, however, not much is known about this variety — e.g., when it is nonsingular, what its intersection ring is.

The other candidate is the Hilbert scheme

$$\mathrm{Hilb}^{d,g}(\mathbf{P}^n) = \{C | C \subset \mathbf{P}^n \quad \text{1-dimensional subscheme}$$
$$\text{of degree } d \text{ and arithmetic genus } g\}.$$

In general, not so much is known about this scheme either (but see [**47**]). The advantage of the Hilbert scheme as parameter space is that it comes equipped with a universal family:

$$\begin{array}{ccc} \mathcal{F}_{d,g} & \subset & \mathrm{Hilb}^{d,g}(\mathbf{P}^n) \times \mathbf{P}^n \\ & \searrow & \downarrow \\ & & \mathrm{Hilb}^{d,g}(\mathbf{P}^n) \end{array}$$

just as in the case of plane curves.

Example. For $n = 2$ we have

$$\mathrm{Chow}^d(\mathbf{P}^2) = \mathrm{Hilb}^{d,\frac{1}{2}(d-1)(d-2)}(\mathbf{P}^2) = \mathcal{C}_d \cong \mathbf{P}^{\frac{1}{2}d(d+3)}.$$

Twisted cubics. Consider the simplest example of non-plane curves, namely the *twisted cubics*. By definition, a twisted cubic is a nonsingular curve in $\mathbf{P}^3$ of degree 3 and genus 0. One can show that any such curve is projectively equivalent to the curve given as the image of the Veronese morphism

$$\mathbf{P}^1 \to \mathbf{P}^3$$

sending (u, v) to (u^3, u^2v, uv^2, v^3). The set of twisted cubics can therefore be identified with the quotient

$$T = PGL(4)/PGL(2),$$

which is a homogeneous space of dimension 12.

The enumerative theory of twisted cubics was considered already by Cremona [**34**] and Schubert [**82**].

As opposed to the case of plane curves, the parameter space for twisted cubics has no easily understood compactification. Since all questions in enumerative geometry translate into problems about computing the intersection ring of a parameter space (preferably, but not necessarily, compact), there has been a natural quest for understanding compactified parameter spaces and their intersection rings. Since 1981 this problem has again been attacked, eventually by many people ([**1**], [**36**], [**37**], [**72**], [**73**], [**74**], [**79**], [**91**]). In particular it was shown that the component H_3 of the Hilbert scheme containing the twisted cubics is nonsingular, hence gives a nice compactification of the homogeneous space T.

Theorem 2. (Piene–Schlessinger [**74**]) $\mathrm{Hilb}^{3,0}(\mathbf{P}^3)$ *consists of two irreducible components,* H_3 *and* H_3'*, both nonsingular and rational, of dimension* 12 *and* 15 *respectively. The intersection* $H_3 \cap H_3'$ *is nonsingular, of dimension* 11.

Note that a general point of H_3' corresponds to the union of a plane nonsingular cubic curve and a point in $\mathbf{P}^3$, whereas a general point of $H_3 \cap H_3'$ corresponds to a singular, plane cubic with an embedded point at the singularity, "sticking out of" the plane. It is furthermore known that H_3 is the blowup of a minimal nonsingular compactification of T in a nonsingular subvariety (isomorphic to the point–plane flag variety) [**36**], and that the Chow ring is computable ([**36**], [**37**], [**38**]).

Kleiman–Strømme–Xambó [**61**] were able to verify some of the characteristic numbers of twisted cubics found by Schubert in 1879, by using complete 1-parameter families contained in a locally closed subset of the Hilbert scheme.

Rational normal curves. More generally, consider *rational normal curves* of degree d, i.e., curves projectively equivalent to the image of the Veronese embedding of $\mathbf{P}^1$ in $\mathbf{P}^d$. Let

$$H_d \subset \mathrm{Hilb}^{d,0}(\mathbf{P}^d)$$

denote the irreducible component containing (as a dense, open subset) the points corresponding to the rational normal curves. Then, as in the case $d = 3$, one can compute

$$\dim H_d = \dim PGL(d+1) - \dim PGL(2) = (d-1)(d+3).$$

Unfortunately, the situation for $d \geq 4$ does not seem to be as nice as for $d \leq 3$, namely we have the following (see [**22**]):

Conjecture. *For* $d \geq 4$, H_d *is singular along a subvariety of dimension* $2(d-1)$.

The evidence for this conjecture is the following observation: Let $L \subset \mathbf{P}^d$ be a line, $\mathcal{I}_L$ its sheaf of ideals, and let $\tilde{L}$ be the subscheme defined by $\mathcal{I}_L^2$. Then $\tilde{L} \in H_d$ and

$$\dim T_{Hilb,\tilde{L}} = d(d-1)^2.$$

Moreover, $\tilde{L}$ is arithmetically Cohen–Macaulay, and Christophersen [**22**] proved that *every* arithmetically Cohen–Macaulay curve $C \in \mathrm{Hilb}^{d,0}(\mathbf{P}^d)$ specializes to some $\tilde{L}$. To prove the conjecture, it suffices to show that $\tilde{L}$ is not contained in any other component of $\mathrm{Hilb}^{d,0}(\mathbf{P}^d)$, or, equivalently, that all arithmetically Cohen–Macaulay curves are contained in H_d, or that the open subset of arithmetically Cohen–Macaulay curves is irreducible.

Another approach to finding a parameter space for rational (not necessarily normal) curves in a projective space, is to consider *parametrized curves*, i.e., maps from $\mathbf{P}^1$ to $\mathbf{P}^n$ ([**88**], [**93**], [**94**], [**75**]). In addition, there have also been made some attempts to find enumerative results for such curves without using parameter spaces at all ([**32**], [**33**]).

Concerning curves of genus 1, Avritzer and Vainsencher constructed a compactification of the space of elliptic quartic curves in $\mathbf{P}^3$ [**15**], by blowing up the Grassmann variety of pencils of quadrics twice, and they were able to determine formally the intersection ring of this space.

3. Rational curves on projective threefolds

Suppose now that we want to consider curves lying on a given 3-dimensional nonsingular variety (or *threefold*) $X \subset \mathbf{P}^n$ — in other words, suppose that one of the conditions we impose is "to lie on a given threefold". It turns out that this situation occurs naturally for rational curves, in at least the three following situations:

A) The study of codimension 2 cycles on varieties.
B) The classification of threefolds.
C) String theory in theoretical physics.

Codimension 2 cycles. Let $X \subset \mathbf{P}^n$ be a threefold, and set

$$H^{i,j} := H^j(X, \Omega^i_X).$$

Define the *Griffiths group* of X to be the group $G(X)$ of algebraic codimension 2 cycles on X homologous to 0, modulo those which are rationally equivalent to 0. There is a natural map, the *Abel–Jacobi map*,

$$\Phi : G(X) \to J(X) := \frac{(H^{3,0} + H^{2,1})^*}{H_3(X; \mathbf{Z})}$$

from this group to the intermediate Jacobian $J(X)$ ([**45**],[**23**], [**24**], [**25**], [**26**]).

Suppose $X \subset \mathbf{P}^4$ is a generic hypersurface of degree 5. Let $G_a(X) \subset G(X)$ denote the subgroup generated by cycles algebraically equivalent to 0. Griffiths [**45**, Cor.14.2, p.508] proved that $\Phi(G_a(X)) = 0$ and that $G(X)/G_a(X)$ has elements of infinite order. Later Clemens [**23**] proved that $\Phi(G(X)/G_a(X)) \otimes \mathbf{Q}$ is infinite dimensional.

The proof of the first result uses the existence of isolated *lines* on X, the proof of the second the existence of isolated *rational curves* of arbitrarily large degree on X — we shall return to the quintic threefold below.

A different proof of Clemens' theorem has been given by Voisin [**95**], using an infinitesimal approach. This method can probably also be applied to some other complete intersection threefolds with trivial canonical bundle (i.e., threefolds X such that $\Omega^3_X \cong \mathcal{O}_X$), as well as to the double cover of $\mathbf{P}^3$ ramified along an octic surface. Moreover, Voisin ([**96**], [**97**]) has generalized Griffiths' result by proving that if X is non-rigid and has trivial canonical bundle, then for a general deformation of X the image of the Abel–Jacobi map Φ is not contained in the torsion part of the intermediate Jacobian.

The differential of the Abel–Jacobi map is called the infinitesimal Abel–Jacobi map and has been studied as well ([**25**], [**26**], [**27**], [**28**]).

The differential of another map, the socalled *period map* ([**20**], [**45**], [**69**]), induces a map

$$H^1(X,T_X) \otimes H^1(X,\Omega^2_X) \to H^2(X,\Omega^1_X),$$

where $T_X = (\Omega^1_X)^*$ denotes the tangent sheaf. If X has trivial canonical sheaf, i.e., $\Omega^3_X \cong \mathcal{O}_X$, then $\Omega^2_X \cong T_X$ and $H^2(X,\Omega^1_X)$ is dual to $H^1(X,T_X)$, so that this map induces a cubic form

$$H^1(X,T_X) \otimes H^1(X,T_X) \otimes H^1(X,T_X) \to \mathbf{C},$$

to which we shall return when discussing string theory.

Classification of threefolds. One of the most important problems in algebraic geometry, is the problem of classification of varieties, or the search for general structure theorems. Curves and surfaces are well understood — for varieties of dimension 3 there are still a number of problems, though enormous progress has been made recently (see [**62**], [**30**]). In particular, there is a program — due essentially to Mori — for constructing a (unique) *minimal model* for each class of birationally equivalent threefolds. A threefold is called minimal if it has no subvarieties that can be contracted (even allowing the contracted variety to acquire singularities of certain kinds). A threefold which is *not* minimal contains rational curves, and it turns out that understanding rational curves on the variety gives a key to understanding the variety — the variety is "easier" to understand if it contains no rational curves, and the more rational curves it contains, the more complicated its birational geometry is. In this connection consider the following three situations involving the existence of rational curves on a variety. Here X is a threefold, and we let K_X denote a canonical divisor (i.e., a divisor corresponding to the invertible sheaf Ω_X^3).

a) Extremal contraction. If K_X is not *nef* (hence there exists a curve $C \subset X$ s.t. $K_X.C < 0$), then there exists a morphism $X \to Y$ contracting those rational curves on X that generate the "extremal rays" of the cone of positive curves on X (Mori, see [**30**], [**62**]).

b) Small resolution. A *node* (i.e., a quadratic singularity) $x \in X$ can be resolved by replacing x by a curve isomorphic to $\mathbf{P}^1$ (Atiyah [**14**]) — though the resolved variety need not be algebraic.

c) Flops. Suppose $C \subset X$, $C \cong \mathbf{P}^1$ is a rational curve on X which can be contracted to a point. Assume $K_X = 0$. Then X can be "flopped" along C: there exists a threefold X^+ containing a rational curve C^+ such that $X - C \cong X^+ - C^+$, $K_{X^+} = 0$, and X and X^+ are not (in general) even diffeomorphic (Reid, see [**30**], [**62**]).

In order to start a birational classification of threefolds, one considers the Kodaira dimension $\kappa(X)$, defined as one less than the transcendence degreee of the canonical ring $\oplus H^0(X, K_X^n)$ over the base field k. It is known that $\kappa(X) = -1$ if and only if X can be covered by rational curves (see [**63**] and [**31**]). If $\kappa(X) \geq 1$, one can use pluricanonical maps to obtain a "stable canonical variety" and use these for classification. In the remaining case $\kappa(X) = 0$, however, one can only classify those threefolds that have first Betti number $b_1(X) \neq 0$ — otherwise very little is known.

Definition. A Calabi–Yau threefold is a projective, non-singular variety X of dimension 3, with $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$.

The simplest examples of Calabi–Yau threefolds are the quintic hypersurfaces in $\mathbf{P}^4$. The only other complete intersections in projective space that are Calabi-Yau threefolds are: those of types (3,3) and (2,4) in $\mathbf{P}^5$, those of type (2,2,3) in $\mathbf{P}^6$, and those of type (2,2,2,2) in $\mathbf{P}^7$. Note that, for X a Calabi-Yau threefold, also $H^2(X, \mathcal{O}_X) = 0$ (by Serre duality) and $H^0(X, \Omega_X^1) = 0$ and $H^0(X, \Omega_X^2) = H^0(X, T_X) = 0$ (by Hodge duality).

An important problem is to describe a moduli space for Calabi–Yau threefolds. The 2-dimensional analogue to Calabi–Yau threefolds are the $K3$ surfaces. Inspired by the classification problem for these, Reid has the following speculation [**76**]: If X has enough contractible rational curves to span $H_2(X;\mathbf{Z})$, then X contracts to a (non-Kähler!) space with a "simple" structure — from these one can make a moduli space where the projective threefolds correspond to singular spaces.

String theory. In the string theory of theoretical physics, the compactification of superstrings leads to an effective field theory for which the "space–time manifold" is a product

$$\mathbf{R}^4 \times X,$$

where $\mathbf{R}^4$ is ordinary space–time and X is a *very* small "curled up" manifold of real dimension 6. Further constraints in the theory imply that X has the structure of a complex Kähler variety with $K_X = 0$, and that in fact only the Calabi–Yau threefolds give potentially interesting models (see [**50**] and references therein).

In this theory, the two families (a family and its "anti-family") of massless particles correspond to the elements of

$$H^{1,1} := H^1(X, \Omega^1_X) \quad \text{and } H^{2,1} := H^1(X, \Omega^2_X),$$

and they come equipped with "Yukawa couplings"

$$H^{1,1} \otimes H^{1,1} \otimes H^{1,1} \to \mathbf{C}$$

and

$$H^{2,1} \otimes H^{2,1} \otimes H^{2,1} \to \mathbf{C}.$$

We observe that, for a Calabi–Yau threefold, the tangent bundle $T_X := (\Omega^1_X)^*$ is isomorphic to Ω^2_X — in fact, the latter coupling is the differential of the period map encountered earlier in this section.

Finally, the theory also implies that the absolute value of the Euler number $\chi(X) = 2(h^{1,1} - h^{2,1})$ of X should be equal to twice the number of "generations" (groups of elementary particles). Since it is now widely agreed that there are only three generations, one would want the Euler number of X to be plus or minus 6. Thus the obvious problem becomes: Describe all Calabi–Yau threefolds with Euler number 6 or -6. The very first step was to show that such varieties do exist.

Example. (Tian–Yau [**89**]) Set

$$\tilde{X} = S_1 \times S_2 \cap H \subset \mathbf{P}^3 \times \mathbf{P}^3,$$

where $S_1 \subset \mathbf{P}^3$ is the Fermat cubic surface $\sum_{i=0}^{3} X_i^3 = 0$ in the first factor, similarly S_2 is given by $\sum_{i=0}^{3} Y_i^3 = 0$ in the second, and H is the hypersurface $\sum_{i=0}^{3} X_i Y_i = 0$. There is a free action of the group $\mathbf{Z}_3$ on $\tilde{X}$, and the quotient $X := \tilde{X}/\mathbf{Z}_3$ is a Calabi–Yau threefold with Euler number -6.

One can show, for example, that there are exactly 162 "lines" (i.e., curves of bidegree (1,0) or (0,1)) on $\tilde{X}$, 567 rational curves of bidegree (1,1), 81 of bidegree (2,0), 918 of bidegree (2,1), and none of bidegree (3,0) [**85**]. This gives at least 2727 isolated rational curves on $\tilde{X}$ — and hence X has at least $2727 : 3 = 909$ isolated rational curves. Whenever one or several of these curves are contractible, they can be "flopped", and one thus obtains other examples of Calabi–Yau threefolds of similar type [**89**].

Other examples of Calabi–Yau threefolds with $|\chi(X)| = 6$ have been given by Hirzebruch [**49**], Werner ([**98**],[**99**]), Schoen [**81**], Borcea [**17**], Schimmrigk [**80**], Candelas and Lynken [**18**].

The existence of rational curves on Calabi–Yau threefolds turns out to be interesting also in the string theory context. Witten [**103**] pointed out that the existence of rational curves on X obstructs the solutions to a certain differential equation. This made it interesting to look for Calabi–Yau threefolds with *no* rational curves — however, there are no known examples of such threefolds, and the general belief seems to be that they do not exist.

In fact, maps $\mathbf{P}^1 \to X$, for X as above, are interpreted as *instantons* (or instanton corrections) in the physical theory — and physicists would like there to be few of these. So again one inquires whether there are only finitely many, and — if so — how many. This leads to the following problem: Describe all rational curves on a given Calabi–Yau threefold — are they isolated (hence only finitely many of each type — if so, how many), what are their normal bundles, describe the families if there are infinitely many, etc. All known examples of Calabi–Yau threefolds contain rational curves, but the only general results one has is the following. Note that for a Calabi–Yau threefold X, the rank $\rho(X)$ of the Picard group $\mathrm{Pic}(X)$ is equal to $h^{1,1}(X)$, since $h^{2,0} = h^{0,2} = 0$.

Theorem 3. (Peternell [**71**]) *Let X be a Calabi–Yau threefold with Picard number $\rho(X)$. If $\rho(X) \geq 3$, then X contains rational curves.*

There are also other interesting results on the structure of Calabi–Yau threefolds by Wilson ([**100**], [**101**], [**102**]), and by Heath-Brown and Wilson:

Theorem 4. (Heath-Brown–Wilson [**48**]) *Let X be a Calabi–Yau threefold with Picard number $\rho(X)$. If $\rho(X) \geq 14$, then there exists a contraction $X \to Y$ such that $\rho(Y) < \rho(X)$ and the exceptional locus is covered by rational curves.*

Rational curves on the quintic threefold. The Calabi–Yau threefolds that have been studied the most regarding the question of existence of rational curves, are certainly the quintic hypersurfaces. In addition to these, only a few other special Calabi–Yau threefolds have been considered in this context ([**52**], [**84**], [**85**]).

Let $X \subset \mathbf{P}^4$ be a nonsingular quintic threefold. Then X has Picard number $\rho(X) = 1$ and Euler number

$$\chi(X) = 2(h^{1,1} - h^{2,1}) = 2(1 - 101) = -200.$$

Clemens' Conjecture. ([**25**], [**27**]) *A generic quintic threefold contains only finitely many rational curves of each degree d. Each rational curve is nonsingular, with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and they are mutually disjoint.*

The first part was proved by Katz [**53**] in the case $d \leq 7$. He also proved: For all d there exists a rational curve $C \subset X$ of degree d, with normal bundle $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

The evidence for the conjecture is given by counting dimensions: the space of maps from $\mathbf{P}^1$ to $\mathbf{P}^4$ of degree d, modulo automorphisms of $\mathbf{P}^1$, has dimension $5(d+1)-4 = 5d+1$, whereas the condition "to be contained in a given quintic hypersurface" has codimension $5d+1$.

The number 2875 of lines on a generic quintic threefold was found by Schubert ([**83**], [**46**], [**51**]), whereas the number 609, 250 of conics was found by Katz ([**53**], [**54**]).

To the surprise of many algebraic geometers, a group of physicists [**19**] were recently able to make a computation which predicted the number of rational curves on X of each degree (provided these numbers were finite). The ingredients in their computation are a *q-expansion principle* for functions on the moduli space of Calabi–Yau threefolds and a *mirror symmetry* for (at least some) Calabi–Yau threefolds. Their idea is the following (see [**10**], [**11**], [**12**], [**13**], [**44**], [**69**], [**70**], [**104**], [**105**]):

We start with the 1-dimensional family $\{X_\lambda\}$, where $X_\lambda \subset \mathbf{P}^4$ is given by the equation

$$\sum_{i=0}^{4} X_i^5 - 5\lambda X_0 \dots X_4 = 0.$$

Note that these quintic threefolds are not "generic" — for example, they contain infinitely many lines ([**2**], [**3**]). There is an action of the group $G := (\mathbf{Z}_5)^3$ on $\{X_\lambda\}$, and each quotient variety $\{X_\lambda\}/G$ admits a small resolution $\{Y_\lambda\}$, which is again a Calabi–Yau threefold. We call $\{Y_\lambda\}$ the *mirror family* of a (generic) quintic $X \subset \mathbf{P}^4$. The correspondence between X and Y is summarized in the following table.

$$\begin{array}{rr} \chi(Y) = 200 & \chi(X) = -200 \\ H^{2,1}(Y) & H^{1,1}(X) \\ H^{1,1}(Y) & H^{2,1}(X) \\ (H^{2,1}(Y))^{\otimes 3} \to \mathbf{C} & (H^{1,1}(X))^{\otimes 3} \to \mathbf{C} \end{array}$$

The coupling on Y is computable as a function on the given 1-dimensional moduli space:

$$f(q) = 5 + 2875q + 4876875q^2 + \dots$$

The coupling on X can be expressed in terms of instantons:

$$f(q) = a_0 + \sum_{i \geq 1} a_i i^3 q^i (1 - q^i)^{-1} = a_0 + a_1 q + (2^3 a_2 + a_1) q^2 + \dots$$

where a_i denotes the number of rational curves on X of degree i (if finite). Hence the predicted numbers are as follows:

$$\begin{aligned} a_0 &= \deg X = 5 \\ a_1 &= \#\text{lines on } X = 2875 \\ a_2 &= \#\text{conics on } X = 609,250 \\ a_3 &= \#\text{twisted cubics on } X = 317,206,375 \\ a_4 &= \#\text{twisted quartics on } X = 242,467,530,000 \\ & \textit{etc.} \end{aligned}$$

The predicted number $317,206,375$ of twisted cubics was recently verified by Ellingsrud and Strømme [**38**], using essentially the same methods as in the conics case, though things are much more complicated. The idea of their proof is as follows:

Consider the component

$$\mathcal{H}_d \subset \mathrm{Hilb}^{d,0}(\mathbf{P}^4)$$

containing (as a dense, open subset) the points corresponding to the rational curves of degree d. It has dimension $5d + 1$. Given a (generic) quintic threefold $X \subset \mathbf{P}^4$, the condition "to be contained in X" has codimension $5d + 1$ in $\mathcal{H}_d$; we want to compute the degree of this cycle. Let

$$\begin{array}{ccc} \mathcal{F}_d & \subset & \mathbf{P}^4 \times \mathcal{H}_d \\ & \overset{\phi}{\searrow} & \downarrow \\ & & \mathcal{H}_d \end{array}$$

denote the universal family and consider the exact sequence

$$0 \to \mathcal{I}_{\mathcal{F}_d} \to \mathcal{O}_{\mathbf{P}^4 \times \mathcal{H}_d} \to \mathcal{O}_{\mathcal{F}_d} \to 0.$$

Twisting with $\mathcal{O}_{\mathbf{P}^4}(5)$ and applying ϕ_* we obtain

$$0 \to \phi_* \mathcal{I}_{\mathcal{F}_d}(5) \to H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(5))_{\mathcal{H}_d} \stackrel{\nu}{\to} \phi_* \mathcal{O}_{\mathcal{F}_d}(5) \to R^1 \phi_* \mathcal{I}_{\mathcal{F}_d}(5) \to 0.$$

Let $F \in H^0(\mathbf{P}^4, \mathcal{O}_{\mathbf{P}^4}(5))$ correspond to $X \subset \mathbf{P}^4$. Then $\nu(F)$ is zero at those points C of $\mathcal{H}_d$ s.t. $F \in \mathcal{I}_{\mathcal{F}_d}(5)$, i.e., such that $C \subset X$. Moreover, if $C \in \mathcal{H}_d$, then

$$\mathrm{rk}\phi_* \mathcal{O}_{\mathcal{F}_d}(5) = \mathrm{dim} H^0(C, \mathcal{O}_C(5)) = 5d + 1 = \mathrm{dim}\mathcal{H}_d.$$

Therefore we conclude that if

$$R^1 \phi_* \mathcal{I}_{\mathcal{F}_d}(5) = 0$$

— in particular, if $d \leq 3$ — then the number of rational curves on a generic quintic $X \subset \mathbf{P}^4$ is finite and is given as the degree of the top Chern class $c_{5d+1}(\phi_* \mathcal{O}_{\mathcal{F}_d}(5))$. This degree can be computed "by hand" for $d = 1, 2$ and with the aid of a computer for $d = 3$. The computation in the last case relies on the knowledge of H_3, $\mathcal{H}_3$, and their intersection rings ([**74**], [**36**], [**37**], [**38**]).

For the moment, there seems to be no hope of using these methods to verify the numbers a_d for $d \geq 4$. As we have seen, even for $d = 4$ we don't know what the corresponding component of the Hilbert scheme is like.

Complete intersection Calabi–Yau threefolds. Morrison [**70**] has computed the q-expansion for the other Calabi–Yau threefolds that can be obtained as hypersurfaces of a (weighted) projective 4-space. It is interesting that he is able to check one special case of these computations with a formula due to Schubert [**83**]: There are 14,752 lines that are 4 times tangent to a general octic surface in $\mathbf{P}^3$.

Libgober and Teitelbaum [**64**] have looked at the cases of the remaining complete intersection Calabi–Yau threefolds in a projective space. Under suitable hypotheses, they give predictions for the number of rational curves of each degree on these varieties, which coincide with the known numbers in the case of lines [**52**].

4. Open problems

To sum up, we see that there are still many interesting open problems in the theory of enumeration of algebraic curves — here are some that we have discussed:

Problem 1. Determine the characteristic numbers for plane curves of degree $d \geq 4$.

Problem 2. Determine the characteristic numbers for cubic plane curves in characteristic 2 and 3 — and, more generally, for curves of degree d in positive characteristic $p \leq d$.

Problem 3. Describe the Hilbert scheme of rational normal curves of degree d in $\mathbf{P}^d$ and determine its intersection ring — and similarly for rational curves of degree d in $\mathbf{P}^n$, for $n \leq d$.

Problem 4. Prove the finiteness of rational curves of each degree d on a generic quintic threefold. More generally, find the number of rational curves of given "type" — if finite — on a given (general) Calabi–Yau threefold.

Problem 5. Prove that every Calabi–Yau threefold X contains a rational curve.

By Peternell's theorem quoted above, the last problem is solved for X with Picard number $\rho(X) \geq 3$, and, moreover, it is true for all *known* Calabi–Yau threefolds. As observed by Peternell [**71**], a solution of Problem 5 would prove Kobayashi's conjecture for hyperbolic threefolds: A complex manifold X is called *hyperbolic* if every holomorphic map $\mathbf{C} \to X$ is constant. Kobayashi conjectured that every hyperbolic manifold has ample canonical sheaf.

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