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by

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Introduction.

Let $\mathcal{A}$ be a C*-subalgebra of $\mathcal{B}(\mathcal{H})$, the bounded linear operators on some non-zero Hilbert space $\mathcal{H}$. By a *hypertrace* on $\mathcal{A}$ we mean a state ϕ on $\mathcal{B}(\mathcal{H})$ which contains $\mathcal{A}$ in its centralizer, i.e. $\mathcal{A} \subseteq C_\phi$ where C_ϕ is the C*-algebra given by

$$C_\phi = \{Y \in \mathcal{B}(\mathcal{H}) | \phi(XY) = \phi(YX) \quad \text{for all} \quad X \in \mathcal{B}(\mathcal{H})\}.$$

This concept was introduced by Connes in his fundamental paper on the classification of injective factors ([C1]) as an important tool in establishing that any injective II_1-factor $\mathcal{N}$ acting on a separable Hilbert space is $*$-isomorphic to the hyperfinite II_1-factor, the analogy between the existence of a hypertrace on $\mathcal{N}$ and the existence of an invariant mean on $l^\infty(G)$ (where G denotes a discrete group), i.e. the amenability of G being nicely exposed in [C2]. The key observation was that from the existence of a hypertrace on $\mathcal{N}$ one may deduce that $\mathcal{N}$ satisfies a property analogous to Følner's characterization of the amenability of a group. The reader may also consult [Po1] or [Po2] to see how the hyperfiniteness of $\mathcal{N}$ may be obtained from this Følner property.

The concept of hypertrace for C*-algebras has also been of some interest, though in a somewhat unexplicit form. For example, Connes uses it to rule out the existence of finitely summable unbounded Fredholm modules on some C*-algebras in [C4]. Further, an operator $T \in \mathcal{B}(\mathcal{H})$ is finite in the sense of Williams if there exists a hypertrace on the C*-algebra generated by T (cf. [Wi; Th.4]), while a unitary representation U of a locally compact group G is amenable in the sense of Bekka ([Bek]) it there exists a hypertrace on the C*-algebra generated by $U(G)$. Finally, it has also been useful in the recent work of Kirchberg ([Ki1],[Ki2]).

In the first section of these notes, we point out how [Bek] may be used to characterize the existence of a "Følner net" for $\mathcal{A}$ of non-zero finite dimensional projections in $\mathcal{B}(\mathcal{H})$. Then we explain how the notion of a Følner net for $\mathcal{A}$ is related to [A1] and [Bed 2] where the problem of approximating the spectrum of some self-adjoint operators in $\mathcal{B}(\mathcal{H})$

is considered.

In the second section, we explore in some details the following notion, inspired by [Bek]: if $\mathcal{A}$ is a C*-algebra and π is a non-degenerate representation of $\mathcal{A}$, we say that π is hypertracial if $\pi(\mathcal{A})$ has a hypertrace, and we say that $\mathcal{A}$ is weakly hypertracial if all faithful non-degenerate representations of $\mathcal{A}$ are hypertracial (it is in fact enough that $\mathcal{A}$ has one hypertracial non-degenerate representation). We show that the class of weakly hypertracial C*-algebras is quite large and includes many familiar examples. Also, it behaves nicely with respect to crossed products by amenable discrete groups and tensor products.

In many aspects, the class of weakly hypertracial C*-algebras is too large and in the third and final section, we study the smaller class of hypertracial C*-algebras: a C*-algebra is hypertracial if all its non-degenerate representations are hypertracial. This class contains all strongly amenable C*-algebras (as defined in [J]) and many nuclear C*-algebras satisfying some finiteness condition. In many cases hypertraciality is quite easy to establish compared to nuclearity. An interesting open question is the following: is a separable unital hypertracial C*-algebra necessarily nuclear? (the answer being no for non-separable C*-algebras).

As general references on operator algebras, we refer to [D], [M], [Pe], [S] and [Ta]. Our notation shall mainly be as in [M]. Especially if $\mathcal{A}$ is a C*-algebra, then $\tilde{\mathcal{A}}$ denotes its unitization (defined even if A is unital). If $\mathcal{A}$ is unital, then $\mathcal{U}(\mathcal{A})$ denotes its unitary group.

Concerning amenability of groups, the reader should consult [Pa] or [Pi]. Nuclearity and amenability of C*-algebras are equivalent notions ([C3], [H1]). For an overview, see [Pa; 1.31] and also [L]. A review of injectivity of von Neumann algebras and its equivalent formulations is given in [Pa; 2.35].

Although not essential in this paper which deals mostly with C*-algebras, we conclude

this introduction by gathering for completeness some results on hypertraces and finite von Neumann algebras.

Let $\mathcal{N}$ denote a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. If $\mathcal{N}$ is injective, i.e. there exists a conditional expectation $E : \mathcal{B}(\mathcal{H}) \to \mathcal{N}$, then $\tau \circ E$ is a hypertrace on $\mathcal{N}$ for any tracial state τ on $\mathcal{N}$. Conversely, if $\mathcal{N}$ is countably decomposable and $\mathcal{N}$ has a hypertrace ϕ such that $\phi|_{\mathcal{N}}$ is faithful, then $\mathcal{N}$ is injective ([C2], [S; 10.27]). Further, if $\mathcal{N}$ has a separable predual and $\mathcal{N}$ has a hypertrace, then $\mathcal{N}$ is approximately finite dimensional (this follows from [Po1] or [Po2] if one uses the fact that the existence of a hypertrace on $\mathcal{N}$ is independent of the Hilbert space on which $\mathcal{N}$ acts, cf. Theorem 2.1, so that one may assume that $\mathcal{H} = L^2(\mathcal{N}, \tau)$ where τ is a faithful normal trace on $\mathcal{N}$ as in [Po1] or [Po2]). Finally, $\mathcal{N}$ is of course injective whenever $\mathcal{N}$ is approximately finite dimensional.

1. Hypertraces and Følner nets.

In this section, $\mathcal{A}$ denotes a C*-algebra of bounded operators acting on a Hilbert space $\mathcal{H}$. We assume that $\mathcal{A}$ contains the identity operator $I = I_{\mathcal{H}}$ on $\mathcal{H}$ (if $I \notin \mathcal{A}$, one may just replace $\mathcal{A}$ with $\mathcal{A} + \mathbf{C} \cdot I$). If T is an operator on $\mathcal{H}$, $\|T\|_1$ (resp. $\|T\|_2$) denotes its trace-class norm (resp. its Hilbert-Schmidt norm).

Based on the work of Bekka ([Bek]) (and therefore implicitely on [C1]), the existence of a hypertrace on $\mathcal{A}$ may be characterized by properties analogous to Reiter's and Følner's characterizations of the amenability of a group:

Theorem 1.1: Consider the following conditions on $\mathcal{A}$, where $p = 1, 2$.

(H) There exists a hypertrace on $\mathcal{A}$.

(P_p) There exists a net $\{S_\alpha\}$ of operators on $\mathcal{H}$ such that $\|S\|_p = 1$, $S \geq 0$ and

$$\lim_\alpha \|S_\alpha A - A S_\alpha\|_p = 0 \quad \text{for all} \quad A \in \mathcal{A}.$$

(F_p) There exists a net $\{Q_\alpha\}$ of non-zero finite dimensional projections in $\mathcal{B}(\mathcal{H})$ such that

$$\lim_\alpha \frac{\|Q_\alpha A - AQ_\alpha\|_p}{\|Q_\alpha\|_p} = 0 \quad \text{for all} \quad A \in \mathcal{A}.$$

Then conditions $(H), (P_1), (P_2), (F_1), (F_2)$ are all equivalent. If $\mathcal{A}$ is separable, then the nets $\{S_\alpha\}$ and $\{Q_\alpha\}$ may be chosen as sequences.

Proof: The equivalences $(P_1) \Leftrightarrow (P_2)$ and $(F_1) \Leftrightarrow (F_2)$ are consequences of the Powers-Størmer inequalities (just as in [Bek], [C1] and [C2]), while $(F_p) \Rightarrow (P_p)$ and $(P_1) \Rightarrow (H)$ are quite obvious.

Now set $G = \mathcal{U}(\mathcal{A})$, considered as a discrete group. Since $G \subseteq \mathcal{U}(\mathcal{B}(\mathcal{H}))$, the identity representation of $\mathcal{A}$ on $\mathcal{H}$ induces a unitary representation of G on $\mathcal{H}$ which we denote by i. As every element of $\mathcal{A}$ is a linear combination of elements in G, it is clear that (H) is equivalent to the fact that i is an amenable representation of G in the sense of Bekka. Therefore, by [Bek; Th. 6.2], (H) implies that given $\varepsilon > 0$ and $U_1, U_2, \cdots, U_n \in G$, there exists a non-zero finite dimensional projection Q in $\mathcal{B}(\mathcal{H})$ such that

$$\|U_j Q U_j^* - Q\|_1 = \|i(U_j)Qi(U_j^{-1}) - Q\|_1 < \varepsilon \|Q\|_1,$$

$j = 1, 2, \cdots, n$, from which (F_1) easily follows and also that $\{Q_\alpha\}$ and $\{P_\alpha\}$ may be chosen as sequences if $\mathcal{A}$ is separable.

□

Remarks 1.2.

1) By invoking the Jordan decomposition, we have $(H) \Leftrightarrow$ there exists a non-zero bounded linear functional ψ on $\mathcal{B}(\mathcal{H})$ such that $\psi(AX) = \psi(XA)$ for all $A \in \mathcal{A}, X \in \mathcal{B}(\mathcal{H})$ (cf. [Bu2; Prop. 5]).

2) By taking into account the generalized Powers-Størmer inequalities ([K]), one may show that $(H) \Leftrightarrow (P_p) \Leftrightarrow (F_p)$ for any $p \geq 1$.

3) If $\{\mathcal{H}, D\}$ is a finitely summable unbounded Fredholm module over $\mathcal{A}$, Connes shows in [C4; Lem. 9] the existence of a net satisfying (P_1).

4) In the case when $\mathcal{A}$ is separable and $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$, Kirchberg shows in [Ki2; Proposition 3.2] that the existence of a hypertrace on $\mathcal{A}$ is equivalent to the fact that $\mathcal{A}$ has a "liftable" tracial state.

A net $\{Q_\alpha\}$ satisfying (F_1) shall be called *a Følner net for* $\mathcal{A}$.

Inspired by Arveson's paper [A1] on C*-algebras and numerical linear algebra, we introduced in [Bed2] the following notion:

Let $\mathcal{F} = \{\mathcal{H}_n\}_{n\leq 1}$ be a filtration of $\mathcal{H}$, i.e. each $\mathcal{H}_n$ is a non-zero finite dimensional subspace of $\mathcal{H}$, $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$ and $\mathcal{H} = \overline{\bigcup_{n\leq 1} \mathcal{H}_n}$. Then $\mathcal{F}$ is called a weak $\mathcal{A}$-filtration if

$$(*) \qquad \lim_{n\to+\infty} \frac{\|P_n A - A P_n\|_1}{\|P_n\|_1} = 0 \qquad \text{for all } A \in \mathcal{A}$$

where P_n denotes the projection of $\mathcal{H}$ onto $\mathcal{H}_n$. Our motivation was that we could give an extended version of [A1; Th. 4.5], cf. [Bed2; Th. 1], to the case when $\mathcal{A}$ has a unique tracial state and has a weak $\mathcal{A}$-filtration, and that for some interesting examples of $\mathcal{A}$'s with a unique tracial state there exists a natural weak $\mathcal{A}$-filtration, thus making it possible to put some numerical approximations of the spectrum of self-adjoint operators in $\mathcal{A}$ on a somewhat firm ground.

However, an inspection of the proof of [Bed 2; Th. 1] makes it clear that the projections P_n in fact don't have to be associated to a filtration of $\mathcal{H}$: it suffices that they satisfy condition $(*)$ for the proof to go through. Further, the proof is easily adapted to handle nets instead of sequences of projections. This means that the following result holds:

Theorem 1.3: Suppose that $\mathcal{A}$ has a hypertrace φ such that $\varphi|\mathcal{A}$ is the only tracial state on $\mathcal{A}$. Let $\{Q_\alpha\}$ be a Følner net for $\mathcal{A}$ (such a net exists by Theorem 1.1). Then we have

$$\lim_\alpha \frac{1}{d_\alpha}\Big[f(\lambda_{1,\alpha}) + \cdots + f(\lambda_{d_\alpha,\alpha})\Big] = \varphi(f(A)) = \int_{\mathbf{R}} f(t) d\mu_A(t)$$

for all $f \in C_0(\mathbf{R})$, where $d_\alpha = \dim Q_\alpha$, A is a self-adjoint operator in $\mathcal{A}$, $\{\lambda_{1,\alpha}, \ldots, \lambda_{d_\alpha,\alpha}\}$ is the list of eigenvalues (repeated according to multiplicity) of $Q_\alpha A|_{Q_\alpha \mathcal{H}}$ and μ_A is the spectral measure of A associated with φ. If $\varphi|\mathcal{A}$ is faithful, then $\mathrm{sp}(A) = \mathrm{supp}(\mu_A)$.

Remarks 1.4:

1) In a concrete situation where one wants to approximate numerically the spectrum of a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$, the problem is of course to find if possible an appropriate $\mathcal{A}$ containing A and an explicit sequence $\{Q_n\}$ satisfying the above assumptions.

2) If $\mathcal{A}$ has a faithful unique tracial state and $\mathcal{A}$ is infinite dimensional, one gets easily from [A2: Prop. 2.2] that the spectrum of a self-adjoint operator in $\mathcal{A}$ coincides with its essential spectrum.

3) If $\mathcal{A} = \mathcal{B}(\mathbf{C}^n)$ for some $n \geq 1$, then any net $\{Q_\alpha\}$ of non-zero projections in $\mathcal{B}(\mathbf{C}^n)$ is a Følner net for $\mathcal{A}$ if and only if $Q_\alpha = I$ for all $\alpha \geq \alpha_0$ (for some α_0) and $\mathcal{A}$ has only one hypertrace given by the normalized trace on $\mathcal{B}(\mathbf{C}^n)$. The statement in Theorem 1.3 is then just an obvious corollary of the spectral theorem.

4) Another quite trivial example, but more instructive, is the following: Suppose now $\mathcal{H}$ is infinite dimensional and let $\mathcal{A} = \mathcal{K}(\mathcal{H}) + \mathbf{C} \cdot I$, where $\mathcal{K}(\mathcal{H})$ denotes the C*-algebra of compact operators on $\mathcal{H}$. Then $\mathcal{A}$ has a hypertrace. Indeed, a state φ on $\mathcal{B}(\mathcal{H})$ is a hypertrace on $\mathcal{A}$ if and only if φ is singular (since φ is singular $\Leftrightarrow$ $\varphi|\mathcal{K}(\mathcal{H}) = 0$, cf. [Ta]). On the other hand, let $\{Q_\alpha\}$ be any increasing net of non-zero finite dimensional projections in $\mathcal{B}(\mathcal{H})$ converging strongly to I. Then $\{Q_\alpha\}$ is a Følner net for $\mathcal{A}$, since

$$\begin{aligned} \frac{\|Q_\alpha A - A Q_\alpha\|_1}{\|Q_\alpha\|_1} &\leq \frac{\mathrm{rank}(Q_\alpha A - A Q_\alpha)\|Q_\alpha A - A Q_\alpha\|}{\|Q_\alpha\|_1} \\ &\leq 2\|Q_\alpha A - A Q_\alpha\| \to 0 \end{aligned}$$

for all $A \in \mathcal{K}(\mathcal{H})$.

Now, $\mathcal{A}$ has clearly a unique tracial state τ, so Theorem 1.3 applies. Let A be a

self-adjoint compact operator on $\mathcal{H}$. For any $f \in C_0(\mathbf{R})$ we have $\tau(f(A)) = f(0)$, hence $\mu_A = \delta_0$ (the Dirac measure at 0). If $\{\xi_j\}_{j\in J}$ is an orthornormal basis for $\mathcal{H}$ consisting of eigenvectors of A with associated eigenvalues $\{\lambda_j\}$, by choosing Q_F as the projection onto the linear span of $\{\xi_j\}_{j\in F}$ when F is a non-empty finite subset of J, we obtain

$$\lim_F \frac{1}{\#(F)} \sum_{j\in F} f(\lambda_j) = f(0) \qquad \text{for all } f \in C_0(\mathbf{R}).$$

In the spirit of [A1], this might be seen as an indication of the well-known fact that the essential spectrum of A reduces to $\{0\}$.

Let now G denote a locally compact group, λ its left regular representation acting on $L^2(G)$, $C^*(\lambda(G))$ the C*-algebra generated by $\lambda(G)$ (so that $C^*(\lambda(G)) = C^*_r(G)$ when G is discrete) and $vN(G) = \lambda(G)''$, the group von Neumann algebra of G. The following result is essentially well-known:

Proposition 1.5: Consider the four conditions

(1) G is amenable

(2) $C^*(U(G))$ has a hypertrace for all continuous unitary representations U of G

(3) $C^*(\lambda(G))$ has a hypertrace

(4) $vN(G)$ has a hypertrace.

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftarrow$ (4). If G is discrete, all four conditions are equivalent.

Proof: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is merely a rephrasing of [Bek; Th. 2.2], while (4) $\Rightarrow$ (3) is trivial. If G is discrete, then $vN(G)$ is a finite von Neumann algebra which is injective whenever G is amenable ([C1; Prop. 6.8]), hence (1) $\Rightarrow$ (4) follows. □

We also mention a C*-algebraic analog to [C1; Prop. 6.8]. Let G denote a right amenable discrete semigroup, i.e. there exists a state M on $\ell^\infty(G)$ such that

$$M(f_h) = M(f) \qquad \text{for all } f \in \ell^\infty(G),\ h \in G,$$

where $f_h(g) = f(gh)$, $g \in G$.

We shall use the suggestive notation $\int_G f(g)dM(g)$ for $M(f)$.

Suppose there exists a map $U : G \to \mathcal{U}(\mathcal{B}(\mathcal{H}))$ satisfying $U_g \mathcal{A} U_g^* \subseteq \mathcal{A}$ and $U_g U_h U_{gh}^* \in \mathcal{U}(\mathcal{A})$ for all $g, h \in G$. Let $C^*(\mathcal{A}, U(G))$ denote the C*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{A}$ and $U(G)$. Then we have:

Proposition 1.6: If there exists a hypertrace φ on $\mathcal{A}$ and G is right-amenable, then there exists a hypertrace on $C^*(\mathcal{A}, U(G))$, also.

Proof: For each $X \in \mathcal{B}(\mathcal{H})$, define

$$\tilde{\varphi}(X) = \int_G \varphi(U_g X U_g^*) dM(g)\,,$$

which is well-defined since

$$|\varphi(U_g X U *_g)| \leq \|\varphi\| \, \|U_g X U_g^*\| \leq \|X\| \qquad \text{for all } g \in G\,.$$

One checks easily that $\tilde{\varphi}$ is a state on $\mathcal{B}(\mathcal{H})$. Set $V(g,h) = U_g U_h U_{gh}^* \in \mathcal{U}(\mathcal{A})$, $g, h \in G$. For $X \in \mathcal{B}(\mathcal{H})$, $h \in G$, $A \in \mathcal{A}$, we have

$$\begin{aligned}
\tilde{\varphi}(U_h X U_h^*) &= \int_G \varphi(U_g U_h X U_h^* U_g^*) dM(g) \\
&= \int_G \varphi(V(g,h) U_{gh} X U_h^* U_g^*) dM(g) \\
&= \int_G \varphi(U_{gh} X U_h^* U_g^* V(g,h)) dM(g) \quad \text{(since } \varphi \text{ is a hypertrace on } \mathcal{A}) \\
&= \int_G \varphi(U_{gh} X U_{gh}^*) dM(g) \\
&= \int_G \varphi(U_g X U_g^*) dM(g) \quad \text{(since } M \text{ is right-invariant)} \\
&= \tilde{\varphi}(X)
\end{aligned}$$

and

$$\begin{aligned}\widetilde{\varphi}(AX) &= \int_G \varphi(U_g A X U_g^*) dM(g) \\ &= \int_G \varphi(U_g A U_g^* U_g X U_g^*) dM(g) \\ &= \int_G \varphi(U_g X U_g^* U_g A U_g^*) dM(g) \\ &= \int_G \varphi(U_g X A U_g^*) dM(g) \\ &= \widetilde{\varphi}(XA)\end{aligned}$$

Hence it follows that $C^*(\mathcal{A}, U(G))$ is contained in the centralizer of $\widetilde{\varphi}$ and the result follows.

□

Of course, one may also state a corresponding "left"-version of this proposition. If G is a group, right- and left-amenability are equivalent notions; this is not generally true in the semigroup case (cf. [Pa]). The semi-group case might be of interest when dealing with twisted crossed products by semigroups of automorphisms, but we shall only consider the group case in the next sections. We just mention here:

Corollary 1.7: Let G denote a discrete group and $Z^2(G,\mathbf{T})$ the set of (normalized) 2-cocycles of G with values in the circle group $\mathbf{T}$. If $u \in Z^2(G,\mathbf{T})$, we denote by λ_u the associated projective (left or right) regular representation of G on $l^2(G)$ and we set $C_r^*(G,u) = C^*(\lambda_u(G)), vN(G,u) = \lambda_u(G)''$. Then the following four conditions are equivalent:

(1) G is amenable

(2) $C^*(U(G))$ has a hypertrace for all projective unitary representations of G

(3) $C_r^*(G,u)$ has a hypertrace for some $u \in Z^2(G,\mathbf{T})$

(4) $vN(G,u)$ has a hypertrace for some $u \in Z^2(G,\mathbf{T})$.

Proof: (1)⇒(2) follows from Proposition 1.6 by taking $\mathcal{A} = \mathbf{C}$. (2)⇒(3) and (4)⇒(3) are trivial. (1)⇒(4) follows easily for example from [AD; Proposition 3.12]. Finally, (3)⇒(1) follows in the "classical" way: if ϕ is a hypertrace on $C_r^*(G,u) \subseteq \mathcal{B}(l^2(G))$, then a (left

or right) invariant mean M on $l^\infty(G)$ is given by $M(f) = \phi(m_f)$, where m_f denotes the multiplication operator by $f \in l^\infty(G)$ on $l^2(G)$. An easy computation shows indeed that if λ_u is for example the right regular representation, then one has $\lambda_u(h) m_f \lambda_u(h)^* = m_{f_h}$ for all $h \in G, f \in l^\infty(G)$ as usual. □

Finally, there is another situation where the existence of hypertraces is easily shown:

Proposition 1.8: Suppose $\mathcal{A}$ has a non-zero multiplicative linear functional ψ (e.g. $\mathcal{A}$ is abelian). Then $\mathcal{A}$ has a hypertrace.

Proof: Since ψ is a state on $\mathcal{A}$, we may extend it to a state on $\mathcal{B}(\mathcal{H})$ which is easily seen to be a hypertrace on $\mathcal{A}$ by making use of the Cauchy-Schwartz inequality for states. □

2. Weakly hypertracial C*-algebras

Let $\mathcal{A}$ be a C*-algebra. We denote by $\mathrm{Rep}(\mathcal{A})$ the set of all non-degenerate $*$-representations of of $\mathcal{A}$ on non-zero Hilbert spaces and if $\pi \in \mathrm{Rep}(\mathcal{A})$, we denote by $\mathcal{H}_\pi$ the associated Hilbert space.

We say that $\pi \in \mathrm{Rep}(\mathcal{A})$ is *hypertracial* if there exists a hypertrace on $\pi(\mathcal{A})$, i.e. there exists a state φ on $\mathcal{B}(\mathcal{H}_\pi)$ such that $\pi(\mathcal{A}) \subseteq C_\varphi$.

Recall that when $\rho, \pi \in \mathrm{Rep}(\mathcal{A})$, one says that ρ is weakly contained in π (resp. is weakly equivalent to π) whenever $\ker \pi \subseteq \ker \rho$ (resp. $\ker \pi = \ker \rho$). The following result is inspired by [Bu2; Proposition 7]:

Theorem 2.1: Suppose $\rho, \pi \in \mathrm{Rep}(\mathcal{A})$, ρ is weakly contained in π and ρ is hypertracial. Then π is hypertracial.

Proof: Set $\mathcal{B} = \pi(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H}_\pi)$. Since $\ker \pi \subseteq \ker \rho$ we may define $\psi : \mathcal{B} \to \mathcal{B}(\mathcal{H}_\rho)$ by

$$\psi(\pi(A)) = \rho(A), \qquad A \in \mathcal{A}.$$

It is clear that $\psi \in \mathrm{Rep}(\mathcal{B})$ with $\mathcal{H}_\psi = \mathcal{H}_\rho$. By [M; Th. 5.5.1], we may extend ψ to a non-degenerate $*$-representation $\tilde{\psi} : \mathcal{B}(\mathcal{H}_\pi) \to \mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is a Hilbert space such that

there exists a closed subspace $\mathcal{H}'_\rho$ of $\mathcal{K}$ which is invariant for $\tilde{\psi}(\mathcal{B})$ and a unitary operator U from $\mathcal{H}_\rho$ onto $\mathcal{H}'_\rho$ satisfying

$$\tilde{\psi}(B)|_{\mathcal{H}'_\rho} = U\psi(B)U^* \qquad \text{for all } B \in \mathcal{B}\,,$$

i.e. $\tilde{\psi}(\pi(A))|_{\mathcal{H}'_\rho} = U\rho(A)U^*$ for all $A \in \mathcal{A}$.

We let P denote the projection of $\mathcal{K}$ onto $\mathcal{H}'_\rho$. Since ρ is hypertracial, there exists a state φ on $\mathcal{B}(\mathcal{H}_\rho)$ such that

$$\varphi(\rho(A)X) = \varphi(X\rho(A)) \qquad \text{for all } A \in \mathcal{A},\ X \in \mathcal{B}(\mathcal{H}_\rho).$$

Define ω on $\mathcal{B}(\mathcal{H}_\pi)$ by

$$\omega(Y) = \varphi(U^*P\tilde{\psi}(Y)|_{\mathcal{H}'_\rho}U), \qquad Y \in \mathcal{B}(\mathcal{H}_\pi).$$

Then ω is a state on $\mathcal{B}(\mathcal{H}_\pi)$ (since $\|\omega\| = \omega(I) = 1$ is easily verified).

Further, for $A \in \mathcal{A}$, $Y \in \mathcal{B}(\mathcal{H}_\pi)$, we have

$$\begin{aligned}
\omega(Y\pi(A)) &= \varphi[U^*P\tilde{\psi}(Y\pi(A))|_{\mathcal{H}'_\rho}U] \\
&= \varphi[U^*P\tilde{\psi}(Y)|_{\mathcal{H}'_\rho}UU^*\tilde{\psi}(\pi(A))|_{\mathcal{H}'_\rho}U] \\
&= \varphi[U^*P\tilde{\psi}(Y)|_{\mathcal{H}'_\rho}U\rho(A)] \\
&= \varphi[\rho(A)U^*P\tilde{\psi}(Y)|_{\mathcal{H}'_\rho}U] \\
&= \varphi[U^*\tilde{\psi}(\pi(A))|_{\mathcal{H}'_\rho}P\tilde{\psi}(Y)|_{\mathcal{H}'_\rho}U] \\
&= \varphi[U^*P\tilde{\psi}(\pi(A))\tilde{\psi}(Y)|_{\mathcal{H}'_\rho}U] \\
&= \omega(\pi(A)Y)\,.
\end{aligned}$$

Hence ω is a hypertrace on $\pi(\mathcal{A})$, i.e. π is hypertracial. □

Corollary 2.2: Hypertraciality (in $\mathrm{Rep}(\mathcal{A})$) is invariant under weak equivalence.

Corollary 2.3: The following conditions are equivalent.

(1) There exists $\rho \in \mathrm{Rep}(\mathcal{A})$ which is hypertracial

(2) There exists a faithful $\pi \in \mathrm{Rep}(\mathcal{A})$ which is hypertracial

(3) All faithful elements in $\mathrm{Rep}(\mathcal{A})$ are hypertracial.

We say that $\mathcal{A}$ is *weakly hypertracial* if $\mathcal{A}$ satisfies one of the conditions in Corollary 2.3. This property is clearly invariant under $*$-isomorphism.

Remark 2.4: The following facts are easily deduced:

1) Any non-zero finite dimensional C*-algebra is weakly hypertracial.

2) If J is a proper ideal in $\mathcal{A}$ and $\mathcal{A}/J$ is weakly hypertracial, then $\mathcal{A}$ is weakly hypertracial.

3) Any C*-algebra possessing a non-zero finite dimensional representation is weakly hypertracial. Especially, any non-zero abelian C*-algebra is weakly hypertracial (cf. also Prop. 1.8).

4) Any quasidiagonal C*-algebra (cf. [Wa 2]) is weakly hypertracial.

5) Any injective finite von Neumann algebra is weakly hypertracial.

Before giving more examples of weakly hypertracial C*-algebras, we point out that this notion is of interest only for unital C*-algebras.

Proposition 2.5: Any non-zero non-unital C*-algebra $\mathcal{B}$ is weakly hypertracial.

Proof: Let $\pi \in \mathrm{Rep}(\mathcal{B})$ be faithful. Then $I = I_{\mathcal{H}_\pi} \notin \pi(\mathcal{B})$, hence $\pi(\mathcal{B}) + \mathbf{C} \cdot I \simeq \widetilde{\pi(\mathcal{B})}$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H}_\pi)$ which has a hypertrace φ induced by the obvious non-zero multiplicative linear functional on $\pi(\mathcal{B}) + \mathbf{C} \cdot I$ (cf. Proposition 1.8). This implies that π is hypertracial, so $\mathcal{B}$ is weakly hypertracial. □

In the notation of the above proof, we have $\varphi|\pi(\mathcal{B}) = 0$. This cannot happen with unital C*-algebras: if $\mathcal{A}$ is unital, $\pi \in \mathrm{Rep}(\mathcal{A})$ and φ is a hypertrace on $\pi(\mathcal{A})$, then $\tau = \varphi|\pi(\mathcal{A})$ is a tracial state on $\pi(\mathcal{A})$ (since $I_{\mathcal{H}_\pi} = \pi(1) \in \pi(\mathcal{A})$ and $\tau(I_{\mathcal{H}_\pi}) = 1$). The next proposition follows readily.

Proposition 2.6: If $\mathcal{A}$ is a unital weakly hypertracial C*-algebra, then $\mathcal{A}$ has at least

one tracial state.

Thus any unital C*-algebra without any tracial state, such as the Cuntz algebras $\mathcal{O}_n$ ($n \geq 2$) or the type III-factors of countable type or $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ infinitely dimensional, are not weakly hypertracial. We shall soon see that the converse of proposition 2.6 is not true.

Proposition 2.7: The following C*-algebras are weakly hypertracial.

1) All AF-algebras.

2) All inductive limits of postliminal (= type I) C*algebras.

3) All strongly amenable (in the sense of [J]) C*-algebras.

4) All nuclear (= amenable) C*-algebras with at least one tracial state.

Proof: We may assume that the C*-algebras in consideration are unital. 1) is a special case of 2) which itself is a special case of 3) (cf. [J]), which again is a special case of 4). However, 3) follows immediately from [Bu1; Prop. 1]. If one uses the amenability definition of a C*-algebra, 4) follows from [Bu1; Prop. 2]. It may also be deduced from remark 2.4.4: If τ is a tracial state on a unital nuclear C*-algebra $\mathcal{A}$, then $M = \pi_\tau(\mathcal{A})'' \subseteq \mathcal{B}(\mathcal{H}_\tau)$ is an injective finite von Neumann algebra (cf. [L] or [Pa]), hence M (and therefore $\pi_\tau(\mathcal{A})$) has a hypertrace, so π_τ is hypertracial. □

Note that by [H2] all stably finite nuclear unital C*-algebras have at least one tracial state. Combined with remark 2.4.1, 1) may also be obtained from:

Proposition 2.8: Let $\mathcal{S}$ be a non-empty set of weakly hypertracial C*-subalgebras of a C*-algebra $\mathcal{A}$ which is upwards directed by inclusion and such that $\bigcup_{S \in \mathcal{S}} S$ is dense in $\mathcal{A}$. Then $\mathcal{A}$ is weakly hypertracial also.

Proof: We may assume that $\mathcal{A}$ and all elements in $\mathcal{S}$ are unital with the same unit. Let $\pi \in \mathrm{Rep}(\mathcal{A})$ be faithful. Then for each $S \in \mathcal{S}$ we have $\pi|_S : S \to \mathcal{B}(\mathcal{H}_\pi)$ is a faithful element in $\mathrm{Rep}(S)$, so that there exists a hypertrace φ_S on $\pi(S)$. But clearly, any weak*-limit point of the net $\{\varphi_S\}_{S \in \mathcal{S}}$ is a hypertrace on $\pi(\mathcal{A})$, i.e. π is hypertracial as desired. □

The class of weakly hypertracial C*-algebras has the following interesting property:

Proposition 2.9: Let $\mathcal{A}$ be weakly hypertracial C*-algebra and $\mathcal{B}$ be a C*-subalgebra of $\mathcal{A}$. Assume that $\mathcal{A}$ is unital and $\mathcal{B}$ contains the unit of $\mathcal{A}$. Then $\mathcal{B}$ is weakly hypertracial.

Proof: Choose $\pi \in \mathrm{Rep}(\mathcal{A})$, π hypertracial. Under the above assumption, we have $\pi|\mathcal{B} \in \mathrm{Rep}(\mathcal{B})$. As $\pi|\mathcal{B}$ is obviously hypertracial, the result follows. □

Remarks 2.10:

1) The assertion in Proposition 2.9 is not necessarily true if the assumption is not satisfied: consider for example $\mathcal{A} = \mathcal{O}_2 \oplus \mathbf{C}$, $\mathcal{B} = \mathcal{O}_2 \oplus 0$.
2) Let $\mathcal{A}$ be a unital AF-algebra and $\mathcal{B}$ a C*-subalgebra of $\mathcal{A}$ containing the unit of $\mathcal{A}$. Blackadar has shown in [Bl; Th. 1] that any non type I C*-algebra $\mathcal{S}$ contains a non-nuclear C*-subalgebra which may be chosen to contain the unit of $\mathcal{S}$ is $\mathcal{S}$ is unital. Especially, this means that $\mathcal{B}$ is not necessarily nuclear if $\mathcal{A}$ is infinite dimensional. However, $\mathcal{B}$ is weakly hypertracial by Proposition 2.10.
3) The fact that the class of unital weakly hypertracial C*-algebras is larger than the class of unital nuclear C*-algebras possessing at least one tracial state may also be seen by considering the hyperfinite II_1-factor on a separable Hilbert space, which is known to be non-nuclear ([Wa1; Cor. 1.9]), or by considering the group C*-algebra $C^*(G)$ of a non-amenable discrete group G possessing a non-zero finite dimensional representation (e.g. $G = \mathbf{F}_2$ = the free group on two generators) since $C^*(G)$ is then non-nuclear ([L]).

When G is a discrete group and $u \in Z^2(G,\mathbf{T})$, we obtain from Corollary 1.7 that G is amenable $\Leftrightarrow$ $C_r^*(G,u)$ is weakly hypertracial $\Leftrightarrow$ $vN(G,u)$ is weakly hypertracial, and these conditions imply that $C^*(G,u)$ is weakly hypertracial. When combined with Proposition 2.9 and by taking into account remark 1.2.3, the following result which is a slight generalization of [C4; Th. 12] (see also [Bed2; Th. 7]) is obtained:

Corollary 2.11: Let $\mathcal{A}$ be a unital C*-algebra containing a unitally embedded copy of

$C_r^*(G,u)$ for some non-amenable discrete group G and some $u \in Z^2(G,\mathbf{T})$. Then $\mathcal{A}$ is not weakly hypertracial. Especially, there exists no finitely summable unbounded Fredholm module over $\mathcal{A}$.

Let now $(\mathcal{A}, G, \alpha, u)$ denote a twisted C*-dynamical system as considered in [PR] and in [Bed1], where $\mathcal{A}$ is a unital C*-algebra and G is a discrete group. Note that since G is discrete, no separability assumptions are required (cf. [Z-M] in the case when the two-cocycle u takes values in the unitary group of the center at $\mathcal{A}$). We denote by $C^*(\mathcal{A}, G, \alpha, u)$ (resp. $C_r^*(\mathcal{A}, G, \alpha, u)$) the associated (resp. reduced) twisted C*-crossed product. Recall that these two C*-algebras coincide when G is amenable (cf. [PR; Th. 3.11]).

Proposition 2.12: Let $(\mathcal{A}, G, \alpha, u)$ be as above.

1) Suppose $\mathcal{A}$ is weakly hypertracial and G is amenable.

   Then $C^*(\mathcal{A}, G, \alpha, u) \simeq C_r^*(\mathcal{A}, G, \alpha, u)$ is weakly hypertracial.

2) If $C_r^*(\mathcal{A}, G, \alpha, u)$ or $C^*(\mathcal{A}, G, \alpha, u)$ is weakly hypertracial, then $\mathcal{A}$ is weakly hypertracial.

3) If $C_r^*(\mathcal{A}, G, \alpha, u)$ is weakly hypertracial and $u \in Z^2(G,\mathbf{T})$, then G is amenable.

Proof: 1) Suppose $\mathcal{A}$ is weakly hypertracial and G is amenable. Pick some faithful $\pi \in \mathrm{Rep}(\mathcal{A})$. Then the associated regular representation $\tilde{\pi} \times R$ of $C^*(\mathcal{A}, G, \alpha, u)$ is a non-degenerate $*$-representation of $C^*(\mathcal{A}, G, \alpha, u)$ on $\ell^2(G, \mathcal{H}_\pi)$ such that $C_r^*(\mathcal{A}, G, \alpha, u) \simeq \tilde{\pi} \times R(C^*(\mathcal{A}, G, \alpha, u)) = C^*(\tilde{\pi}(\mathcal{A}), R(G))$ (cf. [PR; 3.10, 3.12]). Further $\tilde{\pi} \in \mathrm{Rep}(\mathcal{A})$ is faithful, so $\tilde{\pi}$ is hypertracial. By Proposition 1.6 we get that there exists a hypertrace on $C^*(\tilde{\pi}(\mathcal{A}), R(G))$, hence $C_r^*(\mathcal{A}, G, \alpha, u)$ is weakly hypertracial.

2) Follows from Proposition 2.9 since $\mathcal{A}$ is unitally embedded in $C_r^*(\mathcal{A}, G, \alpha, u)$ and in $C^*(\mathcal{A}, G, \alpha, u)$.

3) When $u \in Z^2(G,\mathbf{T})$, then $C_r^*(G, u)$ is unitally embedded in $C_r^*(\mathcal{A}, G, \alpha, u)$, so the result follows from Corollary 2.11. □

Remark that a crossed product $C^*(\mathcal{A}, G, \alpha)$ may be weakly hypertracial for non-amenable G (take for example $\mathcal{A} = \mathbf{C}$, $G = \mathbf{F}_2$, $\alpha = id.$). It is a simple consequence of Proposition 2.12.1 that the rotation algebras $\mathcal{A}_\theta$ are all weakly hypertracial.

Tensor products behave nicely with respect to weak hypertraciality:

Proposition 2.13: Let $\mathcal{A}, \mathcal{B}$ denote two unital C*-algebras and let γ denote a C*-norm on the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$. Then $\mathcal{A} \otimes_\gamma \mathcal{B}$ is weakly hypertracial $\Leftrightarrow$ $\mathcal{A}$ and $\mathcal{B}$ are weakly hypertracial.

Proof: ($\Rightarrow$) Both $\mathcal{A}$ and $\mathcal{B}$ embedd unitally in $\mathcal{A} \otimes_\gamma \mathcal{B}$, so that this implication is a consequence of Proposition 2.9.

($\Leftarrow$) Suppose $\mathcal{A}$ and $\mathcal{B}$ are weakly hypertracial. Let σ denote the spatial (= minimal) C*-norm on $\mathcal{A} \odot \mathcal{B}$. Since $\mathcal{A} \otimes_\sigma \mathcal{B}$ is a quotient of $\mathcal{A} \otimes_\gamma \mathcal{B}$, it suffices to show that $\mathcal{A} \otimes_\sigma \mathcal{B}$ is weakly hypertracial. Let $\pi_1 \in \mathrm{Rep}(\mathcal{A}), \pi_2 \in \mathrm{Rep}(\mathcal{B})$, π_1 and π_2 faithful. Then we have

$$\begin{aligned}\pi_1 \otimes_\sigma \pi_2(\mathcal{A} \otimes_\sigma \mathcal{B}) &= C^*(\{\pi_1(A) \otimes \pi_2(B) | A \in \mathcal{A}, B \in \mathcal{B}\}) \\ &= C^*(\{U \otimes V | U \in \mathcal{U}(\pi_1(\mathcal{A})), V \in \mathcal{U}(\pi_2(\mathcal{B}))\}) \\ &\subseteq \mathcal{B}(\mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}).\end{aligned}$$

Let $G_1 = \mathcal{U}(\pi_1(\mathcal{A}))$, $G_2 = \mathcal{U}(\pi_2(\mathcal{A}))$ considered as discrete groups, and denote by i_1 (resp. i_2) the identity unitary representation of G_1 on $\mathcal{H}_{\pi_1}$ (resp. G_2 on $\mathcal{H}_{\pi_2}$). Since π_1 and π_2 are hypertracial this means that i_1 and i_2 are amenable in the sense of Bekka ([Bek]). By [Bek; Cor. 5.4], we get that $i_1 \otimes i_2$ is amenable, hence that there exists a state φ on $\mathcal{B}(\mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2})$ such that

$$\{U \otimes V | U \in G_1,\ V \in G_2\} \subseteq C_\varphi\,.$$

This implies that $\pi_1 \otimes \pi_2(\mathcal{A} \otimes_\sigma \mathcal{B}) \subseteq C_\varphi$, i.e. that $\pi_1 \otimes \pi_2$ is hypertracial, which shows that $\mathcal{A} \otimes_\sigma \mathcal{B}$ is weakly hypertracial as desired. □

3. Hypertracial C*-algebras

We say that a C*-algebra $\mathcal{A}$ is *hypertracial* if π is hypertracial for all $\pi \in \mathrm{Rep}(\mathcal{A})$. Hypertraciality is clearly preserved under $*$-isomorphism. Further, a simple C*-algebra is hypertracial if and only if it is weakly hypertracial. Especially, a simple non-unital C*-algebra is hypertracial. So it seems that the concept of hypertraciality will be mostly of interest for unital C*-algebras.

The class of hypertracial C*-algebras is obviously smaller than the class of weakly hypertracial C*-algebras. For example, we have:

Proposition 3.1: Suppose $\mathcal{A}$ is hypertracial and $\mathcal{J}$ is a non-zero proper ideal in $\mathcal{A}$. Then $\mathcal{J}$ and $\mathcal{A}/\mathcal{J}$ are hypertracial.

Proof: As we may extend any $\pi \in \mathrm{Rep}(\mathcal{J})$ to a $\tilde{\pi} \in \mathrm{Rep}(\mathcal{A})$ satisfying $\mathcal{H}_{\tilde{\pi}} = \mathcal{H}_\pi$, it follows readily from the hypertraciality of $\mathcal{A}$ that $\mathcal{J}$ is hypertracial. Further, let $\pi \in \mathrm{Rep}(\mathcal{A}/\mathcal{J})$ and set $\dot{\pi} = \pi \circ \psi$ where ψ denotes the canonical $*$-homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\mathcal{J}$. Then $\dot{\pi} \in \mathrm{Rep}(\mathcal{A})$ with $\mathcal{H}_{\dot{\pi}} = H_\pi$ and $\dot{\pi}$ is hypertracial since $\mathcal{A}$ is hypertracial. As $\dot{\pi}(\mathcal{A}) = \pi(\mathcal{A}/\mathcal{J})$, we get that π is hypertracial. Hence, $\mathcal{A}/\mathcal{J}$ is hypertracial. □

We don't know whether the converse of Proposition 3.1 is true.

Corollary 3.2: $\mathcal{A}$ is hypertracial

$\Leftrightarrow$ $\mathcal{A}/\mathcal{J}$ is weakly hypertracial for all proper ideals $\mathcal{J}$ in $\mathcal{A}$

$\Leftrightarrow$ $\mathcal{A}/\mathcal{J}$ is hypertracial for all proper ideals $\mathcal{J}$ in $\mathcal{A}$.

In light of Proposition 2.5, it is enough to consider modular ideals $\mathcal{J}$ in $\mathcal{A}$ (i.e. such that $\mathcal{A}/\mathcal{J}$ is unital) in the above corollary.

Corollary 3.3: $\mathcal{A}$ is hypertracial

$\Leftrightarrow \widetilde{\mathcal{A}}$ is hypertracial.

Proof: ($\Rightarrow$) Let $\rho \in \mathrm{Rep}(\widetilde{\mathcal{A}})$, so that $\rho(I) = I_{\mathcal{H}_\rho}$ where I denotes the unit of $\widetilde{\mathcal{A}}$. Set $\pi = \rho|\mathcal{A}$. If $I_{\mathcal{H}_\rho} \in \pi(\mathcal{A})$, then $\pi(\mathcal{A}) = \rho(\widetilde{\mathcal{A}})$ and π is non-degenerate, hence ρ is hypertracial

if $\mathcal{A}$ is hypertracial. Suppose now that $I_{\mathcal{H}_\rho} \notin \pi(\mathcal{A})$. Then we have $\rho(\tilde{\mathcal{A}}) \simeq \pi(\mathcal{A})\tilde{}$. But $\pi(\mathcal{A})\tilde{}$ has a non-zero 1-dimensional representation, thus it follows from remark 2.4.3 that $\rho(\tilde{\mathcal{A}})$ is weakly hypertracial, i.e. ρ is hypertracial. So we have shown that ρ is hypertracial for all $\rho \in \mathrm{Rep}(\tilde{\mathcal{A}})$ as desired.

($\Leftarrow$) This implication follows from Proposition 3.1. □

To check the hypertraciality of $\mathcal{A}$, it is enough to consider irreducible representations of $\mathcal{A}$:

Proposition 3.4: $\mathcal{A}$ is hypertracial

$\Leftrightarrow$ ρ is hypertracial for all irreducible $\rho \in \mathrm{Rep}(\mathcal{A})$.

Proof: The forward implication is trivial. So assume that ρ is hypertracial for all irreducible $\rho \in \mathrm{Rep}(\mathcal{A})$. Let $\pi \in \mathrm{Rep}(\mathcal{A})$. Choose some irreducible $\rho' \in \mathrm{Rep}(\pi(\mathcal{A}))$ and set $\rho = \rho' \circ \pi \in \mathrm{Rep}(\mathcal{A})$. Then ρ is irreducible and ρ is weakly contained in π. Hence we get from Theorem 2.1 that π is hypertracial since ρ is hypertracial. This shows that $\mathcal{A}$ is hypertracial. □

An immediate consequence of Proposition 3.4 is that all liminal C*-algebras are hypertracial. But in fact, since any quotient of a finite dimensional (resp. abelian) (resp. postliminal) (resp. AF-algebra) (resp. strongly amenable) C*-algebra inherits the respective property (cf. [M] and [J; 7.3]), we get from Corollary 3.2 and the results in section 2 that the class of hypertracial C*-algebras includes all finite dimensional C*-algebras, all abelian C*-algebras, all postliminal C*-algebras and more generally all strongly amenable C*-algebras. It also includes the irrational rotation algebras and the hyperfinite II_1-factor (since these are simple weakly hypertracial C*-algebras). Further, we have:

Proposition 3.5: Suppose $\mathcal{A}$ is a nuclear C*-algebra satisfying the following finiteness condition: for all proper (modular) ideals $\mathcal{J}$ in $\mathcal{A}$, $\mathcal{A}/\mathcal{J}$ has a tracial state. Then $\mathcal{A}$ is hypertracial.

Proof: Let $\mathcal{J}$ be a proper (modular) ideal in $\mathcal{A}$. By [ChEf; Cor. 4], $\mathcal{A}/\mathcal{J}$ is nuclear, and it has a tracial state by assumption. Hence $\mathcal{A}/\mathcal{J}$ is weakly hypertracial by Proposition 2.7. By Corollary 3.2 (and its accompanying remark), this shows that $\mathcal{A}$ is hypertracial.

□

Any strongly amenable C*-algebra satisfies the assumptions in Proposition 3.5. We don't know whether the converse is true. On the other hand, the converse of Proposition 3.5 is not true, at least for non-separable C*-algebras: indeed, the hyperfinite II_1-factor is hypertracial, but as pointed out before, it is not nuclear. However, it would be interesting to know whether any (unital) separable hypertracial C*-algebra is nuclear.

Let us also remark that a C*-subalgebra of a hypertracial C*-algebra is not necessarily hypertracial: Blackadar shows in [Bl; Th. 2] that any non type I C*-algebra $\mathcal{A}$ has a C*-subalgebra $\mathcal{B}$ (containing the unit of $\mathcal{A}$ if $\mathcal{A}$ is unital) which has $\mathcal{O}_2$ as a quotient, so that $\mathcal{B}$ is not hypertracial by Corollary 3.2.

The class of hypertracial C*-algebras behaves nicely with respect to inductive limits and crossed products by amenable discrete groups.

Proposition 3.6:

1) Let $\mathcal{S}$ be a non-empty set of hypertracial C*-subalgebras of a C*-algebra $\mathcal{A}$. Suppose that $\mathcal{S}$ is upwards directed by inclusion and $\bigcup_{S\in\mathcal{S}} S$ is dense in $\mathcal{A}$.

   Then $\mathcal{A}$ is hypertracial also.

2) Suppose that $(\mathcal{A}_n, \varphi_n)_{n=1}^{\infty}$ is a direct sequence of hypertracial C*-algebras. Then the direct limit $\varinjlim \mathcal{A}_n$ is hypertracial also.

Proof: 1) Let $\pi \in \mathrm{Rep}(\mathcal{A})$. Then $\mathcal{S}' = \{\pi(S) | S \in \mathcal{S}\}$ is upwards directed by inclusion and $\overline{\bigcup_{S'\in\mathcal{S}'} S'} = \pi(\mathcal{A})$.

Since every element in $\mathcal{S}'$ is weakly hypertracial by assumption, it follows from Proposition 2.8 that $\pi(\mathcal{A})$ is weakly hypertracial, hence that π is hypertracial. So $\mathcal{A}$ is hypertracial as

asserted.

2) Let $\varphi^n : \mathcal{A}_n \to \mathcal{A}$ be the natural map, where $\mathcal{A} = \varinjlim \mathcal{A}_n$ (cf. [M]). Then $\mathcal{S} = \{\varphi^n(\mathcal{A}_n) | n \geq 1\}$ is an upwards directed family of hypertracial C*-subalgebras of $\mathcal{A}$ whose union is dense in $\mathcal{A}$, so that $\mathcal{A}$ is hypertracial by 1). □

Let G denote a discrete group. It is not difficult to deduce from [Bek; Th. 2.2] that G is amenable $\Leftrightarrow$ $C^*(G)$ is hypertracial $\Leftrightarrow$ $C_r^*(G)$ is hypertracial. More generally, we have:

Proposition 3.7: Let $\mathcal{A}, G, \alpha, u)$ denote a twisted C*-dynamical system as in Proposition 2.12, but where we don't assume that $\mathcal{A}$ is unital if u is trivial. Consider the following conditions:

(1) G is amenable and $\mathcal{A}$ is hypertracial

(2) $C^*(\mathcal{A}, G, \alpha, u)$ is hypertracial

(3) $C_r^*(\mathcal{A}, G, \alpha, u)$ is hypertracial

(4) G is amenable.

Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If $u \in Z^2(G,\mathbf{T})$, then we have (3) $\Rightarrow$ (4).

Proof: (1) $\Rightarrow$ (2) Suppose G is amenable and $\mathcal{A}$ is hypertracial. Let $\mathcal{B} = C^*(\mathcal{A}, G, \alpha, u)$ and assume first that $\mathcal{A}$ is unital. Let $\rho \in \mathrm{Rep}(\mathcal{B})$. Recall from [PR] that we may write $\rho = \pi \times U$, where $\pi \times U$ is the non-degenerate representation of $\mathcal{B}$ associated to a covariant representation (π, U) of $(\mathcal{A}, G, \alpha, u)$ on a Hilbert space $\mathcal{H}$, i.e. $\pi \in \mathrm{Rep}(\mathcal{A})$, $\mathcal{H}_\pi = \mathcal{H}$ and $U : G \to \mathcal{U}(\mathcal{B}(\mathcal{H}))$ is a map such that

$$U_g U_h = \pi(u(g,h))U_{gh} \qquad \text{and} \qquad \pi(\alpha_g(A)) = U_g \pi(A) U_g^*$$

for all $g, h \in G$, $A \in \mathcal{A}$, and that we have $\rho(\mathcal{B}) = \pi \times U(\mathcal{B}) = C^*(\pi(A), U(G))$. We may now apply Proposition 1.6 and deduce that ρ is hypertracial. This shows that $\mathcal{B}$ is hypertracial. Now, if u is trivial and $\mathcal{A}$ is non-unital, then we may extend α to an action of G on $\tilde{\mathcal{A}}$ in the obvious way. As $\tilde{\mathcal{A}}$ is hypertracial by Corollary 3.3, we get from the first part that $C^*(\tilde{\mathcal{A}}, G, \alpha)$ is hypertracial. But $C^*(\mathcal{A}, G, \alpha)$ is an ideal of $C^*(\tilde{\mathcal{A}}, G, \alpha)$, so that

$C_r^*(\mathcal{A}, G, \alpha)$ is hypertracial by Corollary 3.2.

(2) $\Rightarrow$ (3) follows from Corollary 3.2 since $C_r^*(\mathcal{A}, G, \alpha, u)$ is a quotient of $C^*(\mathcal{A}, G, \alpha, u)$.

Finally, suppose that $u \in Z^2(G,\mathbf{T})$ and that $C_r^*(\mathcal{A}, G, \alpha, u)$ is hypertracial. Then $C_r^*(\mathcal{A}, G, \alpha, u)$ is especially weakly hypertracial, so that G is amenable by Proposition 2.12.3. □

Corollary 3.8: Let $(\mathcal{A}, G, \alpha, u)$ denote a C*-dynamical system where G is a discrete group, $\mathcal{A}$ is a unital C*-algebra, and $u \in Z^2(G,\mathbf{T})$.

1) Suppose $\mathcal{A}$ is hypertracial. Then G is amenable

$\Leftrightarrow$ $C^*(\mathcal{A}, G, \alpha, u)$ is hypertracial

$\Leftrightarrow$ $C_r^*(\mathcal{A}, G, \alpha, u)$ is hypertracial.

2) Suppose $\mathcal{A}$ is simple. Then G is amenable and $\mathcal{A}$ is weakly hypertracial

$\Leftrightarrow$ $C^*(\mathcal{A}, G, \alpha, u)$ is hypertracial

$\Leftrightarrow$ $C_r^*(\mathcal{A}, G, \alpha, u)$ is hypertracial.

Proof: 1) follows from Proposition 3.7. 2) follows from Proposition 3.7 combined with Proposition 2.9. □

The behaviour of hypertraciality with respect to tensor products seems more difficult to handle. We can only show the following.

Proposition 3.9: Let $\mathcal{A}$ be a hypertracial C*-algebra and $\mathcal{B}$ a unital C*-algebra such that there exists a subgroup G of $\mathcal{U}(\mathcal{B})$ which generates $\mathcal{B}$ as a C*-algebra and which is amenable as a discrete group. Let γ denote a C*-norm on $\mathcal{A} \odot \mathcal{B}$. Then $\mathcal{A} \otimes_\gamma \mathcal{B}$ is hypertracial (especially, $\mathcal{B}$ is hypertracial under the above condition).

Proof: As $\mathcal{A} \otimes_\gamma \mathcal{B}$ is an ideal in $\tilde{\mathcal{A}} \otimes_\gamma \mathcal{B}$, we may assume that $\mathcal{A}$ is unital (cf. Corollaries 3.2 and 3.3). Now, let $\pi \in \mathrm{Rep}(\mathcal{A} \otimes_\gamma \mathcal{B})$. Then there exist $\pi_1 \in \mathrm{Rep}(\mathcal{A}), \pi_2 \in \mathrm{Rep}(\mathcal{B})$ with $\mathcal{H}_{\pi_1} = \mathcal{H}_{\pi_2} = \mathcal{H}_\pi$ such that

$$\pi(A \otimes B) = \pi_1(A)\pi_2(B) = \pi_2(B)\pi_1(A), \quad A \in \mathcal{A},\ B \in \mathcal{B},$$

and

$$\pi(\mathcal{A} \otimes_\gamma \mathcal{B}) = C^*(\pi_1(\mathcal{A}), \pi_2(\mathcal{B})).$$

Let $G' = \pi_2(G)$. Then G' is a subgroup of $\mathcal{U}(\mathcal{B}(\mathcal{H}_\pi))$ which is amenable as a discrete group since it is a quotient of the amenable discrete group G. Further, as G generates $\mathcal{B}$ as a C*-algebra, it is clear that G' generates $\pi_2(\mathcal{B})$ as a C*-algebra, so that we have

$$\pi(\mathcal{A} \otimes_\gamma \mathcal{B}) = C^*(\pi_1(\mathcal{A}), G').$$

As π_1 is hypertracial since $\mathcal{A}$ is hypertracial, we may invoke Proposition 1.6 and obtain that π is hypertracial. This shows that $\mathcal{A} \otimes_\gamma \mathcal{B}$ is hypertracial. □

An analog assumption on $\mathcal{B}$ has been considered in [ChEc] in the case when $\mathcal{B}$ is a von Neumann algebra.

Corollary 3.10: Suppose $\mathcal{A}$ is a hypertracial C*-algebra. Then

1) $\mathcal{A} \otimes M_n(\mathbf{C})$ is hypertracial for all $n \geq 1$

2) $\mathcal{A} \otimes \mathcal{B}$ is hypertracial whenever $\mathcal{B}$ is an AF-algebra.

3) $\mathcal{A} \otimes \mathcal{B}$ is hypertracial whenever $\mathcal{B}$ is an abelian C*-algebra.

Proof: 1) It is not difficult to show that there exists a discrete amenable subgroup of $\mathcal{U}(M_n(\mathbf{C}))$ which generates $M_n(\mathbf{C})$ as a C*-algebra for any $n \geq 1$, so that this assertion follows from Proposition 3.9. However, this may also be shown directly; if $\pi \in \mathrm{Rep}(\mathcal{A} \otimes M_n(\mathbf{C}))$, then $\pi(\mathcal{A} \otimes M_n(\mathbf{C})) \simeq \pi(\mathcal{A} \otimes 1) \otimes M_n(\mathbf{C})$, so that π is hypertracial (by Proposition 2.13 if $\pi(\mathcal{A} \otimes 1)$ is unital, trivially otherwise).

2) This follows easily from Proposition 3.6 combined with 1).

3) Assume $\mathcal{B}$ is abelian and unital. Then $\mathcal{U}(\mathcal{B})$ is abelian, hence amenable as a discrete group, so that the result follows from Proposition 3.9. The non-unital case follows by unitization. □

There has been recently considerable interest in the study of nuclear C*-algebras which may be written as an inductive limit $\mathcal{A} = \varinjlim \mathcal{A}_n$ where each $\mathcal{A}_n$ is a finite direct sum of

C*-algebras of the form $C_0(X) \otimes M_k(\mathbf{C})$, X being a locally compact Hausdorff space and $k \in \mathbf{N}$.

Corollary 3.11: Let $\mathcal{A}$ be as above. Then $\mathcal{A}$ is hypertracial.

Proof: Each direct summand of $\mathcal{A}_n$ being hypertracial by Corollary 3.10, it is clear that each $\mathcal{A}_n$ is hypertracial. Hence $\mathcal{A}$ is hypertracial by Proposition 3.6. □

In some cases, such $\mathcal{A}$'s may also be written as a twisted crossed product of an abelian C*-algebra by an abelian discrete group (cf. [Th]) and the hypertraciality of these $\mathcal{A}$'s may then also be deduced from Proposition 3.7.

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