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An introduction to stochastic analysis

by

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An Introduction to Stochastic Analysis

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Introduction

The following pages are the notes from a seminar that I have given during the spring and some portion of the summer of 1993 at the Mathematics Institute of Oslo University. The aim of the seminars was to give a rapid but rigorous introduction for the graduate students to the Analysis on the Wiener space, a subject which has grown up very quickly these recent years under the impulse of the Stochastic Calculus of Variations of Paul Malliavin.

Although some concepts are given in the first chapter, I assumed that the students had already acquired the notions of stochastic calculus with semimartingales, Brownian motion and some rudiments of the theory of Markov processes. A small portion of the exposed material is our own research, the rest has been taken from the works given at the bibliography. Although we avoided to quote them as much as they deserve to be quoted, the reason is to give a homogeneity to the text and the warned reader will realize immediately the impact of all these works all along the following lines.

I have had the chance of having an ideal environment for working and a very careful audience. These notes have particularly profited from the serious criticism of my colleagues and friends Bernt Øksendal, Tom Lindstrøm, Ya-Zhong Hu, and the graduate students of the Mathematics department. It remains to me to express also my gratitude to Nina Haraldsson for her careful typing, and, the last but not the least, to Laurent Decreusefond for correcting so many errors .

Ali Süleyman Üstünel

Chapter

Preliminaries

This chapter is devoted to the basic results about the Wiener measure, Brownian motion, construction of the Ito stochastic integral and the chaos decomposition associated to it.

1 The Brownian Motion and the Wiener Measure

1) Let $W = C_0([0, 1])$, $\omega \in W$, $t \in [0, 1]$, define $W_t(\omega) = \omega(t)$ (the coordinate functional). If we note by $\mathcal{B}_t = \sigma\{W_s; s \leq t\}$, then there is one and only one measure μ on W such that

$$\text{i) } \mu\{W_0(\omega) = 0\} = 1,$$

ii) $\forall f \in C_b^\infty(\mathbb{R})$, $\left(f(W_t(\omega)) - \frac{1}{2} \int_0^t f''(W_s(\omega)) ds\right)_{t \in [0, 1]}$ is a (\mathcal{B}_t, μ) -martingale. μ is called the canonical Wiener measure.

2) From the construction we see that for $t > s$, $E_\mu[\exp i\alpha(W_t - W_s) | \mathcal{B}_s] = \exp -\alpha^2(t-s)$, hence $t \mapsto W_t$ is a continuous additive process and $(W_t; t \in [0, 1])$ is also a continuous martingale.

3) Stochastic Integration

Let $K : W \times [0, 1] \rightarrow \mathbb{R}$ be a step process :

$$K_t(\omega) = \sum_{i=1}^n a_i(\omega) \cdot 1_{[t_i, t_{i+1}[}(t), \quad a_i(\omega) \in L^2(\mathcal{B}_{t_i}).$$

Define $I(K) = \int_0^1 K_s dW_s(\omega)$ as $\sum_{i=1}^n a_i(\omega) \cdot (W_{t_{i+1}}(\omega) - W_{t_i}(\omega))$. Then we have

$$E \left[\left(\int_0^1 K_s dW_s \right)^2 \right] = E \int_0^1 K_s^2 ds,$$

i.e. I is an isometry from the adapted step processes into $L^2(\mu)$, hence it has a unique extension as an isometry from

$$L^2([0, 1] \times W, \mathcal{A}, dt \times d\mu) \xrightarrow{I} L^2(\mu)$$

where \mathcal{A} denotes the sigma algebra on $[0, 1] \times W$ generated by the adapted, left (or right) continuous processes. $I(K)$ is called the stochastic integral of K and it is sometimes denoted as $\int_0^1 K_s dW_s$. With some localization techniques, I can be extended to any adapted process K such that $\int_0^1 K_s^2(\omega) ds < \infty$ a.s.

Application: a) If $f \in C_1^2(\mathbb{R})$ and $M_t = \int_0^t K_r dW_r$, we have

$$\boxed{f(M_t) = f(0) + \int_0^t f'(M_s) K_s dW_s + \frac{1}{2} \int_0^t f''(M_s) K_s^2 ds.} \quad (\text{Ito formula})$$

b) $\mathcal{E}_t(I(h)) = \exp(\int_0^t h_s dW_s - \frac{1}{2} \int_0^t h_s^2 ds)$ is a martingale for any $h \in L^2[0, 1]$.

4) Alternative Constructions

A) Let $(\gamma_i; i \in \mathbb{N})$ be an independent sequence of $N_1(0, 1)$ Gaussian random variables. Let (g_i) be a complete, orthonormal basis of $L^2[0, 1]$. Then W_t defined by

$$W_t(\omega) = \sum_{i=1}^{\infty} \gamma_i(\omega) \cdot \int_0^t g_i(s) ds$$

is a Brownian motion.

Remark: If $(g_i; i \in \mathbb{N})$ is a complete, orthonormal basis of $L^2([0, 1])$, then $(\int_0^1 g_i(s) ds; i \in \mathbb{N})$ is a complete orthonormal basis of $H_1([0, 1])$ (i. e., the first order Sobolev functionals on $[0, 1]$).

B) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be any abstract probability space and let H be any separable Hilbert space. If $L : H \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a linear operator such that for any $h \in H$, $E[\exp iL(h)] = \exp -\frac{1}{2}|h|_H^2$, then there exists a Banach space with dense injection $H \xrightarrow{j} W$ dense, hence $W^* \xrightarrow{j^*} H$ is also dense and a probability measure μ on W such that

$$\int \exp\langle \omega^*, \omega \rangle d\mu(\omega) = \exp -\frac{1}{2} |j^*(\omega^*)|_H^2$$

and

$$L(j^*(\omega^*))(\omega) = \langle \omega^*, \omega \rangle$$

almost surely. (W, H, μ) is called an Abstract Wiener space and μ is the Wiener measure. If $H = H_1([0, 1])$ then μ is the classical Wiener measure and W can be taken as $C_0([0, 1])$.

Remark: In the case of the classical Wiener space, any element λ of W^* is a signed measure on $[0, 1]$, and its image in $H = H_1([0, 1])$ can be represented as $j^*(\lambda)(t) = \int_0^t \lambda([s, 1]) ds$.

5) Let us come back to the classical Wiener space:

- i) It follows from the martingale convergence theorem and the monotone class theorem that the set of random variables

$$\{f(W_{t_1}, \dots, W_{t_n}); t_i \in [0, 1], f \in \mathcal{S}(\mathbb{R}^n); n \in \mathbb{N}\}$$

is dense in $L^2(\mu)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the space of infinitely differentiable, rapidly decreasing functions on \mathbb{R}^n .

- ii) It follows from (i), via the Fourier transform that the linear span of the set $\{\exp \int_0^1 h_s dW_s - \frac{1}{2} \int_0^1 h_s^2 ds; h \in L^2([0, 1])\}$ is dense in $L^2(\mu)$.

- iii) Because of the analyticity of the characteristic function of the Wiener measure, the elements of the set in (ii) can be approached by the polynomials, hence the polynomials are dense in $L^2(\mu)$.

5.1 Cameron-Martin Theorem:

For any bounded Borel measurable function F , $h \in L^2[0, 1]$, we have

$$E_\mu[F(\omega + \int_0^1 h_s ds) \cdot \exp[-\int_0^1 h_s dW_s - \frac{1}{2} \int_0^1 h_s^2 ds]] = E_\mu[F].$$

This means that the process $W_t(\omega) + \int_0^t h_s ds$ is again a Brownian motion under the new probability measure

$$\exp(-\int_0^1 h_s dW_s - \frac{1}{2} \int_0^1 h_s^2 ds) d\mu.$$

Proof: It is sufficient to show that the new probability has the same characteristic function as μ : if $x^* \in W^*$, then x^* is a measure on $[0, 1]$ and

$$\begin{aligned} w^* \langle x^*, w \rangle_W &= \int_0^1 W_s(\omega) x^*(ds) \\ &= W_t(\omega) \cdot x^*([0, t]) \Big|_0^1 - \int_0^1 x^*([0, t]) dW_t(\omega) \\ &= W_1 x^*([0, 1]) - \int_0^1 x^*([0, t]) \cdot dW_t \\ &= \int_0^1 x^*(]t, 1]) dW_t. \end{aligned}$$

Consequently

$$\begin{aligned}
& E[\exp i \int_0^1 x^*([t, 1]) dW_t (w + \int_0^1 h_s ds) \cdot \mathcal{E}(-I(h))] \\
&= E[\exp i \int_0^1 x^*([t, 1]) dW_t + i \int_0^1 x^*([t, 1]) h_t dt - \int_0^1 h_t dW_t - \frac{1}{2} \int_0^1 h_t^2 dt] \\
&= E[\exp i \int_0^1 (i x^*([t, 1]) - h_t) dW_t \cdot \exp i \int_0^1 x^*([t, 1]) h_t dt - \frac{1}{2} \int_0^1 h_t^2 dt] \\
&= \exp \frac{1}{2} \int_0^1 (i x^*([t, 1]) - h_t)^2 dt + i \int_0^1 x^*([t, 1]) h_t dt - \frac{1}{2} \int_0^1 h_t^2 dt \\
&= \exp -\frac{1}{2} \int_0^1 (x^*([t, 1]))^2 dt \\
&= \exp -\frac{1}{2} |j(x^*)|_{H_1}^2 .
\end{aligned}$$

QED

Corollary (Paul Lévy's Theorem) Suppose that (M_t) is a continuous martingale such that $M_0 = 0$, $M_t^2 - t$ is again a martingale. Then (M_t) is a Brownian motion.

Proof: We have the Ito formula

$$f(M_t) = f(0) + \int_0^t f'(M_s) \cdot dM_s + \frac{1}{2} \int_0^t f''(M_s) \cdot ds .$$

Hence the law of $\{M_t : t \in [0, 1]\}$ is μ .

QED

5.2 The Ito Representation Theorem:

Any $\varphi \in L^2(\mu)$ can be represented as

$$\varphi = E[\varphi] + \int_0^1 K_s dW_s$$

where $K \in L^2([0, 1] \times W)$, adapted.

Proof: Since the Wick exponentials

$$\mathcal{E}(I(h)) = \exp \int_0^1 h_s dW_s - 1/2 \int_0^1 h_s^2 ds$$

can be represented as claimed, the proof follows by density.

QED

5.3 Wiener chaos representation

Let $K_1 = \int_0^1 h_s dW_s$, $h \in L^2([0, 1])$. Then, from the Ito formula, we can write

$$\begin{aligned} K_1^p &= p \int_0^1 K_s^{p-1} h_s dW_s + \frac{p(p-1)}{2} \int_0^1 K_s^{p-2} h_s^2 ds \\ &= p \int_0^1 \left[(p-1) \int_0^{t_1} K_{t_2}^{p-2} h_{t_2} dW_{t_2} + \frac{(p-1)(p-1)}{2} \int_0^{t_1} K_{t_2}^{p-3} h_{t_2}^2 dt_2 \right] dW_{t_1} \\ &\quad + \dots \end{aligned}$$

iterating this procedure we end up on one hand with $K_{t_p}^0 = 1$, on the other hand with the multiple integrals of deterministic integrands of the type

$$J_p = \int_{0 < t_p < t_{p-1} < \dots < t_1 < 1} h_{t_1} h_{t_2} \dots h_{t_p} dW_{t_1}^{i_1} \dots dW_{t_p}^{i_p},$$

$i_j = 0$ or 1 with $dW_t^0 = dt$ and $dW_t^1 = dW_t$.

Let now $\varphi \in L^2(\mu)$, then we have from the Ito representation theorem

$$\varphi = E[\varphi] + \int_0^1 K_s dW_s$$

by iterating the same procedure for the integrand of the above stochastic integral:

$$\begin{aligned} \varphi &= E[\varphi] + \int_0^1 E[K_s] dW_s + \int_0^1 \int_0^{t_1} E[K_{t_1, t_2}^{1,2}] dW_{t_2} dW_{t_1} + \\ &\quad + \int_0^1 \int_0^{t_1} \int_0^{t_2} K_{t_1 t_2 t_3}^{1,2,3} dW_{t_3} dW_{t_2} dW_{t_1}. \end{aligned}$$

After N iterations we end up with

$$\varphi = \sum_0^N J_p(K^p) + \varphi_{N+1}$$

and each element of the sum is orthogonal to the other one. Hence $(\varphi_N; N \in \mathbb{N})$ is bounded in $L^2(\mu)$. Let (φ_{N_k}) be a weakly convergent subsequence and $\varphi_\infty = \lim_{k \rightarrow \infty} \varphi_{N_k}$. Then it is easy from the first part that φ_∞ is orthogonal to the polynomials, therefore $\varphi_\infty = 0$

and $w - \lim_{N \rightarrow \infty} \sum_0^N J_p(K_p)$ exists, moreover $\sup_N \sum_1^N \|J_p(K_p)\|_2^2 < \infty$, hence $\sum_1^\infty J_p(K_p)$ converges in $L^2(\mu)$. Let now \widehat{K}_p be an element of $\widehat{L}^2[0,1]^p$ (i.e. symmetric), defined as $\widehat{K}_p = K_p$ on $C_p = \{t_1 < \dots < t_p\}$. We define $I_p(\widehat{K}_p) = p! J_p(K_p)$ in such a way that

$$E[|I_p(\widehat{K}_p)|^2] = (p!)^2 \int_{C_p} K_p^2 dt_1 \dots dt_p = p! \int_{[0,1]^p} |\widehat{K}_p|^2 dt_1 \dots dt_p.$$

Let $\varphi_p = \frac{\widehat{K}_p}{p!}$, then we have

$$\boxed{\varphi = E[\varphi] + \sum_1^\infty I_p(\varphi_p)} \quad (\text{Wiener chaos decomposition})$$

Chapter I

Gross-Sobolev Derivative, Divergence and Ornstein-Uhlenbeck Operator

Motivations

Let $W = C_0([0, 1], \mathbb{R}^d)$ be the classical Wiener space equipped with μ the Wiener measure. We want to construct on W a Sobolev type analysis in such a way that we can apply it to the random variables that we encounter in the applications. Mainly we want to construct a differentiation operator and to be able to apply it to practical examples. The Fréchet derivative is not satisfactory. In fact the most frequently encountered Wiener functionals, as the multiple (or single) Wiener integrals or the solutions of stochastic differential equations with smooth coefficients are not even continuous with respect to the Fréchet norm of the Wiener space. Therefore, what we need is in fact to define a derivative on the $L^p(\mu)$ -spaces of random variables, but in general, to be able to do this, we need the following property which is essential: if $F, G \in L^p(\mu)$, and if we want to define their directional derivative, in the direction, say $\tilde{w} \in W$, we write $\frac{d}{dt}F(w+t\tilde{w})|_{t=0}$ and $\frac{d}{dt}G(w+t\tilde{w})|_{t=0}$. If $F = G$ μ -a.s., it is natural to ask that their derivatives are also equal

a.s. For this, the only way is to choose \tilde{w} in some specific subspace of W , namely, the Cameron-Martin space H_1 :

$$H_1 = \left\{ h : [0, 1] \rightarrow \mathbb{R}^d / h(t) = \int_0^t \dot{h}(s) ds, \quad |h|_{H_1}^2 = \int_0^1 |\dot{h}(s)|^2 ds \right\}.$$

In fact, the theorem of Cameron-Martin says that for any $F \in L^p(\mu)$, $p > 1$, $h \in H_1$

$$E_\mu[F(w+h) \exp[-\int_0^1 \dot{h}(s) \cdot dW_s - \frac{1}{2}|h|_{H_1}^2]] = E_\mu[F],$$

or equivalently

$$E_\mu[F(w+h)] = E[F(w) \cdot \exp \int_0^1 \dot{h}_s \cdot dW_s - \frac{1}{2}|h|_{H_1}^2].$$

That is to say, if $F = G$ a.s., then $F(\cdot + h) = G(\cdot + h)$ a.s. for all $h \in H_1$.

1 The Construction of ∇ and its properties

If $F : W \rightarrow \mathbb{R}$ is a function of the following type (called cylindrical):

$$F(w) = f(W_{t_1}(w), \dots, W_{t_n}(w)), \quad f \in \mathcal{S}(\mathbb{R}^n),$$

we define, for $h \in H$,

$$\nabla_h F(w) = \frac{d}{d\lambda} F(w + \lambda h)|_{\lambda=0}.$$

Noting that $W_t(w+h) = W_t(w) + h(t)$, we obtain

$$\nabla_h F(w) = \sum_{i=1}^n \partial_i f(W_{t_1}(w), \dots, W_{t_n}(w)) h(t_i),$$

in particular

$$\nabla_h W_t(w) = h(t) = \int_0^t \dot{h}(s) ds = \int_0^1 1_{[0,t]}(s) \dot{h}(s) ds.$$

If we denote by U_t the element of H_1 defined as $U_t(s) = \int_0^s 1_{[0,t]}(r) dr$, we have $\nabla_h W_t(w) = (U_t, h)_{H_1}$. Looking at the linear map $h \mapsto \nabla_h F(w)$ we see that it defines a random element with values in H_1 , i.e. ∇F is an H_1 -valued random variable. Now we can prove:

Prop. I.1: ∇ is a closable operator on any $L^p(\mu)$ ($p > 1$).

Proof: This means that if $(F_n : n \in \mathbb{N})$ are cylindrical functions on W , such that $F_n \rightarrow 0$ in $L^p(\mu)$ and if $(\nabla F_n; n \in \mathbb{N})$ is Cauchy in $L^p(\mu, H_1)$, then its limit is zero. Hence suppose that $\nabla F_n \rightarrow \xi$ in $L^p(\mu; H_1)$.

To prove $\xi = 0$ μ -a.s., we use the Cameron-Martin theorem: Let φ be any cylindrical function. Since such φ 's are dense in $L^p(\mu)$, it is sufficient to prove that $E[(\xi, h)_{H_1} \cdot \varphi] = 0$ a.s. for any $h \in H_1$. But we have

$$\begin{aligned} E[(\nabla F_n, h)\varphi] &= \frac{d}{d\lambda} E[F_n(w + \lambda h) \cdot \varphi] |_{\lambda=0} \\ &= \frac{d}{d\lambda} E[F_n(w)\varphi(w - \lambda h) \exp(\lambda \int_0^1 \dot{h}(s) dW_s - \frac{\lambda^2}{2} \int_0^1 |\dot{h}_s|^2 ds)] |_{\lambda=0} \\ &= E[F_n(w)(-\nabla_h \varphi(w) + \varphi(w) \int_0^1 \dot{h}(s) dW_s)] \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

by the fact that $F_n \rightarrow 0$ in $L^p(\mu)$.

QED

This result tells us that we can define $\text{Dom}_p(\nabla)$ as

$$F \in \text{Dom}_p(\nabla) \Leftrightarrow \exists (F_n) \text{ cylindrical such that } F_n \rightarrow F \text{ in } L^p \text{ and } (\nabla F_n) \text{ is Cauchy in } L^p(\mu, H).$$

Then we define $\boxed{\nabla F = \lim_{n \rightarrow \infty} \nabla F_n}$.

We will denote by $D_{p,1}$ the linear space $\text{Dom}_p(\nabla)$ equipped with the norm $\|F\|_{p,1} = \|F\|_p + \|\nabla F\|_{L^p(\mu, H)}$.

Remarks: 1) If \mathcal{X} is a separable Hilbert space we can define $D_{p,1}(\mathcal{X})$ exactly in the same way as before, the only difference is that we take $\mathcal{S}_{\mathcal{X}}$ instead of \mathcal{S} , i.e., the rapidly decreasing functions with values in \mathcal{X} . Then the same closability result holds (exercise!).

2) Hence we can define $D_{p,k}$ by iteration:

- i) We say that $F \in D_{p,2}$ if $\nabla F \in D_{p,1}(H)$, then write $\nabla^2 F = \nabla(\nabla F)$.

ii) $F \in D_{p,k}$ if $\nabla^{k-1}F \in D_{p,1}(H^{\otimes(k-1)})$.

3) Note that, for $F \in D_{p,k}$, $\nabla^k F$ is in fact with values $H^{\widehat{\otimes}k}$ (i.e. symmetric tensor product).

4) From the proof we have that if $F \in D_{p,1}$, $h \in H_1$ and φ is cylindrical, we have

$$E[\nabla_h F \cdot \varphi] = -E[F \cdot \nabla_h \varphi] + E[I(h) \cdot F \cdot \varphi],$$

where $I(h)$ is the first order Wiener integral of the (Lebesgue) density of h . If $\varphi \in D_{q,1}$ ($q^{-1} + p^{-1} = 1$), by a limiting argument, the same relation holds again. Let us note that this limiting procedure shows in fact that if $\nabla F \in L^p(\mu, H)$ then $F \cdot I(h) \in L^p(\mu)$, i.e., F is more than p -integrable. This observation gives rise to the logarithmic Sobolev inequality.

1.1 Relations with the stochastic integration

Let $\varphi = f(W_{t_1}, \dots, W_{t_n})$, $t_i \leq t$, f smooth. Then we have

$$\nabla_h \varphi(w) = \sum_{i=1}^n \partial_i f(W_{t_1}, \dots, W_{t_n}) h(t_i),$$

hence $\nabla \varphi$ is again a random variable which is \mathcal{B}_t -measurable. In fact this property is satisfied by a larger class of Wiener functionals:

Proposition II.1 Let $\varphi \in D_{p,1}$, $p > 1$ and suppose that φ is \mathcal{B}_t -measurable for a given $t \geq 0$. Then $\nabla \varphi$ is also \mathcal{B}_t -measurable and furthermore, for any $h \in H_1$, whose support is in $[t, 1]$, $\nabla_h \varphi = (\nabla \varphi, h)_H = 0$ a.s.

Proof: Let (φ_n) be a sequence of cylindrical random variable converging to φ in $D_{p,1}$. If φ_n is of the form $f(W_{t_1}, \dots, W_{t_k})$, it is easy to see that, even if φ_n is not \mathcal{B}_t -measurable, $E[\varphi_n | \mathcal{B}_t]$ is another cylindrical random variable, say $\theta_n(W_{t_1 \wedge t}, \dots, W_{t_k \wedge t})$. In fact, suppose that $t_k > t$

and $t_1, \dots, t_{k-1} \leq t$. We have

$$\begin{aligned} E[f(W_{t_1}, \dots, W_{t_k}) | \mathcal{B}_t] &= E[f(W_{t_1}, \dots, W_{t_{k-1}}, W_{t_k} - W_t + W_t) | \mathcal{B}_t] \\ &= \int_{\mathbb{R}} f(W_{t_1}, \dots, W_{t_{k-1}}, W_t + x) p_{t_k-t}(x) dx \\ &= \theta(W_{t_1}, \dots, W_{t_{k-1}}, W_t), \end{aligned}$$

and $\theta \in \mathcal{S}$ if $f \in \mathcal{S}(\mathbb{R}^k)$, where p_t denotes the heat kernel. Hence we can choose a sequence (φ_n) converging to φ in $D_{p,1}$ such that $\nabla \varphi_n$ is \mathcal{B}_t -measurable for each $n \in \mathbb{N}$. Hence $\nabla \varphi$ is also \mathcal{B}_t -measurable.

If $h \in H_1$ has its support in $[t, 1]$, then, for each n , we have $\nabla_h \varphi_n = 0$ a.s., because $\nabla \varphi_n$ has its support in $[0, t]$ as one can see from the explicit calculation for $\nabla \varphi_n$. Taking an a.s. convergent subsequence, we see that $\nabla_h \varphi = 0$ a.s. also. QED.

Let now K be a step process:

$$K_t(w) = \sum_{i=1}^n a_i(w) 1_{]t_i, t_{i+1}]}(t)$$

where $a_i \in D_{p,1}$ and \mathcal{B}_{t_i} -measurable for any i . Then we have

$$\int_0^1 K_s dW_s = \sum_i a_i(W_{t_{i+1}} - W_{t_i})$$

and

$$\begin{aligned} \nabla_h \int_0^1 K_s dW_s &= \sum_1^n \nabla_h a_i(W_{t_{i+1}} - W_{t_i}) + a_i(h(t_{i+1}) - h(t_i)) \\ &= \int_0^1 \nabla_h K_s dW_s + \int_0^1 K_s \dot{h}(s) ds. \end{aligned}$$

Hence

$$\left| \nabla \int_0^1 K_s dW_s \right|_H^2 \leq 2 \left\{ \left| \int_0^1 \nabla K_s dW_s \right|_H^2 + \int_0^1 |K_s|^2 ds \right\}$$

and

$$E \left[\left(\left| \nabla \int_0^1 K_s dW_s \right|_H^2 \right)^{p/2} \right] \leq 2^p E \left[\left(\left| \int_0^1 \nabla K_s dW_s \right|_H^p \right) \right]$$

$$+ \int_0^1 |K_s|^2 ds \Big)^{p/2} \Big].$$

Using the Burkholder-Davis-Gundy inequality for the Hilbert space valued martingales, the above quantity is majorized by

$$\begin{aligned} & 2c_p E \left(\left[\left(\int_0^1 |\nabla K_s|_H^2 ds \right)^{p/2} \right] + E \left[\left(\int_0^1 |K_s|^2 ds \right)^{p/2} \right] \right) \\ &= \tilde{c}_p \|\nabla \tilde{K}\|_{L^p(\mu, H \otimes H)}^p + \|\tilde{K}\|_{L^p(\mu, H)}^p, \quad \text{where } \tilde{K} \cdot = \int_0^\cdot K_r dr. \end{aligned}$$

Thanks to this majoration, we have proved:

Proposition II.2 Let $\tilde{K} \in D_{p,1}(H)$ such that $K_t = \frac{d\tilde{K}(t)}{dt}$ be \mathcal{B}_t -measurable for almost all t . Then we have

$$\boxed{\nabla \int_0^1 K_s dW_s = \int_0^1 \nabla \cdot K_s dW_s + \tilde{K} \text{ a.s.}}$$

Corollary 1: If $\varphi = I_n(f_n)$, $f_n \in \hat{L}^2([0, 1]^n)$, then we have, for $h \in H_1$,

$$\nabla_h I_n(f_n) = n \int_{[0,1]^n} f(t_1, \dots, t_n) dW_{t_1}, \dots, dW_{t_{n-1}} \cdot \dot{h}(t_n) \cdot dt_n.$$

Proof: Apply the above proposition n -times to the case in which, first f_n is $C^\infty([0, 1]^n)$, then pass to the limit in $L^2(\mu)$. QED

The following result will be extended in the sequel to much larger classes of random variables:

Corollary 2: Let $\varphi : W \rightarrow \mathbb{R}$ be analytic in H -direction. Then we have

$$\varphi = E[\varphi] + \sum_{n=1}^{\infty} I_n \left(\frac{E[\nabla^n \varphi]}{n!} \right),$$

i.e., the kernel $\varphi_n \in \hat{L}^2[0, 1]^n$ of the Wiener chaos decomposition of φ is equal to

$$\frac{E[\nabla^n \varphi]}{n!}.$$

Proof: We have, on one hand, for any $h \in H$,

$$E[\varphi(w + h)] = E\left[\varphi \cdot \exp\left(\int_0^1 \dot{h}_s dW_s - \frac{1}{2} \int_0^1 \dot{h}_s^2 ds\right)\right] = E[\varphi \cdot \mathcal{E}(I(h))].$$

On the other hand, from Taylor's formula:

$$\begin{aligned} E[\varphi(w + h)] &= E[\varphi] + \sum_1^\infty E\left[\frac{(\nabla^n \varphi(w), h^{\otimes n})}{n!}\right] \\ &= E[\varphi] + \sum_1^\infty \frac{(E[\nabla^n \varphi], h^{\otimes n})}{n!} H^{\otimes n} \\ &= E[\varphi] + \sum_1^\infty \frac{1}{n!} \frac{E[I_n(E[\nabla^n \varphi]) \cdot I_n(h^{\otimes n})]}{n!} \\ &= E[\varphi] + \sum_1^\infty E\left[\frac{I_n(E[\nabla^n \varphi])}{n!} \frac{I_n(h^{\otimes n})}{n!}\right] \end{aligned}$$

hence, from the symmetry, we have

$$I_n(\varphi_n) = \frac{1}{n!} I_n(E[\nabla^n \varphi]),$$

where we have used the notation $I_1(h) = I(h) = \int_0^1 \dot{h}_s dW_s$ and

$$I_n(\varphi_n) = \int_{[0,1]^n} \frac{\partial^n \varphi_n}{\partial t_1, \dots, \partial t_n}(t_1, \dots, t_n) dW_{t_1}, \dots, dW_{t_n}.$$

QED

Definition II.1: Let $\xi : W \rightarrow H$ be a random variable. We say that $\xi \in \text{Dom}_p(\delta)$, if for any $\varphi \in D_{p,1}$ ($q^{-1} + p^{-1} = 1$), we have

$$E[(\nabla \varphi, \xi)_H] \leq c_{p,q}(\xi) \cdot \|\varphi\|_1,$$

and in this case we define $\delta \xi$ by

$$E[\delta \xi \cdot \varphi] = E[(\xi, \nabla \varphi)_H],$$

i.e., $\delta = \nabla^*$ with respect to the measure μ , it is called the divergence operator. Let us give some properties of it:

1.) Let $a : W \rightarrow \mathbb{R}$ be “smooth”, $\xi \in \text{Dom}_p(\delta)$. Then we have, for any $\varphi \in D_{q,1}$,

$$\begin{aligned} E[\delta(a\xi)\varphi] &= E[(a\xi, \nabla\varphi)] \\ &= E[(\xi, a\nabla\varphi)] \\ &= E[(\xi, \nabla(a\varphi) - \varphi.\nabla a)] \\ &= E[\delta\xi.a\varphi - \varphi.(\nabla a, \xi)], \end{aligned}$$

hence $\boxed{\delta(a\xi) = a\delta\xi - (\nabla a, \xi)}$.

2.) Let $h \in H_1$, then we pretend that $\boxed{\delta h = \int_0^1 \dot{h}(s)dW_s}$. To see this, it is sufficient to test this relation on the exponential martingales: if $k \in H_1$, we have

$$\begin{aligned} E[\delta h \cdot \exp \cdot \int_0^1 k_s dW_s - \frac{1}{2} \int_0^1 k_s^2 ds] &= \\ &= E[(h, \nabla \mathcal{E}(I(k))_{H_1})] \\ &= E[(h, k) \cdot \mathcal{E}(I(k))] \\ &= (h, k)_{H_1}. \end{aligned}$$

On the other hand, supposing first $h \in W^*$,

$$\begin{aligned} E[I(h) \cdot \mathcal{E}(I(k))] &= E[I(h)(w + k)] \\ &= E[I(h)] + (h, k)_{H_1} \\ &= (h, k)_{H_1}. \end{aligned}$$

Hence in particular, if we denote by $\tilde{1}_{[s,t]}$ the element of H such that $\tilde{1}_{[s,t]}(r) = \int_0^r 1_{[s,t]}(u)du$, we have that

$$\boxed{\delta(\tilde{1}_{[s,t]}) = W_t - W_s.}$$

3.) Let now K be a step process

$$K_t(v) = \sum_1^n a_i(w) \cdot 1_{[t_i, t_{i+1}[}(t),$$

where $a_i \in D_{p,1}$ and \mathcal{B}_{t_i} -measurable for each i . Let \tilde{K} be $\int_0^1 K_s ds$. Then from the property 1, we have

$$\delta \tilde{K} = \delta \left(\sum_1^n a_i \cdot \tilde{1}_{[t_i, t_{i+1}[} \right) = \sum_1^n \left\{ a_i \delta(\tilde{1}_{[t_i, t_{i+1}[}) - (\nabla a_i, \tilde{1}_{[t_i, t_{i+1}[}) \right\}.$$

From the property 2., we have $\delta(\tilde{1}_{[t_i, t_{i+1}[}) = W_{t_{i+1}} - W_{t_i}$, furthermore, from the proposition II.1, the support of ∇a_i is in $[0, t_i]$, consequently, we obtain

$$\delta \tilde{K} = \sum_{i=1}^n a_i (W_{t_{i+1}} - W_{t_i}) = \int_0^1 K_s dW_s.$$

Hence we have the important result which says, with some abuse of notation that

Theorem II.1: $\text{Dom}_p(\delta)$ ($p > 1$) contains the adapted stochastic processes (in fact their primitives) such that

$$E \left[\left(\int_0^1 K_s^2 ds \right)^{p/2} \right] < \infty$$

and on this class δ coincides with the Ito stochastic integral.

Remark: To be translated as: the stochastic integral of K is being equal to the divergence of \tilde{K} !

We will come back to the notion of divergence later.

2 The Ornstein-Uhlenbeck Operator

For a nice function f on W , $t \geq 0$, we define

$$P_t f(x) = \int_W f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \mu(dy).$$

Since $\mu(dx)\mu(dy)$ is invariant under the rotations of $W \times W$, i.e., $(\mu \times \mu)(dx, dy)$ is invariant under the transformation

$$T_t(x, y) = (e^{-t}x + y\sqrt{1 - e^{-2t}}, y\sqrt{1 - e^{-2t}} - xe^{-t}),$$

we have obviously

$$\begin{aligned} \|P_t f(x)\|_{L^p(\mu)}^p &\leq \iint |(f \otimes 1)(T_t(x, y))|^p \mu(dx) \mu(dy) \\ &= \iint |(f \otimes 1)(x, y)|^p \mu(dx) \mu(dy) \\ &= \int |f(x)|^p \mu(dx), \end{aligned}$$

for any $p \geq 1$, $\|P_t f\|_{L^p} \leq \|f\|_{L^p}$; hence also for $p = \infty$ by duality. A straightforward calculation gives that, for any $h \in H \cap W^* (= W^*)$,

$$\begin{aligned} P_t(\mathcal{E}(I(h))) &= \mathcal{E}(e^{-t} I(h)) \\ &= \sum_{n=0}^{\infty} e^{-nt} \frac{I_n(h^{\otimes n})}{n!}. \end{aligned}$$

Hence, by homogeneity, we have

$$P_t(I_n(h^{\otimes n})) = e^{-nt} I_n(h^{\otimes n})$$

and by density, we obtain

$$P_t I_n(f_n) = e^{-nt} I_n(f_n),$$

for any $f_n \in \hat{L}^2([0, 1]^n)$. Consequently $P_s \circ P_t = P_{s+t}$, i.e., (P_t) is a measure preserving Markov semi-group. Its infinitesimal generator is denoted by \mathcal{L} and is called the Ornstein-Uhlenbeck or the number operator. Evidently, we have $\mathcal{L} I_n(f_n) = n J_n(f_n)$; i.e., the Wiener chaos are its eigenspace. From the definition, it follows directly that (for a_i being \mathcal{F}_{t_i} -measurable)

$$P_t(\sum a_i(W_{t_{i+1}} - W_{t_i})) = e^{-t} \sum (P_t a_i)(W_{t_{i+1}} - W_{t_i}),$$

that is to say

$$P_t \int_0^1 H_s dW_s = e^{-t} \int_0^1 P_t H_s dW_s,$$

and by differentiation

$$\mathcal{L} \int_0^1 H_s dW_s = \int_0^1 (I + \mathcal{L}) H_s dW_s.$$

Also we have

$$\nabla P_t \varphi = e^{-t} P_t \nabla \varphi.$$

Lemma: We have $\delta \circ \nabla = \mathcal{L}$.

Proof: Let $\varphi = \mathcal{E}(I(h))$, then

$$\begin{aligned} (\delta \circ \nabla)\varphi &= \delta(h \cdot \mathcal{E}(I(h))) \\ &= (I(h) - |h|^2)\mathcal{E}(I(h)) \\ &= \mathcal{L}\mathcal{E}(I(h)) \end{aligned}$$

QED

Let us define for the smooth functions φ , a semi-norm

$$\|\varphi\|_{p,k} = \|(I + \mathcal{L})^{k/2}\varphi\|_{L^p(\mu)}.$$

At first glance, these semi-norms (in fact norms), seem different from the one define by $\|\varphi\|_{p,k} = \sum_0^k \|\nabla^j \varphi\|_{L^p(\mu, H^{\otimes j})}$. We will show that they are equivalent. Before that we need

Proposition We have the following identity:

$$\delta \circ \nabla = \mathcal{L}.$$

Proof: It is sufficient to prove, for the moment that result, on the exponential martingales; if $h \in H_1$,

$$\begin{aligned} \mathcal{L}\mathcal{E}(I(h)) &= -\frac{dP_t}{dt}\mathcal{E}(I(h))\Big|_{t=0} \\ &= -\frac{d}{dt}\mathcal{E}(e^{-t}I(h))\Big|_{t=0} \\ &= (e^{-t}I(h)) - e^{-2t}|h|_{H_1}^2)\mathcal{E}(e^{-t}I(h))\Big|_{t=0} \\ &= (I(h) - |h|^2)\mathcal{E}(I(h)). \end{aligned}$$

On the other hand:

$$\nabla\mathcal{E}(I(h)) = h \cdot \mathcal{E}(I(h))$$

and

$$\begin{aligned}\delta(\nabla\mathcal{E}(I(h))) &= \delta(h \cdot \mathcal{E}(I(h))) \\ &= \delta h \cdot \mathcal{E}(I(h)) - (\nabla\mathcal{E}(I(h)), h) \\ &= \delta h \mathcal{E}(I(h)) - |h|^2 \mathcal{E}(I(h)).\end{aligned}$$

QED

Chapter II

Meyer Inequalities

Meyer Inequalities and Distributions

Meyer inequalities are essential to control the Sobolev norms defined with the Sobolev derivative with the norms defined via the Ornstein-Uhlenbeck operator. The key point is the continuity property of the Riesz transform on $L^p([0, 2\pi], dx)$, i.e., from a totally analytic origin, although the original proof of P. A. Meyer was probabilistic (cf. [5]). Here we develop the proof suggested by [3].

1 Some Preparations

Let f be a function on $[0, 2\pi]$, extended to the whole \mathbb{R} by periodicity. We denote by $\tilde{f}(x)$ the function defined by

$$\tilde{f}(x) = \frac{1}{\pi} p.v. \int_0^\pi \frac{f(x+t) - f(x-t)}{2 \tan t/2} dt \quad (\text{principal value}).$$

then the famous theorem of M. Riesz, cf. [21], asserts that, for any $f \in L^p[0, 2\pi]$, $\tilde{f} \in L^p([0, 2\pi])$, for $1 < p < \infty$ with

$$\|\tilde{f}\|_p \leq A_p \|f\|_p,$$

where A_p is a constant depending only on p . Most of the classical functional analysis of the 20-th century has been devoted to extend

this result to the case where the function f was taking its values in more abstract spaces than the real line. We will show that our problem also can be reduced to this one.

In fact, the main result that we are going to show will be that

$$\|\nabla(I + \mathcal{L})^{-1/2}\varphi\|_p \approx \|\varphi\|_p$$

by rewriting $\nabla(I + \mathcal{L})^{-1/2}$ as an $L^p(\mu, H)$ -valued Riesz transform. For this we need first, the following elementary

Lemma 1: Let K be any function on $[0, 2\pi]$ such that

$$K(\theta) - \frac{1}{2} \cot \frac{\theta}{2} \in L^\infty([0, \pi]),$$

then the operator $f \rightarrow T_K f$ defined by

$$T_K f(x) = \frac{1}{\pi} p.v. \int_0^\pi (f(x+t) - f(x-t))K(t)dt$$

is again a bounded operator on $L^p([0, 2\pi])$ with

$$\|T_K f\|_p \leq B_p \|f\|_p \quad \text{for any } p \in (1, \infty)$$

where B_p depends only on p .

Proof: In fact we have

$$\begin{aligned} |T_K f - \tilde{f}|(x) &\leq \frac{1}{\pi} \int_0^\pi |f(x+t) - f(x-t)| |K(t) - \frac{1}{2} \cot \frac{\theta}{2}| dt \\ &\leq c \|f\|_{L^p} \|K - \frac{1}{2} \cot \frac{\theta}{2}\|_{L^\infty}. \end{aligned}$$

Hence $\|T_K f\|_p \leq (c \|K - \frac{1}{2} \cot \frac{\theta}{2}\|_{L^\infty} + A_p) \|f\|_p$. QED

Remark: If for some $a \neq 0$, $aK(\theta) - \frac{1}{2} \cot \frac{\theta}{2} \in L^\infty([0, 2\pi])$, then we have

$$\begin{aligned} \|T_K f\|_p &= \frac{1}{|a|} \|aT_K f\|_p \leq \frac{1}{|a|} \|aT_K f - \tilde{f}\|_p + \frac{1}{|a|} \|\tilde{f}\|_p \\ &\leq \frac{1}{|a|} \|aK - \frac{1}{2} \cot \frac{\theta}{2}\|_{L^\infty} \|f\|_p + \frac{A_p}{|a|} \|f\|_p \\ &\leq c_p \|f\|_p \end{aligned}$$

with another constant c_p .

Corollary: Let K be a function on $[0, \pi]$ such that $K = 0$ on $\left[\frac{\pi}{2}, \pi\right]$ and $K - \frac{1}{2} \cot \frac{\theta}{2} \in L^\infty\left(\left[0, \frac{\pi}{2}\right]\right)$. Then T_K defined by

$$T_K f(x) = \int_0^{\pi/2} (f(x+t) - f(x-t))K(t)dt$$

is continuous from $L^p([0, 2\pi])$ into itself for any $p \in [1, \infty[$.

Proof: We have $cK(\theta)1_{[0, \frac{\pi}{2}]} - \frac{1}{2} \cot \frac{\theta}{2} \in L^\infty([0, \pi])$ since on the interval $\left[\frac{\pi}{2}, \pi\right]$, $\sin \frac{\theta}{2} \in \left[\frac{\sqrt{2}}{2}, 1\right]$, then the result follows from the Lemma. QED

2 $\nabla(I + \mathcal{L})^{-1/2}$ as the Riesz Transform

Let us denote by $R_\theta(x, y)$ the rotation on $W \times W$ defined by

$$R_\theta(x, y) = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta).$$

Note that $R_\theta \circ R_\phi = R_{\phi+\theta}$. We have also, putting $e^{-t} = \cos \theta$,

$$\begin{aligned} P_t f(x) &= \int_W f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\mu(dy) \\ &= \int_W (f \otimes 1)(R_\theta(x, y))\mu(dy) = P_{-\log \cos \theta} f(x). \end{aligned}$$

Let us now calculate $(I + \mathcal{L})^{-1/2}\varphi$ using this transformation:

$$\begin{aligned} (I + \mathcal{L})^{-1/2}\varphi(x) &= \int_0^\infty t^{-1/2} e^{-t} P_t \varphi(x) dt \\ &= \int_0^{\pi/2} (-\log \cos \theta)^{-1/2} \cos \theta \cdot \int_W (\varphi \otimes 1)(R_\theta(x, y))\mu(dy) \tan \theta d\theta \\ &= \int_W \mu(dy) \left[\int_0^{\pi/2} (-\log \cos \theta)^{-1/2} \sin \theta (\varphi \otimes 1)(R_\theta(x, y)) d\theta \right]. \end{aligned}$$

On the other hand, we have, for $h \in H_1$ (even in $C_0^\infty([0, 1])$)

$$\begin{aligned}
& \nabla_h P_t \varphi(x) \\
&= \frac{d}{d\lambda} P_t \varphi(x + \lambda h) |_{\lambda=0} \\
&= \frac{d}{d\lambda} \int \varphi(e^{-t}(x + \lambda h) + \sqrt{1 - e^{-2t}} y) \mu(dy) |_{\lambda=0} \\
&= \frac{d}{d\lambda} \int \varphi\left(e^{-t}x + \sqrt{1 - e^{-2t}}\left(y + \frac{\lambda e^{-t}}{\sqrt{1 - e^{-2t}}}h\right)\right) \mu(dy) |_{\lambda=0} \\
&= \frac{d}{d\lambda} \int \varphi(e^{-t}x + \sqrt{1 - e^{-2t}} y) \mathcal{E}\left(\frac{\lambda e^{-t}}{\sqrt{1 - e^{-2t}}} I(h)\right)(y) \mu(dy) |_{\lambda=0} \\
&= \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_W \varphi(e^{-t}x + \sqrt{1 - e^{-2t}} y) \delta h(y) \mu(dy).
\end{aligned}$$

Therefore

$$\begin{aligned}
& \nabla_h (I + \mathcal{L})^{-1/2} \varphi(x) \\
&= \int_0^\infty t^{-1/2} e^{-t} \nabla_h P_t \varphi(x) dt \\
&= \int_0^\infty t^{-1/2} \frac{e^{-2t}}{\sqrt{1 - e^{-2t}}} \int_W \delta h(y) \varphi(e^{-t}x + \sqrt{1 - e^{-2t}} y) \mu(dy) dt \\
&= \int_0^{\pi/2} (-\log \cos \theta)^{-1/2} \frac{\cos^2 \theta}{\sin \theta} \tan \theta \int \delta h(y) (\varphi \otimes 1)(R_\theta(x, y)) \mu(dy) d\theta \\
&= \int_0^{\pi/2} (-\log \cos \theta)^{-1/2} \cos \theta \int_W \delta h(y) \cdot (\varphi \otimes 1)(R_\theta(x, y)) \mu(dy) d\theta
\end{aligned}$$

Since $\mu(dy)$ is invariant under the transformation $y \mapsto -y$, we have

$$\int \delta h(y) (\varphi \otimes 1)(R_\theta(x, y)) \mu(dy) = - \int \delta h(y) (\varphi \otimes 1)(R_{-\theta}(x, y)) \mu(dy),$$

therefore:

$$\begin{aligned}
& \nabla_h(I + \mathcal{L})^{-1/2}\varphi(x) \\
&= \int_0^{\pi/2} (-\log \cos \theta)^{-1/2} \cdot \\
&\quad \int \delta h(y) \frac{(\varphi \otimes 1)(R_\theta(x, y)) - (\varphi \otimes 1)(R_{-\theta}(x, y))}{2} \mu(dy) d\theta \\
&= \int_W \delta h(y) \int_0^{\pi/2} K(\theta) ((\varphi \otimes 1)(R_\theta(x, y)) - (\varphi \otimes 1)(R_{-\theta}(x, y))) d\theta \mu(dy),
\end{aligned}$$

where $K(\theta) = \frac{1}{2} \cos \theta (-\log \cos \theta)^{-1/2}$.

Lemma 2: We have $2K(\theta) - \cot \frac{\theta}{2} \in L^\infty(0, \frac{\pi}{2}]$.

Proof: The only problem is when $\theta \rightarrow 0$. To see this let us put $e^{-t} = \cos \theta$, then $\cot \frac{\theta}{2} = \frac{\sqrt{1+e^{-t}}}{\sqrt{1-e^{-t}}} \approx \frac{2}{\sqrt{t}}$ and

$$K(\theta) = \frac{e^{-t}}{\sqrt{t}} \approx \frac{1}{\sqrt{t}}$$

hence $2K(\theta) - \cot \frac{\theta}{2} \in L^\infty([0, \frac{\pi}{2}])$.

QED

Using Lemma 1, the remark following it and the corollary, we see that the map $f \mapsto p.v. \int_0^{\pi/2} (f(x + \theta) - f(x - \theta))K(\theta)d\theta$ is a bounded map from $L^p[0, \pi]$ into itself. Moreover

Lemma 3: Let $F : W \times W \rightarrow \mathbb{R}$ be a measurable, bounded function. Define $TF(x, y)$ as

$$TF(x, y) = p.v. \int_0^{\pi/2} (F \circ R_\theta(x, y) - F \circ R_{-\theta}(x, y))K(\theta)d\theta.$$

Then, for any $p > 1$, there exists some $c_p > 0$ such that

$$\|TF\|_{L^p(\mu \times \mu)} \leq c_p \|F\|_{L^p(\mu \times \mu)}.$$

Proof: We have

$$(TF)(R_\beta(x, y)) = p.v. \int_0^{\pi/2} (F(R_{\beta+\theta}(x, y)) - F(R_{\beta-\theta}(x, y)))K(\theta)d\theta,$$

this is the Riesz transform for fixed $(x, y) \in W \times W$, hence we have

$$\int_0^{\pi/2} |TF(R_\beta(x, y))|^p d\beta \leq c_p \int_0^\pi |F(R_\beta(x, y))|^p d\beta,$$

taking the expectation with respect to $\mu \times \mu$, which is invariant under R_β , we have

$$\begin{aligned} E_{\mu \times \mu} \int_0^\pi |TF(R_\beta(x, y))|^p d\beta &= E_{\mu \times \mu} \int_0^\pi |TF(x, y)|^p d\beta \\ &= \frac{\pi}{2} E[|TF|^p] \\ &\leq c_p E \int_0^\pi |F(R_\beta(x, y))|^p d\beta \\ &= \pi c_p E[|F|^p]. \end{aligned}$$

QED

We have

Theorem 1: $\nabla \circ (I + \mathcal{L})^{-1/2} : L^p(\mu) \rightarrow L^p(\mu, H)$ is continuous for any $p > 1$.

Proof: With the notations of Lemma 3, we have

$$\nabla_h(I + \mathcal{L})^{-1/2}\varphi = \int_W \delta h(y) T(\varphi \otimes 1)(x, y)\mu(dy).$$

From Schwarz inequality:

$$|\nabla(I + \mathcal{L})^{-1/2}\varphi|_H^2 \leq \int_W |T(\varphi \otimes 1)(x, y)|^2 \mu(dy)$$

hence, for $p \geq 2$,

$$\begin{aligned} E[|\nabla(I + \mathcal{L})^{-1/2}\varphi|_H^p] &\leq E\left[\left(\int_W |T(\varphi \otimes 1)(x, y)|^2 \mu(dy)\right)^{p/2}\right] \\ &\leq E \int_W |T(\varphi \otimes 1)(x, y)|^p \mu(dy) \\ &\leq \iint |(\varphi \otimes 1)(x, y)|^p \mu(dy) \mu(dx) = \|\varphi\|_{L^p(\mu)}^p. \end{aligned}$$

For the case $1 < q < 2$, let φ and ψ be smooth (i.e., cylindrical), since $\delta \circ \nabla = \mathcal{L}$, we have, for $p^{-1} + q^{-1} = 1$ (hence $p > 2$!):

$$\begin{aligned} E[\varphi\psi] &= \\ &E[(\nabla(I + \mathcal{L})^{-1/2}\varphi, \nabla(I + \mathcal{L})^{-1/2}\psi)] \\ &+ E[(I + \mathcal{L})^{-1/2}\varphi \cdot (I + \mathcal{L})^{-1/2}\psi], \end{aligned}$$

hence

$$E[(\nabla(I + \mathcal{L})^{-1/2}\varphi, \nabla(I + \mathcal{L})^{-1/2}\psi)_H] = E[\varphi\psi] - E[(I + \mathcal{L})^{-1}\varphi \cdot \psi].$$

Since $(I + \mathcal{L})^{-1}$ is continuous on $L^p(\mu)$ (it is a contraction), we have

$$\sup_{\|\varphi\|_p \leq 1} |E[(\nabla(I + \mathcal{L})^{-1/2}\varphi, \nabla(I + \mathcal{L})^{-1/2}\psi)_H]| \leq c\|\psi\|_q,$$

hence $\|\nabla(I + \mathcal{L})^{-1/2}\psi\|_q \leq \tilde{c}_q\|\psi\|_q$. QED

Corollary 1: We have

$$\|(I + \mathcal{L})^{-1/2}\delta\xi\|_p \leq c_p\|\xi\|_p,$$

for any $\xi \in L^p(\mu; H)$ for $p \in]1, \infty[$.

Proof: Just take the adjoint of $\nabla(I + \mathcal{L})^{-1/2}$. QED

Corollary 2: We have

- i) $\|\nabla\varphi\|_p \leq c_p\|(I + \mathcal{L})^{1/2}\varphi\|_p$
- ii) $\|(I + \mathcal{L})^{1/2}\varphi\|_p \leq \tilde{c}_p(\|\varphi\|_p + \|\nabla\varphi\|_p)$.

Proof:

$$\text{i) } \|\nabla\varphi\|_p = \|\nabla(I + \mathcal{L})^{-1/2}(I + \mathcal{L})^{1/2}\varphi\|_p \leq c_p\|(I + \mathcal{L})^{1/2}\varphi\|_p.$$

$$\begin{aligned} \text{ii) } \|(I + \mathcal{L})^{1/2}\varphi\|_p &= \|(I + \mathcal{L})^{-1/2}(I + \mathcal{L})\varphi\|_p \\ &= \|(I + \mathcal{L})^{-1/2}(I + \delta\nabla)\varphi\|_p \\ &\leq \|(I + \mathcal{L})^{-1/2}\varphi\|_p + \|(I + \mathcal{L})^{-1/2}\delta\nabla\varphi\|_p \\ &\leq \|\varphi\|_p + c_p\|\nabla\varphi\|_p \quad (\text{from Corollary 1}). \end{aligned}$$

QED

Chapter III

Hypercontractivity

Hypercontractivity

We know that the semi-group of Ornstein-Uhlenbeck is a bounded operator on $L^p(\mu)$, for any $p \in [1, \infty]$. In fact for $p \in]1, \infty[$, it is more than bounded. It increases the degree of integrability, this property is called *hypercontractivity*. It has been first discovered by E. Nelson, here we follow the proof given by [6].

In the sequel we shall show that this result can be proved using the Ito formula. Let (Ω, \mathcal{A}, P) be a probability space with $(\mathcal{B}_t; t \in \mathbb{R}_+)$ being a filtration. We take two Brownian motions $(X_t; t \geq 0)$ and $(Y_t; t \geq 0)$ which are not necessarily independent, i.e., X and Y are two continuous, real martingales such that $(X_t^2 - t)$ and $(Y_t^2 - t)$ are again martingales (with respect to (\mathcal{B}_t)) and that $X_t - X_s$ and $Y_t - Y_s$ are independent of \mathcal{B}_s , for $t > s$. Moreover there exists $(\rho_t; t \in \mathbb{R}_+)$, progressively measurable with values in $[-1, 1]$ such that

$$(X_t Y_t - \int_0^t \rho_s ds, t \geq 0)$$

is again a (\mathcal{B}_t) -martingale. Let us denote by

$$\mathcal{X}_t = \sigma(X_s; s \leq t), \quad \mathcal{Y}_t = \sigma(Y_s; s \leq t)$$

i. e., the corresponding filtrations of X and Y and by \mathcal{X} and by \mathcal{Y} their respective supremum.

Lemma 1: 1) For any $\varphi \in L^1(\Omega, \mathcal{X}, P)$, $t \geq 0$, we have

$$E[\varphi|\mathcal{B}_t] = E[\varphi|\mathcal{X}_t] \text{ a.s.}$$

2) For any $\psi \in L^1(\Omega, \mathcal{Y}, P)$, $t \geq 0$, we have

$$E[\psi|\mathcal{B}_t] = E[\psi|\mathcal{Y}_t] \text{ a.s.}$$

Proof: 1) From Lévy's theorem, we have also that (X_t) is an (\mathcal{X}_t) -Brownian motion. Hence

$$\varphi = E[\varphi] + \int_0^\infty H_s dX_s$$

where H is (\mathcal{X}_t) -adapted process. Hence

$$E[\varphi|\mathcal{B}_t] = E[\varphi] + \int_0^t H_s dX_s = E[\varphi|\mathcal{X}_t].$$

QED

Let us look at the operator $T : L^1(\Omega, \mathcal{X}, P) \rightarrow L^1(\Omega, \mathcal{Y}, P)$ which is the restriction of $E[\cdot|\mathcal{Y}]$ to the space $L^1(\Omega, \mathcal{X}, P)$. We know that $T : L^p(\mathcal{X}) \rightarrow L^p(\mathcal{Y})$ is a contraction for any $p \geq 1$. In fact, if we impose some conditions to ρ , then we have more:

Proposition 1: If $|\rho_t(w)| \leq r$ ($dt \times dP$ a.s.) for some $r \in [0, 1]$, then $T : L^p(\mathcal{X}) \rightarrow L^q(\mathcal{Y})$ is a bounded operator, where

$$p - 1 \geq r^2(q - 1).$$

Proof: $p = 1$ is already known. So suppose $p, q \in]1, \infty[$. Since $L^\infty(\mathcal{X})$ is dense in $L^p(\mathcal{X})$, it is enough to prove that $\|TF\|_q \leq \|F\|_p$ for any $F \in L^\infty(\mathcal{X})$. Moreover, since T is a positive operator, we have $|T(F)| \leq T(|F|)$, hence we can work as well with $F \in L_+^\infty(\mathcal{X})$.

From the duality between L^p -spaces, we have to show that

$$E[T(F)G] \leq \|F\|_p \|G\|_{q'}, \quad \left(\frac{1}{q'} + \frac{1}{q} = 1 \right),$$

for any $F \in L_+^\infty(\mathcal{X})$, $G \in L_+^\infty(\mathcal{Y})$. Moreover, we can suppose without loss of generality that $F, G \in [a, b]$ a.s. where $0 < a < b < \infty$ (since such random variables are total in all L_+^p -spaces, i.e., they separate L_+^p for any $p > 1$).

Let

$$M_t = E[F^p | \mathcal{X}_t]$$

$$N_t = E[G^{q'} | \mathcal{Y}_t].$$

Then, from the Ito representation theorem we have

$$M_t = M_0 + \int_0^t \phi_s dX_s$$

$$N_t = N_0 + \int_0^t \psi_s dY_s$$

where ϕ is \mathcal{X} -adapted, ψ is \mathcal{Y} -adapted, $M_0 = E[F^p]$, $N_0 = E[G^{q'}]$. From the Ito formula, we have

$$\begin{aligned} M_t^\alpha N_t^\beta &= M_0^\alpha N_0^\beta + \int_0^t \alpha M_s^{\alpha-1} N_s^\beta dM_s + \beta \int_0^t M_s^\alpha N_s^{\beta-1} dN_s + \\ &+ \frac{1}{2} \int_0^t M_s^\alpha N_s^\beta A_s ds \end{aligned}$$

where

$$A_t = \alpha(\alpha-1) \left(\frac{\phi_t}{M_t} \right)^2 + 2\alpha\beta \frac{\phi_t}{M_t} \frac{\psi_t}{N_t} \rho_t + \beta(\beta-1) \left(\frac{\psi_t}{N_t} \right)^2$$

and $\alpha = \frac{1}{p}$, $\beta = \frac{1}{q'}$.

[To see this we have

$$\begin{aligned} M_t^\alpha &= M_0^\alpha + \alpha \int_0^t M_s^{\alpha-1} \phi_s dX_s + \frac{\alpha(\alpha-1)}{2} \int_0^t M_s^{\alpha-2} \phi_s^2 ds \\ N_t^\beta &= \dots \end{aligned}$$

hence

$$\begin{aligned}
& M_t^\alpha N_t^\beta - M_0^\alpha N_0^\beta \\
&= \int_0^t M_s^\alpha dN_s^\beta + \int_0^t N_s^\beta dM_s^\alpha + \alpha\beta \int_0^t M_s^{\alpha-1} N_s^{\beta-1} \phi_s \psi_s \rho_s ds \\
&= \int_0^t M_s^\alpha (\beta N_s^{\beta-1} \psi_s dY_s + \frac{\beta(\beta-1)}{2} N_s^{\beta-2} \psi_s^2 ds) \\
&\quad + \int_0^t N_s^\beta (\alpha M_s^{\alpha-1} \phi_s dX_s + \frac{\alpha(\alpha-1)}{2} M_s^{\alpha-2} \phi_s^2 ds) \\
&\quad + \alpha\beta \int_0^t M_s^{\alpha-1} N_s^{\beta-1} \phi_s \psi_s \rho_s ds
\end{aligned}$$

then put together all terms with “ ds ”.]

As everything is square integrable, we have

$$\begin{aligned}
E[M_\infty^\alpha N_\infty^\beta] &= E\left[E[X^p|\mathcal{X}_\infty]^\alpha \cdot E[Y^{q'}|\mathcal{Y}_\infty]^\beta\right] \\
&= E[X \cdot Y] \\
&= \frac{1}{2} \int_0^\infty E[N_t^\beta M_t^\alpha A_t] dt + EM_0^\alpha N_0^\beta \\
&= E[X^p]^\alpha E[Y^{q'}]^\beta + \frac{1}{2} \int_0^\infty E[M_t^\alpha N_t^\beta A_t] dt.
\end{aligned}$$

Hence

$$E[XY] - \|X\|_p \|Y\|_{q'} = \frac{1}{2} \int_0^\infty E[M_t^\alpha N_t^\beta A_t] dt$$

Now look at A_t as a polynomial of second degree with respect to $\frac{\phi}{M}$. Then $\frac{\delta}{4} = \alpha^2 \beta^2 \rho_t^2 - \alpha(\alpha-1)\beta(\beta-1)$. If $\delta \leq 0$, then the sign of A_t is same as the sign of $\alpha(\alpha-1) \leq 0$, i.e., if $\rho_t^2 \leq \frac{(\alpha-1)(\beta-1)}{\alpha\beta} = \left(1 - \frac{1}{\alpha}\right)\left(1 - \frac{1}{\beta}\right) = (p-1)(q'-1)$ a.s., then we obtain

$$E[XY] = E[T(X)Y] \leq \|X\|_p \|Y\|_{q'}.$$

QED

Lemma Let $(w, z) = W \times W$ be independent Brownian paths. For $\rho \in [0, 1]$, define $x = \rho w + \sqrt{1 - \rho^2} z$, \mathcal{X}_∞ the σ -algebra associated to the paths x . Then we have

$$E[F(w)|\mathcal{X}_\infty] = \int_W F(\rho x + \sqrt{1 - \rho^2} z) \mu(dz).$$

Proof: For any $G \in L^\infty(\mathcal{X}_\infty)$, we have

$$\begin{aligned} E[F(w) \cdot G(x)] &= E[F(w)G(\rho w + \sqrt{1 - \rho^2} z)] \\ &= E[F(\rho w + \sqrt{1 - \rho^2} z)G(w)] \\ &= \iint F(\rho \bar{w} + \sqrt{1 - \rho^2} \bar{z})G(\bar{w}) \cdot \mu(d\bar{w})\mu(d\bar{z}) \\ &= E[G(x) \int F(\rho x + \sqrt{1 - \rho^2} \bar{z}) \cdot \mu(d\bar{z})] \end{aligned}$$

where \bar{w}, \bar{z} represent the dummy variables of integration. QED

Corollary 1: Under the hypothesis of the above lemma, we have

$$\left\| \int_W F(\rho x + \sqrt{1 - \rho^2} \bar{z}) \mu(d\bar{z}) \right\|_{L^q(d\mu)} \leq \|F\|_{L^p}$$

for any $(p - 1) \geq \rho^2(q - 1)$.

Chapter IV

L^p -Multipliers Theorem, Meyer Inequalities and Distributions

1 L^p -Multipliers Theorem

First let us give some applications of the hypercontractivity:

Theorem 1: Let $F \in L^p(\mu)$ and $F = \sum_n I_n(F_n)$ its Wiener chaos development. Then the map $F \rightarrow I_n(F_n)$ is continuous on $L^p(\mu)$.

Proof: Suppose first $p > 2$. Let t be such that $p = e^{2t} + 1$, then we have

$$\|P_t F\|_p \leq \|F\|_2.$$

Moreover

$$\|P_t I_n(F_n)\|_p \leq \|I_n(F_n)\|_2 \leq \|F\|_2 \leq \|F\|_p$$

but $P_t I_n(F_n) = e^{-nt} I_n(F_n)$, hence

$$\|I_n(F_n)\|_p \leq e^{nt} \|F\|_p.$$

For $1 < p < 2$ we use the duality: let $F \rightarrow I_n(F_n) = J_n(F)$. Then

$$\begin{aligned} \|I_n(F)\|_p &= \sup_{\|G\|_q \leq 1} |\langle G, J_n(F) \rangle| = \sup |\langle J_n(G), F \rangle| \\ &= \sup |\langle J_n G, J_n F \rangle| \leq \sup e^{nt} \|G\|_q \|F\|_p \\ &= e^{nt} \|F\|_p. \end{aligned}$$

QED

Proposition Let $h(x) = \sum_0^\infty a_k x^k$ be an analytic function around the origin with $\sum_k |a_k| \left(\frac{1}{n^\alpha}\right)^k < +\infty$ for $n \geq n_0$, for some $n_0 \in \mathbb{N}$. Let $\phi(x) = h(x^{-\alpha})$ and define T_ϕ on $L^p(\mu)$ as

$$T_\phi F = \sum_{n=0}^{\infty} \phi(n) I_n(F_n).$$

Then the operator T_ϕ is bounded on $L^p(\mu)$ for any $p > 1$.

Proof: Suppose first $\alpha = 1$. Let $T_\phi = T_1 + T_2$ where

$$T_1 F = \sum_{n=0}^{n_0-1} \phi(n) I_n(F_n), \quad T_2 F = (I - T_1)F.$$

From the hypercontractivity, $F \mapsto T_1 F$ is continuous on $L^p(\mu)$. Let $\delta_{n_0} F = \sum_{n=n_0}^\infty I_n(F_n)$. Since $(I - \delta_{n_0})(F) = \sum_0^{n_0-1} I_n(F_n)$, $\delta_{n_0} : L^p \rightarrow L^p$ is continuous, hence $P_t \delta_{n_0} : L^p \rightarrow L^p$ is also continuous. Apply Riesz-Thorin interpolation theorem, which says that if $P_t \delta_{n_0}$ is $L^q \rightarrow L^q$ and $L^2 \rightarrow L^2$ then it is $L^p \rightarrow L^p$ for any p such that $\frac{1}{p}$ is in the interval $\left[\frac{1}{q}, \frac{1}{2}\right]$, in fact we have

$$\|P_t \delta_{n_0}\|_{p,p} \leq \|P_t \delta_{n_0}\|_{2,2}^\theta \|P_t \delta_{n_0}\|_{q,q}^{1-\theta} \leq \|P_t \delta_{n_0}\|_{2,2}^\theta \|\delta_{n_0}\|_{q,q}^{1-\theta}$$

where $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q}$, $\theta \in]0, 1[$. Choose q large enough such that $\theta \approx 1$ (if necessary). Hence we have

$$\|P_t \delta_{n_0}\|_{p,p} \leq e^{-n_0 t \theta} K, \quad K = K(n_0, \theta).$$

Similar argument holds for $p \in]1, 2[$ by duality.

We have

$$\begin{aligned}
T_2(F) &= \sum_{n \geq n_0} \phi(n) I_n(F_n) = \\
&= \sum_{n \geq n_0} \left(\sum_k a_k \left(\frac{1}{n} \right)^k \right) I_n(F_n) \\
&= \sum_k a_k \sum_{n \geq n_0} \left(\frac{1}{n} \right)^k I_n(F_n) \\
&= \sum_k a_k \sum_{n \geq n_0} \mathcal{L}^{-k} I_n(F_n) \\
&= \sum_k a_k \mathcal{L}^{-k} \delta_{n_0} F.
\end{aligned}$$

We also have

$$\begin{aligned}
\|\mathcal{L}^{-1} \delta_{n_0} F\|_p &= \left\| \int_0^\infty P_t \delta_{n_0} F dt \right\|_p \leq K \int_0^\infty e^{-n_0 \theta t} \|F\|_p dt \leq K \cdot \frac{\|F\|_p}{n_0 \theta} \\
\|\mathcal{L}^{-2} \delta_{n_0} F\|_p &= \left\| \int_0^\infty \int_0^\infty P_{t+s} \delta_{n_0} F dt ds \right\|_p \leq K \cdot \frac{\|F\|_p}{(n_0 \theta)^2}, \\
&\dots \\
\|\mathcal{L}^{-k} \delta_{n_0} F\|_p &\leq K \|F\|_p \frac{1}{(n_0 \theta)^k}.
\end{aligned}$$

Therefore

$$\|T_2(F)\|_p \leq \sum_k K \|F\|_p \frac{1}{n_0^k \theta^k} \cong \sum K \|F\|_p \frac{1}{n_0^k}$$

by the hypothesis (take $n_0 + 1$ instead of n_0 if necessary).

For the case $\alpha \in]0, 1[$, let $\theta_t^{(\alpha)}(ds)$ be the measure on \mathbb{R}_+ , defined by

$$\int_{\mathbb{R}_+} e^{-\lambda s} \theta_t^{(\alpha)}(ds) = e^{-t \lambda^\alpha}.$$

Define

$$Q_t^\alpha F = \sum e^{-n^\alpha t} I_n(F_n) = \int_0^\infty P_s F \theta_t^{(\alpha)}(ds).$$

Then

$$\|Q_t^\alpha \delta_{n_0} F\|_p \leq \|F\|_p \int_0^\infty e^{-n_0 \theta s} \theta_t^{(\alpha)}(ds) = \|F\|_p e^{-t(n_0 \theta)^\alpha},$$

the rest of the proof goes as in the case $\alpha = 1$.

QED

Examples of application:

1) Let

$$\begin{aligned} \phi(n) &= \left(\frac{1 + \sqrt{n}}{\sqrt{1+n}} \right)^s & s \in]-\infty, \infty[\\ &= h \left(\sqrt{\frac{1}{n}} \right), & h(x) = \left(\frac{1+x}{\sqrt{1+x^2}} \right)^s. \end{aligned}$$

Then $T_\phi : L^p \rightarrow L^p$ is bounded. Moreover $\phi^{-1}(n) = \frac{1}{\phi(n)} = h^{-1}\left(\sqrt{\frac{1}{n}}\right)$, $h^{-1}(x) = \frac{1}{h(x)}$ is also analytic near the origin, hence $T_{\phi^{-1}} : L^p \rightarrow L^p$ is also a bounded operator.

2) Let $\phi(n) = \frac{\sqrt{1+n}}{\sqrt{2+n}}$ then $h(x) = \sqrt{\frac{x+1}{2x+1}}$ satisfies also the above hypothesis.

3) As an application of (2), look at

$$\begin{aligned} \|(I + \mathcal{L})^{1/2} \nabla \varphi\|_p &= \|\nabla (2I + \mathcal{L})^{1/2} \varphi\|_p \\ &\leq \|(I + \mathcal{L})^{1/2} (2I + \mathcal{L})^{1/2} \varphi\|_p \\ &= \|(2I + \mathcal{L})^{1/2} (I + \mathcal{L})^{1/2} \varphi\|_p \\ &= \|T_\phi (I + \mathcal{L})^{1/2} (I + \mathcal{L})^{1/2} \varphi\|_p \\ &\leq c_p \|(I + \mathcal{L}) \varphi\|_p. \end{aligned}$$

Continuing this way we can show that

$$\begin{aligned} \|\nabla^k \varphi\|_{L^p(\mu, H^{\otimes k})} &\leq c_{p,k} \|\varphi\|_{p,k} (= \|(I + \mathcal{L})^{k/2} \varphi\|_p) \\ &\leq \tilde{c}_{p,k} (\|\varphi\|_p + \|\nabla^k \varphi\|_{L^p(\mu, H^{\otimes k})}) \end{aligned}$$

and this completes the proof of the Meyer inequalities for the scalar-valued Wiener functionals. If \mathcal{X} is a separable Hilbert space, we denote with $D_{p,k}(\mathcal{X})$ the completion of the \mathcal{X} -valued polynomials with respect to the norm

$$\|\alpha\|_{D_{p,k}(\mathcal{X})} = \|(I + \mathcal{L})^{k/2}\|_{L^p(\mu, \mathcal{X})}.$$

We define as in the case $\mathcal{X} = \mathbb{R}$, the Sobolev derivative ∇ , the divergence δ , etc. All we have said for the real case extend trivially to the vector case, including the Meyer inequalities. In fact, in the proof of these inequalities the main step is the Riesz inequality for the Hilbert transform. However this inequality is also true for any Hilbert space (in fact it holds also for a class of Banach spaces which contains Hilbert spaces, called UMD spaces). The rest is almost the transcription of the real case combined with the Khintchine inequalities. We leave hence this passage to the reader. QED

Corollary: For every $p > 1$, $k \in \mathbb{R}$, ∇ has a continuous extension as a map $D_{p,k} \rightarrow D_{p,k-1}(H)$.

Proof: We have

$$\begin{aligned} \|\nabla\varphi\|_{p,k} &= \|(I + \mathcal{L})^{k/2}\nabla\varphi\|_p \\ &= \|\nabla(2I + \mathcal{L})^{k/2}\varphi\|_p \\ &\leq c_p\|(1 + \mathcal{L})^{1/2}(2I + \mathcal{L})^{k/2}\varphi\|_p \\ &\approx \|(I + \mathcal{L}^{(k+1)/2})\varphi\|_p \\ &= \|\varphi\|_{p,k+1}. \end{aligned}$$

QED

Corollary: $\delta = \nabla^* : D_{p,k}(H) \rightarrow D_{p,k-1}$ is continuous $\forall p > 1$ and $k \in \mathbb{R}$.

Proof: The proof follows from the duality. QED

In particular:

Corollary:

i) $\nabla : \bigcap_{p,k} D_{p,k} = D \rightarrow D(H) = \bigcap_{p,k} D_{p,k}(H)$ is continuous and extends continuously as a map

$$\nabla : D' = \bigcup_{p,k} D_{p,k} \rightarrow D'(H) = \bigcup_{p,k} D_{p,k}(H).$$

ii) $\delta : \bigcap_{p,k} D_{p,k}(H) = D(H) \rightarrow D$ is continuous and has a continuous extension $\delta : D'(H) \rightarrow D'$

Proof: Everything follows from the dualities

$$(D)' = D', (D(H))' = D'(H).$$

QED

Definition: For $n \geq 1$, we define δ^n as $(\nabla^n)^*$ with respect to μ .

Proposition: For $\varphi \in L^2(\mu)$, we have

$$\varphi = E[\varphi] + \sum_{n \geq 1} \frac{1}{n!} \delta^n(E[\nabla^n \varphi]).$$

Proof: First suppose that $h \mapsto \varphi(w+h)$ is analytic for almost all w . Then we have

$$\varphi(w+h) = \varphi(w) + \sum_{n \geq 1} \frac{(\nabla^n \varphi(w), h^{\otimes n})_{H^{\otimes n}}}{n!}.$$

Take the expectations:

$$\begin{aligned} E[\varphi(w+h)] &= E[\varphi \cdot \mathcal{E}(\delta h)] \\ &= E[\varphi] + \sum_n \frac{(E[\nabla^n \varphi], h^{\otimes n})}{n!} \\ &= E[\varphi] + \sum_{n \geq 1} E \left[\frac{I_n(E[\nabla^n \varphi])}{n!} \mathcal{E}(\delta h) \right]. \end{aligned}$$

Since the finite linear combinations of the elements of the set $\{\mathcal{E}(\delta h); h \in H\}$ is dense in any $L^p(\mu)$, we obtain the identity

$$\varphi(w) = E[\varphi] + \sum_{n \geq 1} \frac{I_n(E[\nabla^n \varphi])}{n!}.$$

Let $\psi \in D$, then we have (with $E[\psi] = 0$),

$$\begin{aligned}
\langle \varphi, \psi \rangle &= \sum_{n \geq 1} E[I_n(\varphi_n)I_n(\psi_n)] = \\
&= \sum_n E \left[\frac{I_n(E[\nabla^n \varphi])}{n!} \cdot I_n(\psi_n) \right] = \\
&= \sum_n (E[\nabla^n \varphi], \psi_n) = \\
&= \sum_n \frac{1}{n!} (E[\nabla^n \varphi], E[\nabla^n \psi]) \\
&= \sum_n \frac{1}{n!} E[(E[\nabla^n \varphi], \nabla^n \psi)] \\
&= \sum_n \frac{1}{n!} E[\delta^n(E[\nabla^n \varphi]) \cdot \psi]
\end{aligned}$$

hence we obtain that

$$\varphi = \sum_n \frac{1}{n!} \delta^n E[\nabla^n \varphi],$$

in particular $\delta^n E[\nabla^n \varphi] = I_n(E[\nabla^n \varphi])$.

QED

Appendix: Passing from the classical Wiener space to the AWS (or vice-versa):

Let (W, H, μ) be an abstract Wiener space. Since, à priori, there is no notion of time, it seems that we can not define the notion of anticipation, non-anticipation, etc.

This difficulty can be overcome in the following way:

Let $(p_\lambda; \lambda \in \Sigma)$, $\Sigma \subset \mathbb{R}$, be a resolution of identity on the separable Hilbert space H , i.e., each p_λ is an orthogonal projection, increasing to I_H , in the sense that $\lambda \mapsto (p_\lambda h, h)$ is an increasing function. Let us denote by $H_\lambda = \overline{p_\lambda(H)}$

Definition 1: We will denote by \mathcal{F}_λ the σ -algebra generated by the real polynomials φ on W such that $\nabla \varphi \in H_\lambda$ μ -almost surely.

Lemma 1: We have

$$\bigvee_{\lambda \in \Sigma} \mathcal{F}_\lambda = \mathcal{B}(W)$$

up to μ -negligeable sets.

Proof: We have already $\bigvee \mathcal{F}_\lambda \subset \mathcal{B}(W)$. Conversely, if $h \in H$, then $\nabla \delta h = h$. Since $\bigcup_{\lambda \in \Sigma} H_\lambda$ is dense in H , there exists $(h_n) \subset \bigcup_{\lambda} H_\lambda$ such that $h_n \rightarrow h$ in H . Hence $\delta h_n \rightarrow \delta h$ in $L^p(\mu)$, $\forall p \geq 1$. Since each δh_n is $\bigvee \mathcal{F}_\lambda$ -measurable, so does δh . Since $\mathcal{B}(W)$ is generated by $\{\delta h; h \in H\}$ the proof is completed. QED

Definition 2: A random variable $\xi : W \rightarrow H$ is called a simple, adapted vector field if it can be written as

$$\xi = \sum_{i < +\infty} F_i(p_{\lambda_{i+1}} h_i - p_{\lambda_i} h_i)$$

where $h_i \in H$, F_i are \mathcal{F}_{λ_i} -measurable (and smooth for the time being!) random variables.

Proposition 1: For each adapted simple vector field we have

$$\text{i) } \delta \xi = \sum_{i < +\infty} F_i \delta(p_{\lambda_{i+1}} h_i - p_{\lambda_i} h_i)$$

$$\text{ii) } E[(\delta \xi)^2] = E[|\xi|_H^2].$$

Proof: i) We have

$$\delta[F_i(p_{\lambda_{i+1}} - p_{\lambda_i})h_i] = F_i \delta[(p_{\lambda_{i+1}} - p_{\lambda_i})h_i] - (\nabla F_i, (p_{\lambda_{i+1}} - p_{\lambda_i})h_i).$$

Since $\nabla F_i \in H_\lambda$, the second term is null.

(ii) is well-known. QED

Remark: If we note $\Sigma F_{i-1] \lambda_i, \lambda_{i+1}] (\lambda) h_i$ by $\dot{\xi}(\lambda)$, we have the following notations:

$$\delta \xi = \delta \int_{\Sigma} \dot{\xi}(\lambda) dp_{\lambda} \quad \text{with} \quad \|\delta \xi\|_2^2 = E \int_{\Sigma} d(\dot{\xi}_{\lambda}, p_{\lambda} \dot{\xi}_{\lambda}) = \|\xi\|_{L^2(\mu, H)}^2,$$

which are significantly analogous to the things that we have seen before as the Ito stochastic integral.

Now the Ito representation theorem holds in this setting also: suppose $(p_{\lambda}; \lambda \in \Sigma)$ is continuous, then:

Theorem: Let us denote with $D_{2,0}^a(H)$ the completion of adapted simple vector fields with respect to the $L^2(\mu, H)$ -norm. Then we have

$$L_2(\mu) = \mathbb{R} + \{\delta \xi : \xi \in D_{2,0}^a(H)\},$$

i.e., any $\varphi \in L_2(\mu)$ can be written as

$$\varphi = E[\varphi] + \delta \xi$$

for some $\xi \in D_{2,0}^a(H)$. Moreover such ξ is unique up to $L^2(\mu, H)$ -equivalence classes.

The following result explains the reason of the existence of the Brownian motion:

Theorem: Suppose that there exists some $\Omega_0 \in H$ such that the set $\{p_{\lambda} \Omega_0; \lambda \in \Sigma\}$ has a dense span in H (i.e. the linear combinations from it is a dense set). Then the real-valued (\mathcal{F}_{λ}) -martingale defined by

$$b_{\lambda} = \delta p_{\lambda} \Omega_0$$

is a Brownian motion with a deterministic time change and $(\mathcal{F}_{\lambda}; \lambda \in \Sigma)$ is its canonical filtration completed with the negligible sets.

Example: Let $H = H_1([0, 1])$, define A as the operator defined by $Ah(t) = \int_0^t s\dot{h}(s)ds$. Then A is a self-adjoint operator on H with a continuous spectrum which is equal to $[0, 1]$. Moreover we have

$$(p_\lambda h)(t) = \int_0^t 1_{[0, \lambda]}(s)\dot{h}(s)ds$$

and $\Omega_0(t) = \int_0^t 1_{[0, 1]}(s)ds$ satisfies the hypothesis of the above theorem. Ω_0 is called the vacuum vector (in physics).

This is the main example, since all the (separable) Hilbert spaces are isomorphic, we can carry this time structure to any abstract Hilbert-Wiener space as long as we do not need any particular structure of time.

Chapter V

Some applications of the distributions

Some applications of the distributions

In this chapter we give some applications of the extended versions of the derivative and the divergence operators. First we give an extension of the Ito-Clark formula to the space of the scalar distributions. In fact, although, we know from the Ito representation theorem, that each square integrable Wiener functional can be represented as the stochastic integral of an adapted process, without the use of the distributions, we can not calculate this process, since any square integrable random variable is not necessarily in $D_{2,1}$, hence it is not Sobolev differentiable in the ordinary sense. As it will explained, this problem is completely solved using the differentiation in the sense of distributions. Afterwards we give a straightforward application of this result to prove a 0 – 1 law for the Wiener measure. At the second section we construct the composition of the tempered distributions with nondegenerate Wiener functionals as Meyer-Watanabe distributions. This construction carries also the information that the probability density of a nondegenerate random variable is not only infinitely differentiable but also it is rapidly decreasing. The same idea is then applied to prove the regularity of the solutions of the Zakai equation for the filtering of non-linear diffusions.

1 Extension of the Ito-Clark formula

Let F be any integrable random variable. Then we know that F can be represented as

$$F = E[F] + \int_0^1 H_s dW_s,$$

where $(H_s; s \in [0, 1])$ is an adapted process such that, it is unique and

$$\int_0^1 H_s^2 ds < +\infty \text{ a.s.}$$

Moreover, if $F \in L^p$ ($p > 1$), then we also have

$$E\left[\left(\int_0^1 |H_s|^2 ds\right)^{p/2}\right] < +\infty.$$

One question is how to calculate the process H . In fact, below we will extend the Ito representation and answer to the above question for any $F \in D'$ (i.e., the Meyer-Watanabe distributions).

We begin with:

Lemma 1 Let $\xi \in D(H)$, then $\pi\xi$ defined by $\pi\xi(t) = \int_0^t E[\dot{\xi}_s | \mathcal{F}_s] ds$ belongs again to $D(H)$, i.e. $\pi : D(H) \rightarrow D(H)$ is continuous.

Proof: We have $\mathcal{L}\pi\xi = \pi\mathcal{L}\xi$, hence

$$\begin{aligned} \|\pi\xi\|_{p,k} &= E\left[\left(\int_0^1 |(I + \mathcal{L})^{k/2} E[\dot{\xi}_s | \mathcal{F}_s]|^2 ds\right)^{p/2}\right] = \\ &= E\left[\left(\int_0^1 E[(I + \mathcal{L})^{k/2} \dot{\xi}_s | \mathcal{F}_s]|^2 ds\right)^{p/2}\right] \\ &\leq c_p E\left[\left(\int_0^1 |(I + \mathcal{L}^{k/2} \dot{\xi}_s|^2 ds\right)^{p/2}\right) \quad (c_p \cong p) \end{aligned}$$

where the last inequality follows from the convexity inequalities of the dual predictable projections (c.f. Dellacherie-Meyer, Vol. 2). QED

Lemma 2: $\pi : D(H) \rightarrow D(H)$ extends as a continuous mapping to $D'(H) \rightarrow D'(H)$.

Proof: Let $\xi \in D(H)$, then we have, for $k > 0$,

$$\begin{aligned} \|\pi\xi\|_{p,-k} &= \|(I + \mathcal{L})^{-k/2}\pi\xi\|_p \\ &= \|\pi(I + \mathcal{L})^{-k/2}\xi\|_p \leq c_p \|(I + \mathcal{L})^{-k/2}\xi\|_p \\ &\leq c_p \|\xi\|_{p,-k}, \end{aligned}$$

then the proof follows since $D(H)$ is dense in $D'(H)$. QED

Lemma 3: Let $\varphi \in D$, then we have

$$\begin{aligned} \varphi &= E[\varphi] + \int_0^1 E[D_s\varphi|\mathcal{F}_s]dW_s \\ &= E[\varphi] + \delta\pi\nabla\varphi. \end{aligned}$$

Moreover $\pi\nabla\varphi \in D(H)$.

Proof: Let U be an element of $L^2(\mu, H)$ such that $u(t) = \int_0^t \dot{u}_s ds$ with $(\dot{u}_t; t \in [0, 1])$ being an adapted and bounded process. Then we have, from the Girsanov theorem,

$$E[\varphi(w + \lambda u(w)) \cdot \exp(-\lambda \int_0^1 \dot{u}_s dW_s - \frac{\lambda^2}{2} \int_0^1 \dot{u}_s ds)] = E[\varphi].$$

Differentiating both sides at $\lambda = 0$, we obtain:

$$E[(\nabla\varphi(w), u) - \varphi \int_0^1 \dot{u}_s dW_s] = 0,$$

i.e.,

$$E[(\nabla\varphi, u)] = E[\varphi \int_0^1 \dot{u}_s dW_s].$$

Furthermore

$$\begin{aligned}
E\left[\int_0^1 D_s\varphi \dot{u}_s ds\right] &= E\left[\int_0^1 E[D_s\varphi|\mathcal{F}_s]\dot{u}_s ds\right] \\
&= E[(\pi\nabla\varphi, u)_H] \\
&= E\left[\left(\int_0^1 E[D_s\varphi|\mathcal{F}_s]dW_s\right)\left(\int_0^1 \dot{u}_s dW_s\right)\right].
\end{aligned}$$

Since the set $\{\int_0^1 \dot{u}_s dW_s, \dot{u} \text{ as above}\}$ is dense in $L_0^2(\mu) = L^2(\mu) - \langle L^2(\mu), 1 \rangle$, we see that

$$\varphi - E[\varphi] = \int_0^1 E[D_s\varphi|\mathcal{F}_s]dW_s = \delta\pi\nabla\varphi.$$

The rest is obvious from the Lemma 1.

QED

Theorem 1: For any $T \in D'$, we have

$$T = \langle T, 1 \rangle + \delta\pi\nabla T.$$

Proof: Let $(\varphi_n) \subset D$ such that $\varphi_n \rightarrow T$ in D' . Then we have

$$\begin{aligned}
T &= \lim_n \varphi_n \\
&= \lim_n [E[\varphi_n] + \delta\pi\nabla\varphi_n] \\
&= \lim_n E[\varphi_n] + \lim_n \delta\pi\nabla\varphi_n \\
&= \lim_n \langle 1, \varphi_n \rangle + \lim_n \delta\pi\nabla\varphi_n \\
&= \langle 1, T \rangle + \delta\pi\nabla T
\end{aligned}$$

since $\nabla : D' \rightarrow D'(H)$, $\pi : D'(H) \rightarrow D'(H)$ and $\delta : D'(H) \rightarrow D'$ are all linear, continuous mappings. QED

Here is a nontrivial application of the Theorem 1:

Theorem 2: (0-1 law) Let $A \in \mathcal{B}(W)$ such that $A + H = A$. Then $\mu(A) = 0$ or 1.

Proof: $A + H = A$ implies that

$$1_A(w + \lambda h) = 1_A(w) \text{ a.s.}$$

hence $\nabla 1_A = 0$. Consequently, Theorem 1 implies that

$$1_A = \langle 1_A, 1 \rangle = \mu(A) \Rightarrow \mu(A)^2 = \mu(A). \quad \text{QED}$$

2 Lifting of $\mathcal{S}'(\mathbb{R}^d)$ with random variables

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a C_b^1 -function, $F \in D$. Then we know that

$$\nabla(f(F)) = f'(F)\nabla F.$$

Now suppose that $|\nabla F|_H^{-2} \in \cap L^p(\mu)$, then

$$f'(F) = \frac{(\nabla(f(F)), \nabla F)_H}{|\nabla F|_H^2}$$

Even if f is not C^1 , the right hand side of this equality has a sense if we look at $\nabla(f(F))$ as an element of D' . In the following we will develop this idea:

Definition: Let $F : W \rightarrow \mathbb{R}^d$ be a random variable such that $F_i \in D$, $\forall i = 1, \dots, d$, and that $[\det(\nabla F_i, \nabla F_j)]^{-1} \in \cap_{p>1} L^p(\mu)$. Then we say that F is a *non-degenerate* random variable.

Lemma 1 Let us denote by $\sigma_{ij} = (\nabla F_i, \nabla F_j)$ and by $\gamma = \sigma^{-1}$ (as a matrix). Then $\gamma \in D(\mathbb{R}^d \otimes \mathbb{R}^d)$.

Proof: Formally, we have, using the relation $\sigma \cdot \gamma = Id$,

$$\nabla \gamma_{ij} = \sum_{k,l} \gamma_{ik} \gamma_{jl} \nabla \sigma_{kl}.$$

To justify this we define first $\sigma_{ij}^\varepsilon = \sigma_{ij} + \varepsilon \delta_{ij}$, $\varepsilon > 0$. Then we can write $\gamma_{ij}^\varepsilon = f_{ij}(\sigma^\varepsilon)$, where $f : \mathbb{R}^d \otimes \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is a smooth function of polynomial growth. Hence $\gamma_{ij}^\varepsilon \in D$. Then from the dominated convergence theorem we have $\gamma_{ij}^\varepsilon \rightarrow \gamma_{ij}$ in L^p as well as $\nabla^k \gamma_{ij}^\varepsilon \xrightarrow{L^p} \nabla^k \gamma_{ij}$ (this follows again from $\gamma^\varepsilon \cdot \sigma^\varepsilon = Id$). QED

Lemma 2 Let $G \in D$. Then we have, $\forall f \in \mathcal{S}(\mathbb{R}^d)$

$$\text{i) } E[\partial_i f(F).G] = E[f(F).l_i(G)]$$

where $G \mapsto l_i(G)$ is linear and for any $1 < r < g < \infty$,

$$\sup_{\|G\|_{q,1} \leq 1} \|l_i(G)\|_r < +\infty.$$

ii) Similarly

$$E[\partial_{i_1 \dots i_k} f \circ F.G] = E[f(F) \cdot l_{i_1 \dots i_k}(G)]$$

and

$$\sup_{\|G\|_{q,1} \leq 1} \|l_{i_1 \dots i_k}(G)\|_r < \infty.$$

Proof: We have

$$\nabla(f \circ F) = \Sigma \partial_i f(F) \nabla F_i \Rightarrow (\nabla(f \circ F), \nabla F_j) = \Sigma \sigma_{ij} \partial_i f(F).$$

Since σ is invertible, we obtain:

$$\partial_i f(F) = \sum_j \gamma_{ij} (\nabla(f \circ F), \nabla F_j).$$

Then

$$\begin{aligned} E[\partial_i f(F).G] &= \sum_j E[\gamma_{ij} (\nabla(f \circ F), \nabla F_j).G] \\ &= \sum_j E[f \circ F. \delta\{\nabla F_j \gamma_{ij} G\}], \end{aligned}$$

hence we see that $l_i(G) = \sum_j \delta\{\nabla F_j \gamma_{ij} G\}$. We have

$$\begin{aligned} l_i(G) &= - \sum_j [(\nabla(\gamma_{ij} G), \nabla F_j) - \gamma_{ij} G \mathcal{L} F_j] \\ &= - \sum_j [\gamma_{ij} (\nabla G, \nabla F_j) - \sum_{k,l} \gamma_{ik} \gamma_{jl} (\nabla \sigma_{kl}, \nabla F_j) G - \gamma_{ij} G \mathcal{L} F_j]. \end{aligned}$$

Hence

$$\begin{aligned} |l_{ij}(G)| &\leq \sum_j \left[\sum_{kl} |\gamma_{ik} \gamma_{jl}| |\nabla \sigma_{kl}| |\nabla F_j| |G| + |\gamma_{ij}| |\nabla F_j| |\nabla G| + \right. \\ &\quad \left. + |\gamma_{ij}| |G| |\mathcal{L} F_j| \right]. \end{aligned}$$

Choose p such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and apply Hölder's inequality:

$$\begin{aligned} \|l_i(G)\|_r &\leq \sum_{j=1}^d \left[\sum_{k,l} \|G\|_q \|\gamma_{ik}\gamma_{jl}\|\nabla\sigma_{kl}|_H|\nabla F_j|_H\|_p + \right. \\ &\quad \left. + \|\gamma_{ij}\nabla F_j\|_p \|\nabla G\|_q + \|\gamma_{ij}\mathcal{L}F_j\|_p \|G\|_q \right] \\ &\leq \|G\|_{q,1} \left[\sum_{j=1}^d \|\gamma_{ik}\gamma_{jl}\|\nabla F_{kl}\|\nabla F_j\|_p + \right. \\ &\quad \left. + \|\gamma_{ij}\nabla F_j\|_p + \|\gamma_{ij}\mathcal{L}F_j\|_p \right]. \end{aligned}$$

ii) For $i > 1$ we iterate this procedure. QED

Now remember that $\mathcal{S}(\mathbb{R}^d)$ can be written as the intersection (i.e., projective limit) of the following Banach spaces:

Let $A = I - \delta + |x|^2$, $\|f\|_{2k} = \|A^k f\|_\infty$ (the uniform norm) and $S_{2k} =$ completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{2k}$.

Theorem 1 Let $F \in D(\mathbb{R}^d)$ be a non-degenerate random variable. Then we have for $f \in \mathcal{S}(\mathbb{R}^d)$:

$$\|f \circ F\|_{p,-2k} \leq c_{p,k} \|f\|_{-2k}.$$

Proof: Let $\psi = A^{-k}f \in \mathcal{S}(\mathbb{R}^d)$. For $G \in D$, we know that there exists some $\eta_{2k}(G) \in D$ ($G \mapsto \eta_{2k}(G)$ is linear) from the Lemma 2, such that

$$E[A^k \psi \circ F.G] = E[\psi \circ F.\eta_{2k}(G)],$$

i.e.,

$$E[f \circ F.G] = E[(A^{-k}f)(F).\eta_{2k}(G)].$$

Hence

$$|E[f \circ F.G]| \leq \|A^{-k}f\|_\infty \|\eta_{2k}(G)\|_{L^1}$$

and

$$\begin{aligned} \sup_{\|G\|_{q,2k} \leq 1} |E[f \circ F.G]| &\leq \|A^{-k}f\|_\infty \sup_{\|G\|_{q,2k} \leq 1} \|\eta_{2k}(G)\|_1 \\ &= K \|f\|_{-2k}. \end{aligned}$$

Hence $\|f \circ F\|_{p,-2k} \leq K \|f\|_{-2k}$. QED

Corollary 1: The map $f \mapsto f \circ F$ from $\mathcal{S}(\mathbb{R}^d) \rightarrow D$ has a continuous extension to $\mathcal{S}'(\mathbb{R}^d) \rightarrow D'$.

Some applications

If $F : W \rightarrow \mathbb{R}^d$ is a non-degenerate random variable, then we have seen that the map $f \mapsto f \circ F$ from $\mathcal{S}(\mathbb{R}^d) \rightarrow D$ has a continuous extension to $\mathcal{S}'(\mathbb{R}^d) \rightarrow D'$, denoted by $T \mapsto T \circ F$.

For $f \in \mathcal{S}(\mathbb{R}^d)$, let us look at the following Pettis integral:

$$\int_{\mathbb{R}^d} f(x) \mathcal{E}_x dx,$$

where \mathcal{E}_x denotes the Dirac measure at $x \in \mathbb{R}^d$. We have, for any $g \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned} \left\langle \int f(x) \mathcal{E}_x dx, g \right\rangle &= \int \langle f(x) \mathcal{E}_x, g \rangle dx = \int f(x) \langle \mathcal{E}_x, g \rangle dx \\ &= \int f(x) g(x) dx = \langle f, g \rangle. \end{aligned}$$

Hence we have proven:

Lemma 1: The following representation holds in $\mathcal{S}(\mathbb{R}^d)$:

$$f = \int_{\mathbb{R}^d} f(x) \mathcal{E}_x dx.$$

From Lemma 1 we have

Lemma 2: We have

$$\int \langle \mathcal{E}_y(F), \varphi \rangle f(y) dy = E[f(F) \cdot \varphi],$$

for any $\varphi \in D$.

Proof: Let ρ_ϵ be a mollifier. Then $\mathcal{E}_y * \rho_\epsilon \rightarrow \mathcal{E}_y$ in \mathcal{S}' on the other hand

$$\begin{aligned} \int (\mathcal{E}_y * \rho_\epsilon)(F) f(y) dy &= \int \rho_\epsilon(F + y) \cdot f(y) dy = \\ &= \int \rho_\epsilon(y) f(y + F) dy \xrightarrow{\epsilon \rightarrow 0} f(F). \end{aligned}$$

Hence

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int (\mathcal{E}_y * \rho_\epsilon)(F) f(y) dy &= \int \lim_{\epsilon \rightarrow 0} (\mathcal{E}_y * \rho_\epsilon)(F) f(y) dy \\ &= \int \mathcal{E}_y(F) f(y) dy \quad \text{in } D' \\ &= f(F). \end{aligned}$$

QED

Corollary: We have $\langle \mathcal{E}_x(F), 1 \rangle = \frac{d(F^* \mu)}{dx}(x) = p_F(x)$, moreover $p_F \in \mathcal{S}(\mathbb{R}^d)$ (i.e., the probability density of F is not only C^∞ but it is also a rapidly decreasing function).

Proof: We know that the map $T \rightarrow E[T(F) \cdot \varphi]$ is continuous on $\mathcal{S}'(\mathbb{R}^d)$ hence there exists some $p_{F, \varphi} \in \zeta(\mathbb{R}^d)$ such that

$$E[T(F) \cdot \varphi] = \mathcal{S}' \langle T, P_{F, \varphi} \rangle_{\mathcal{S}}.$$

Let $p_{F, 1} = p_F$, then it follows from the Lemma 2 that

$$E[f(F)] = \int \langle \mathcal{E}_y(F), 1 \rangle f(y) dy.$$

QED

Remark: From the disintegration of measures, we have

$$\begin{aligned} \int E[\varphi | F = x] f(x) dx &= E[f(F) \cdot \varphi] \\ &= \int f(x) \langle \mathcal{E}_x(F), \varphi \rangle dx \end{aligned}$$

hence $E[\varphi | F = x] = \langle \mathcal{E}_x(F), \varphi \rangle dx$ -a.s. In fact the right hand side is an everywhere defined version of this conditional probability.

Remark: Let (x_t) be the solution of the following stochastic differential equation:

$$dx_t(w) = b \cdot (x_t(w))dt + \sigma_i(x_t(w))dw_t^i$$

$$x_0 = x \text{ given,}$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are smooth vector fields with bounded derivatives. Let us denote by

$$X_0 = \sum_{i=1}^d \tilde{b}_0^i \frac{\partial}{\partial x_i}, \quad X_j = \sum \sigma_i^j \frac{\partial}{\partial x_j}$$

where $\tilde{b}^i(x) = b^i(x) - \frac{1}{2} \sum_{k,\alpha} \partial_k \sigma_\alpha^i(x) \sigma_\alpha^k(x)$. Then, if the Lie algebra of vector fields generated by $\{X_0, X_1, \dots, X_d\}$ has dimension equal to d at any $x \in \mathbb{R}^d$, then $x_t(w)$ is non-degenerate cf. [20]. In fact it is also uniformly non-degenerate in the following sense:

$$E \int_s^t |Det(\nabla x_r^i, \nabla x_r^j)|^{-p} dr < \infty, \quad \forall 0 < s < t, \forall p > 1.$$

As a corollary of this result, combined with the lifting of \mathcal{S}' to D' , we can show the following:

For any $T \in \mathcal{S}'(\mathbb{R}^d)$, one has the following:

$$T(x_t) - T(x_s) = \int_s^t AT(x_s)ds + \int_s^t \sigma_{ij}(x_s) \cdot \partial_j T(x_s) dW_s^i,$$

where the Lebesgue integral is a Bochner integral, the stochastic integral is as defined at the first section of this chapter and we have used the following notation:

$$A = \sum b^i \partial_i + \frac{1}{2} \sum a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad a(x) = (\sigma \sigma^*)_{ij}, \quad \sigma = [\sigma_1, \dots, \sigma_d].$$

Applications to the filtering of the diffusions

Suppose that $y_t = \int_0^t h(x_s) ds + B_t$ where $h \in C_b^\infty(\mathbb{R}^d) \otimes \mathbb{R}^d$, B is another Brownian motion independent of w above. $(y_t; t \in [0, 1])$ is called an (noisy) observation of (x_t) . Let $\mathcal{Y}_t = \sigma\{y_s; s \in [0, t]\}$ be the observed data till t . The filtering problem consists of calculating the random measure $f \mapsto E[f(x_t)|\mathcal{Y}_t]$. Let P^0 be the probability defined by

$$dP^0 = Z_1^{-1} dP$$

where $Z_t = \exp \int_0^t h(x_s) \cdot dy_s - \frac{1}{2} \int_0^t |h(x_s)|^2 ds$. Then for any bounded, \mathcal{Y}_t -measurable random variable Y_t , we have:

$$\begin{aligned} E[f(x_t) \cdot Y_t] &= E\left[\frac{Z_t}{Z_t} f(x_t) \cdot Y_t\right] = E^0[Z_t f(x_t) Y_t] \\ &= E^0[E[Z_t f(x_t)|\mathcal{Y}_t] \cdot Y_t] = \\ &= E\left[\frac{1}{E^0[Z_t|\mathcal{Y}_t]} E^0[Z_t f(x_t)|\mathcal{Y}_t] \cdot Y_t\right], \end{aligned}$$

hence

$$E[f(x_t)|\mathcal{Y}_t] = \frac{E^0[Z_t f(x_t)|\mathcal{Y}_t]}{E^0[Z_t|\mathcal{Y}_t]}.$$

If we want to study the smoothness of the measure $f \mapsto E[f(x_t)|\mathcal{Y}_t]$, then from the above formula, we see that it is sufficient to study the smoothness of $f \mapsto E^0[Z_t f(x_t)|\mathcal{Y}_t]$. The reason for the use of P^0 is that w and $(y_t; t \in [0, 1])$ are two independent Brownian motions under P^0 (this follows directly from Paul Lévy's theorem of the characterization of the Brownian motion).

After this preliminaries, we can prove the following

Theorem Suppose that the map $f \mapsto f(x_t)$ from $\mathcal{S}(\mathbb{R}^d)$ into D has a continuous extension as a map from $\mathcal{S}'(\mathbb{R}^d)$ into D' . Then the measure $f \mapsto E[f(x_t)|\mathcal{Y}_t]$ has a density in $\mathcal{S}(\mathbb{R}^d)$.

Proof: As explained above, it is sufficient to prove that the (random) measure $f \mapsto E^0[Z_t f(x_t) | \mathcal{Y}_t]$ has a density in $\mathcal{S}(\mathbb{R}^d)$.

Let \mathcal{L}_y be the Ornstein-Uhlenbeck operator on the space of the Brownian motion $(y_t; t \in [0, 1])$. Then we have

$$\mathcal{L}_y Z_t = Z_t \left(- \int_0^t h(x_s) dy_s + \frac{1}{2} \int_0^t |h(x_s)|^2 ds \right) \in \bigcap_p L^p$$

It is also easy to see that

$$\mathcal{L}_w^k Z_t \in \bigcap_p L^p.$$

- Hence $Z_t(w, y) \in D(w, y)$, where $D(w, y)$ denotes the space of test functions defined on the product Wiener space with respect to the laws of w and y .
- The second point is that the operator $E^0[\bullet | \mathcal{Y}_t]$ is a continuous mapping from $D_{p,k}^k(w, y)$ into $D_{p,k}^0(y)$ since \mathcal{L}_y commutes with $E^0[\bullet | \mathcal{Y}_t]$ (for any $p \geq 1, k \in \mathbb{Z}$).
- Hence the map $T \mapsto E^0[T(x_t) Z_t | \mathcal{Y}_t]$ is continuous from $\mathcal{S}'(\mathbb{R}^d) \rightarrow D'(y)$. In particular, for fixed $T \in \mathcal{S}'$, $\exists p > 1$ and $k \in \mathbb{N}$ such that $T(x_t) \in D_{p,-k}(w)$. Since $Z_t \in D(w, y)$, $Z_t T(x_t) \in D_{p,-k}(w, y)$ and $T(x_t) \cdot (I + \mathcal{L}_y)^{k/2} Z_t \in D_{p,-k}(w, y)$. Hence $E^0[T(x_t) \cdot (I + \mathcal{L}_y)^{k/2} Z_t | \mathcal{Y}_t] \in D_{p,-k}(y)$.
- Hence $(I + \mathcal{L})^{-k/2} E^0[T(x_t) (I + \mathcal{L}_y)^{k/2} Z_t | \mathcal{Y}_t] = E^0[T(x_t) Z_t | \mathcal{Y}_t]$ belongs to $L^p(y)$. Therefore we see that:

$$T \mapsto E^0[T(x_t) Z_t | \mathcal{Y}_t]$$

defines a linear, continuous (use the closed graph theorem for instance) map from $\mathcal{S}'(\mathbb{R}^d)$ into $L^p(y)$. Since $\mathcal{S}'(\mathbb{R}^d)$ is a nuclear space, the map $T \overset{\Theta}{\mapsto} E^0[T(x_t) Z_t | \mathcal{Y}_t]$ is a nuclear operator, hence by definition it has a representation:

$$\Theta = \sum_{i=1}^{\infty} \lambda_i f_i \otimes \alpha_i$$

where $(\lambda_i) \in l^1$, $(f_i) \subset \mathcal{S}(\mathbb{R}^d)$ and $(\alpha_i) \subset L^p(y)$ are bounded sequences. Define

$$k_t(x, y) = \sum_i \lambda_i f_i(x) \alpha_i(y) \in \mathcal{S}(\mathbb{R}^d) \tilde{\otimes}_1 L^p(y)$$

where $\tilde{\otimes}_1$ denotes the projective tensor product topology. It is easy now to see that, for $g \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} g(x) k_t(x, y) dx = E^0[g(x_t) \cdot Z_t | \mathcal{Y}_t].$$

QED

Chapter VI

Positive distributions and applications

Positive Meyer-Watanabe distributions

If θ is a positive distribution on \mathbb{R}^d , then a well-known theorem says that θ is a positive measure, finite on the compact sets. We will prove an analogous result for the Meyer-Watanabe distributions in this section, show that they are absolutely continuous with respect to the capacities defined with respect to the scale of the Sobolev spaces on the Wiener space and give an application to the construction of the local time of the Wiener process. We end the chapter by making some remarks about the Sobolev spaces constructed by the second quantization of an elliptic operator on the Cameron-Martin space.

We will work on the classical Wiener space $C_0([0,1]) = W$. First we have the following:

Proposition: Suppose $(T_n) \subset D'$ and each T_n is also a probability on W . If $T_n \rightarrow T$ in D' , then T is also a probability and $T_n \rightarrow T$ in the weak topology of measures (on W).

Proof: It is sufficient to prove that the set of probability measures (ν_n) associated to (T_n) , is tight. In fact, let $S = D \cap C_b(W)$. If the

tightness holds, then we will have, for $\nu = w - \lim \nu_n$,

$$\nu(\varphi) = T(\varphi) \quad \text{on } S.$$

Since S is weakly dense in $C_b(W)$ the proof will be completed (remember $e^{i\langle w, w^* \rangle}$, $w^* \in W^*$, belongs to $S!$).

Let $G(w)$ be defined as

$$G(w) = \int_0^1 \int_0^1 \frac{|w(t) - w(s)|^8}{|t - s|^3} ds dt \Rightarrow G \in D.$$

Then, it is not difficult to show that $A_\lambda = \{G(w) \leq \lambda\}$ is a compact subset of W (cf.[1]).

Moreover, we have $\bigcup_{\lambda \geq 0} A_\lambda = W$ almost surely. Let $\varphi \in C^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$; $\varphi(x) = 1$ for $x \geq 0$, $\varphi(x) = 0$ for $x \leq -1$. Let $\varphi_\lambda(x) = \varphi(x - \lambda)$.

We have

$$\nu_n(A_\lambda^c) \leq \int_W \varphi_\lambda(G(w)) \nu_n(dw).$$

We claim that $\int_W \varphi_\lambda(G) d\nu_n = \langle \varphi_\lambda(G), T_n \rangle$:

For $\varepsilon > 0$, write

$$G_\varepsilon(w) = \int_{[0,1]^2} \frac{|w(t) - w(s)|^8}{(\varepsilon + |t - s|)^3} ds dt.$$

Then $\varphi_\lambda(G_\varepsilon) \in S$ (but not $\varphi_\lambda(G)$, since G is not continuous on $W!$)
Since $\varphi_\lambda(G_\varepsilon) \in S = G_b(W) \cap D$, we have

$$\int \varphi_\lambda(G_\varepsilon) d\nu_n = \langle \varphi_\lambda(G_\varepsilon), T_n \rangle$$

But $\varphi_\lambda(G_\varepsilon) \rightarrow \varphi_\lambda(G)$ in D , hence

$$\lim_{\varepsilon \downarrow 0} \langle \varphi_\lambda(G_\varepsilon), T_n \rangle = \langle \varphi_\lambda(G), T_n \rangle.$$

From the dominated convergence theorem, we have also

$$\lim_{\varepsilon \rightarrow 0} \int \varphi_\lambda(G_\varepsilon) d\nu_n = \int \varphi_\lambda(G) d\nu_n.$$

This proves our claim. Now, since $T_n \rightarrow T$ in D' , exists some $k > 0$ and $p > 1$ such that $T_n \rightarrow T$ in $D_{p,-k}$. Therefore

$$\begin{aligned} \langle \varphi_\lambda(G), T_n \rangle &= \langle (I + \mathcal{L})^{k/2} \varphi_\lambda(G), (I + \mathcal{L})^{-k/2} T_n \rangle \\ &\leq \| (I + \mathcal{L})^{k/2} \varphi_\lambda(G) \|_q \cdot \sup_n \| (I + \mathcal{L})^{-k/2} T_n \|_p. \end{aligned}$$

From the Meyer inequalities, we see that

$$\lim_{\lambda \rightarrow \infty} \| (I + \mathcal{L})^{k/2} \varphi_\lambda(G) \|_q = 0,$$

in fact, it is sufficient to see that $\nabla^i(\varphi_\lambda(G)) \rightarrow 0$ in L^p for all $i \leq [k] + 1$, but this is obvious from the choice of φ_λ .

Therefore we have proven that

$$\lim_{\lambda \rightarrow \infty} \sup_n \mu_n(A_\lambda^c) \leq \sup_n \| (I + \mathcal{L})^{-k} T_n \|_p \lim_{\lambda \rightarrow \infty} \| (I + \mathcal{L})^k \varphi_\lambda(G) \|_p = 0,$$

which is the definition of tightness. QED

Corollary: Let $T \in D'$ such that $\langle T, \varphi \rangle \geq 0$, for all positive $\varphi \in D$. Then T is a Radon measure on W .

Proof: Let $(h_i) \subset H$ be a complete, orthonormal basis of H . Let $V_n = \sigma\{\delta h_1, \dots, \delta h_n\}$. Define T_n as $T_n = E[P_{1/n} T | V_n]$ where $P_{1/n}$ is the Ornstein-Uhlenbeck semi-group on W . Then $T_n \geq 0$ and it is a random variable in some $L^p(\mu)$. Therefore it defines a measure on W (even absolutely continuous with respect to $\mu!$). Moreover $T_n \rightarrow T$ in D' , hence the proof follows from the proposition. QED

1 Capacities and positive Wiener functionals

If $p \in [1, \infty[$, $O \subset W$ is an open set and $k > 0$, we define

- $C_{p,k}(O) = \inf\{\|\varphi\|_{p,k}^p : \varphi \in D_{p,k}, \varphi \geq 1 \text{ } \mu\text{-a.e. on } O\}$.

- If $A \subset W$ is any subset, define

$$C_{p,k}(A) = \inf\{C_{p,k}(O); O \text{ is open } O \supset A\}.$$

- We say that some property takes place (p, k) -quasi everywhere if the set on which it does not hold has (p, k) -capacity zero.
 - We say N is a slim set if $C_{p,k}(N) = 0, \forall p > 1, k > 0$.
 - A function is called (p, k) -**quasi continuous** if $\forall \varepsilon > 0, \exists$ open set O_ε such that $C_{p,k}(O_\varepsilon) < \varepsilon$ and the function is continuous on O_ε^c .
 - It is called ∞ -quasi continuous if it is (p, k) -quasi continuous $\forall (p, k)$.
- The following results are proved by Fukushima & Kanako:

Lemma 1:

- If $\varphi \in D_{p,k}$, then there exists a (p, k) -quasi continuous function \tilde{F} such that $F = \tilde{F}$ μ -a.e. and \tilde{F} is (p, k) -quasi everywhere defined, i.e. if \tilde{G} is another such function, then $C_{p,k}(\{\tilde{F} \neq \tilde{G}\}) = 0$.
- If $A \subset W$ is arbitrary, then

$$C_{p,k}(A) = \inf\{\|\varphi\|_{p,k} : \varphi \in D_{p,k}, \quad \tilde{\varphi} \geq 1 \text{ (} p, r \text{) - } q.e. \text{ on } A\}$$

- There exists a unique element $U_A \in D_{p,k}$ such that $\tilde{U}_A \geq 1$ (p, k) -quasi everywhere on A with $C_{p,k}(A) = \|U_A\|_{p,k}$, and $\tilde{U}_A \geq 0$ (p, k) -quasi everywhere. U_A is called the (p, k) -equilibrium potential of A .

Theorem 1: Let $T \in D'$ be a positive distribution and suppose that $T \in D_{q,-k}$ for some $q > 1, k \geq 0$. Then, if we denote by ν_T the measure associated to T , we have

$$\bar{\nu}_T(A) \leq \|T\|_{q,-k}(C_{p,k}(A))^{1/p},$$

for any set $A \subset W$, where $\bar{\nu}_T$ denotes the outer measure with respect to ν_T . In particular ν_T does not charge the slim sets.

Proof: Let V be an open set in W and let U_V be its equilibrium potential of order (p, k) . We have

$$\begin{aligned}\langle P_{1/n}T, U_V \rangle &= \int P_{1/n}T U_V d\mu \geq \int_V P_{1/n}T U_V d\mu \\ &\geq \int_V P_{1/n}T d\mu = \nu_{P_{1/n}T}(V).\end{aligned}$$

Since V is open, we have, from the fact that $\nu_{P_{1/n}T} \rightarrow \nu_T$ weakly,

$$\liminf_{n \rightarrow \infty} \nu_{P_{1/n}T}(V) \geq \nu_T(V).$$

On the other hand

$$\begin{aligned}\lim_{n \rightarrow \infty} \langle P_{1/n}T, U_V \rangle &= \langle T, U_V \rangle \leq \|T\|_{q,-k} \|U_V\|_{p,k} \\ &= \|T\|_{q,-k} C_{p,k}(V)^{1/p}.\end{aligned}$$

QED

An application

1) Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function from $\mathcal{S}'(\mathbb{R}^d)$ and suppose that (X_t) is a hypoelliptic (i.e., non-degenerate) diffusion on \mathbb{R}^d . We have the Ito formula

$$f(X_t) - f(X_u) = \int_u^t Lf(X_s) ds + \int_u^t \sigma_{ij}(X_s) \partial_i f(X_s) dW_s^j,$$

with the obvious notations. Note that, since we did not make any differentiability hypothesis about f , the above integrals are to be interpreted as the elements of D' . Suppose that Lf is a bounded measure on \mathbb{R}^d , from our result about the positive distributions, we see that $\int_u^t Lf(X_s) ds$ is a measure on W which does not charge the slim sets. By difference, so does the term $\int_u^t \sigma_{ij}(X_s) \partial_i f(X_s) dW_s^j$.

2) Apply this to $d = 1$, $L = \frac{1}{2}\Delta$ (i.e. $\sigma = 1$), $f(x) = |x|$. Then we have

$$|W_t| - |W_u| = \frac{1}{2} \int_u^t \Delta |x|(W_s) ds + \int_u^t \frac{d}{dx} |x|(W_s) dW_s.$$

As $\frac{d}{dx}|x| = \text{sign}(x)$, we have

$$\int_u^t \frac{d}{dx}|x|(W_s) dW_s = \int_u^t \text{sign}(W_s) dW_s = M_t^u$$

is a measure absolutely continuous with respect to μ . Since $\lim_{u \rightarrow 0} M_t^u = N_t$ exists in all L^p , so does

$$\lim_{u \rightarrow 0} \int_u^t \Delta|x|(W_s) ds$$

in L^p for any $p \geq 1$. Consequently $\int_0^t \Delta|x|(W_s) ds$ is absolutely continuous with respect to μ , i.e., it is a random variable. It is easy to see that

$$\Delta|x|(W_s) = 2\mathcal{E}_0(W_s)$$

i.e., we obtain

$$\int_0^t 2\mathcal{E}_0(W_s) ds = \int_0^t \Delta|x|(W_s) ds = 2l_t^0$$

which is the local time of Tanaka. Note that, although $\mathcal{E}_0(W_s)$ is singular with respect to μ , its Pettis integral is absolutely continuous with respect to μ .

2) If $F : W \rightarrow \mathbb{R}^d$ is a non-degenerate random variable, then for any $S \in \mathcal{S}'(\mathbb{R}^d)$ with $S \geq 0$ on $\mathcal{S}_+(\mathbb{R}^d)$, $S(F) \in D'$ is a positive distribution, hence it is a positive Radon measure on W . In particular $\mathcal{E}_x(F)$ is a positive Radon measure.

Distributions associated to $\Gamma(A)$

For a “tentative” generality we suppose that (W, H, μ) is an abstract Wiener space. Let A be a selfadjoint operator on H , we suppose that its spectrum lies in $]1, \infty[$, hence A^{-1} is bounded and $\|A^{-1}\| < 1$. Let

$$H_\infty = \bigcap_n \text{Dom}(A^n),$$

hence H_∞ is dense in H and $\alpha \mapsto (A^\alpha h, h)$ is increasing. Denote by H_α the completion of H_∞ with respect to the norm $|h|_\alpha^2 = (A^\alpha h, h)$; $\alpha \in \mathbb{R}$. Evidently $H'_\alpha \cong H_{-\alpha}$ (isomorphism). If $\varphi : W \rightarrow \mathbb{R}$ is a nice Wiener functional with $\varphi = \sum_{n=0}^{\infty} I_n(\varphi_n)$, define

$$\Gamma(A)\varphi = E[\varphi] + \sum_{n=1}^{\infty} I_n(A^{\otimes n} \varphi_n).$$

Definition: For $p > 1$, $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}$, we define $D_{p,k}^\alpha$ as the completion of polynomials (based on H_∞) with respect to the norm:

$$\|\varphi\|_{p,k;\alpha} = \|(I + \mathcal{L})^{k/2} \Gamma(A^{1/2})\varphi\|_{L^p(\mu)},$$

where $\varphi(w) = \text{polynomial}(\delta h_1, \dots, \delta h_n)$, $h_i \in H_\infty$.

If \mathcal{X} is a separable Hilbert space, $D_{p,k}^\alpha(\mathcal{X})$ is defined likewise.

Remark: i) If $\varphi = \exp(\delta h - \frac{1}{2}|h|^2)$ then we have

$$\Gamma(A)\varphi = \exp \delta(Ah) - \frac{1}{2}|Ah|^2.$$

ii) $D_{p,k}^\alpha$ is decreasing with respect to α , p and k .

Theorem 1: Let $(W^\alpha, H_\alpha, \mu_\alpha)$ be the abstract Wiener space corresponding to the Cameron-Martin space H_α . Let us denote by $D_{p,k}^{(\alpha)}$ the Sobolev space on W^α defined by

$$\|\varphi\|_{D_{p,k}^{(\alpha)}} = \|(I + \mathcal{L})^{k/2} \varphi\|_{L^p(\mu_\alpha, W^\alpha)}$$

Then $D_{p,k}^{(\alpha)}$ and $D_{p,k}^\alpha$ are isomorphic.

Remark: This isomorphism is not algebraic, i.e., it does not commute with the pointwise multiplication.

Proof: We have

$$E[e^{i\delta(A^{\alpha/2}h)}] = \exp \frac{1}{2}|A^{\alpha/2}h|^2 = \exp \frac{|h|_\alpha^2}{2}$$

which is the characteristic function of μ_α on W^α .

QED

Theorem 2: i) For $p > 2$, $\alpha \in \mathbb{R}$, $k \in \mathbb{Z}$, there exists some $\beta > \frac{\alpha}{2}$ such that

$$\|\varphi\|_{D_{p,k}^\alpha} \leq \|\varphi\|_{D_{2,k}^\beta}$$

consequently $\bigcap_{\alpha,k} D_{2,k}^\alpha = \bigcap_{\alpha,p,k} D_{p,k}^\alpha$.

ii) Moreover, for some $\beta > \alpha$ we have

$$\|\varphi\|_{D_{p,k}^\alpha} \leq \|\varphi\|_{D_{2,0}^\beta}.$$

Proof: i) We have

$$\begin{aligned} \|\varphi\|_{D_{p,k}^\alpha} &= \left\| \sum_n (1+n)^{k/2} I_n((A^{\alpha/2})^{\otimes n} \varphi_n) \right\|_{L^p} \\ &= \left\| \sum_n (1+n)^{k/2} e^{nt} e^{-nt} I_n((A^{\alpha/2})^{\otimes n} \varphi_n) \right\|_{L^p}. \end{aligned}$$

From the hypercontractivity of P_t , we can choose t such that $p = e^{2t} + 1$ then

$$\left\| \sum_n (1+n)^{k/2} e^{nt} e^{-nt} I_n(\dots) \right\|_p \leq \left\| \sum_n (1+n)^{k/2} e^{nt} I_n(\dots) \right\|_2.$$

Choose $\beta > 0$ such that $\|A^{-\beta}\| \leq e^{-t}$, hence

$$\begin{aligned} &\left\| \sum_n (1+n)^{k/2} e^{nt} I_n(\dots) \right\|_2 \\ &\leq \left\| \sum_n (1+n)^{k/2} \Gamma(A^\beta) \Gamma(A^{-\beta}) e^{nt} I_n((A^{\alpha/2})^{\otimes n} \varphi_n) \right\|_2 \\ &\leq \sum_n (1+n)^{k/2} \|I_n((A^{\beta+\alpha/2})^{\otimes n} \varphi_n)\|_2 \\ &= \|\varphi\|_{D_{2,k}^{2\beta+\alpha}}. \end{aligned}$$

ii) If we choose $\|A^{-\beta}\| < e^{-t}$ then the difference suffices to absorb the action of the multiplier $(1+n)^{k/2}$ which is of polynomial growth and the former gives an exponential decrease. QED

Corollary 1: We have similar relations for the any separable Hilbert space valued functionals.

Proof: Use the Khintchine inequality.

As another corollary we have

Corollary 2: i) $\nabla : \Phi \rightarrow \Phi(H_\infty) = \bigcap_\alpha \Phi(H_\alpha)$ and $\delta : \Phi(H_\infty) \rightarrow \Phi$ are continuous. Consequently ∇ and δ have continuous extensions as linear operators $\Phi' \rightarrow \Phi'(H_{-\infty})$ and $\Phi'(H_{-\infty}) \rightarrow \Phi'$.

ii) Φ is an algebra.

iii) For any $T \in \Phi'$, there exists some $\zeta \in \Phi'(H_{-\infty})$ such that $T = \langle T, 1 \rangle + \delta\zeta$.

Proof: i) Follows from Theorem 1 and 2.

ii) It is sufficient to show that $\varphi^2 \in \Phi$ if $\varphi \in \Phi$. This follows from the multiplication formula of the multiple Wiener integrals. (left to the reader).

iii) If $T \in \Phi'$, then there exists some $\alpha > 0$ such that $T \in D_{2,0}^{-\alpha}$, i.e., T under the isomorphism of Theorem 1 is in $L^2(\mu_\alpha, W^\alpha)$ on which we have Ito representation (cf. Appendix).

Proposition: Suppose that A^{-1} is p -nuclear, i.e., $\exists p \geq 1$ such that A^{-p} is nuclear. Then Φ is a nuclear Fréchet space.

Proof: This goes as in the white noise case, except that the eigenvectors of $\Gamma(A^{-1})$ are of the form $H_{\vec{\alpha}}(\delta h_{\alpha_1}, \dots, \delta H_n)$ with h_{α_i} are the eigenvectors of A . QED

Applications to positive distributions

Let $T \in \Phi'$ be a positive distribution. Then there exists some $D_{p,-k}^{-\alpha}$ such that $T \in D_{p,-k}^{-\alpha}$ and $\langle T, \varphi \rangle \geq 0$ for any $\varphi \in D_{q,k}^\alpha$, $\varphi \geq 0$. Hence $i_\alpha(T)$ is a positive functional on $D_{1,k}^{(\alpha)}$ (i.e., the Sobolev space on W^α). Therefore $i_\alpha(T)$ is a Radon measure on $W^{-\alpha}$. Hence we find that, in fact the support of T is $W^{-\alpha}$ which is much smaller than $H_{-\infty}$.

Open question: Find the smallest $W^{-\alpha}$?

Chapter VII

Characterization of independence of some Wiener functionals

1 Independence of Wiener functionals

In probability theory, one of the most important and difficult properties is the independence of random variables. In fact, even in the elementary probability, the tests required to verify the independence of three or more random variables get very quickly quite difficult. Hence it is very tempting to try to characterize the independence of random variables via the local operators that we have seen in the preceding chapters.

Let us begin with two random variables: let $F, G \in D_{p,1}$ for some $p > 1$. They are independent if and only if

$$E[e^{i\alpha F} e^{i\beta G}] = E[e^{i\lambda F}]E[e^{i\beta G}]$$

for any $\alpha, \beta \in \mathbb{R}$, which is equivalent to

$$E[a(F)b(G)] = E[a(F)]E[b(G)]$$

for any $a, b \in C_b(\mathbb{R})$.

Let us denote by $\tilde{a}(F) = a(F) - E[a(F)]$, then we have: F and G are independent if and only if

$$E[\tilde{a}(F) \cdot b(G)] = 0, \quad \forall a, b \in C_b(\mathbb{R}).$$

Since $e^{i\alpha x}$ can be approximated pointwise with smooth functions, we can suppose as well that $a, b \in C_b^1(\mathbb{R})$ (or $C_0^\infty(\mathbb{R})$). Since \mathcal{L} is invertible on the centered random variables, we have

$$\begin{aligned}
E[\tilde{a}(F)b(G)] &= E[\mathcal{L}\mathcal{L}^{-1}\tilde{a}(F) \cdot b(G)] \\
&= E[\delta\nabla\mathcal{L}^{-1}\tilde{a}(F) \cdot b(G)] \\
&= E[(\nabla\mathcal{L}^{-1}\tilde{a}(F), \nabla(b(G)))_H] \\
&= E[((I + \mathcal{L})^{-1}\nabla a(F), \nabla(b(G)))_H] \\
&= E[((I + \mathcal{L})^{-1}(a'(F)\nabla F), b'(G)\nabla G)_H] \\
&= E[b'(G) \cdot ((I + \mathcal{L})^{-1}(a'(F)\nabla F), \nabla G)_H] \\
&= E[b'(G) \cdot E[((I + \mathcal{L})^{-1}(a'(F)\nabla F), \nabla G)_H | \sigma(G)]] .
\end{aligned}$$

In particular choosing $a = e^{i\alpha x}$, we find that

Proposition 1: F and G (in $D_{p,1}$) are independent if and only if

$$E[((I + \mathcal{L})^{-1}(e^{i\alpha F}\nabla F), \nabla G)_H | \sigma(G)] = 0 \text{ a.s.}$$

However this result is not very useful, this is because of the non-localness property of the operator \mathcal{L}^{-1} . Let us however look at the case of multiple Wiener integrals:

First recall the following multiplication formula of the multiple Wiener integrals:

Lemma 1: Let $f \in \hat{L}^2([0, 1]^p)$, $g \in \hat{L}^2([0, 1]^q)$. Then we have

$$I_p(f) \cdot I_q(g) = \sum_{m=0}^{p \wedge q} \frac{p!q!}{m!(p-m)!(q-m)!} I_{p+q-2m}(f \otimes_m g),$$

where $f \otimes_m g$ denotes the contraction of order m of the tensor $f \otimes g$ (i.e., the partial scalar product of f and g in $L^2[0, 1]^m$).

By the help of this lemma we will prove:

Theorem 1: $I_p(f)$ and $I_q(g)$ are independent if and only if

$$f \otimes_1 g = 0 \text{ a.s. on } [0, 1]^{p+q-2}.$$

Proof: (\Rightarrow): By independence, we have

$$E[I_p^2 I_q^2] = p! \|f\|^2 q! \|g\|^2 = p! q! \|f \otimes g\|^2.$$

On the other hand

$$I_p(f) I_q(g) = \sum_0^{p \wedge q} m! C_p^m C_q^m I_{p+q-2m}(f \otimes_m g),$$

hence

$$\begin{aligned} & E[(I_p(f) I_q(g))^2] \\ &= \sum_0^{p \wedge q} (m! C_p^m C_q^m)^2 (p+q-2m)! \|f \hat{\otimes}_m g\|^2 \\ &\geq (p+q)! \|f \hat{\otimes} g\|^2 \quad (\text{dropping the terms with } m \geq 1). \end{aligned}$$

We have, by definition:

$$\begin{aligned} \|f \hat{\otimes} g\|^2 &= \left\| \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} f(t_{\sigma(1)}, \dots, t_{\sigma(p)}) g(t_{\sigma(p+1)}, \dots, t_{\sigma(p+q)}) \right\|^2 \\ &= \frac{1}{((p+q)!)^2} \sum_{\sigma, \pi \in S_{p+q}} \lambda_{\sigma, \pi}, \end{aligned}$$

where S_{p+q} denotes the group of permutations of order $p+q$ and

$$\begin{aligned} \lambda_{\sigma, \pi} &= \int_{[0,1]^{p+q}} f(t_{\sigma(1)}, \dots, t_{\sigma(p)}) g(t_{\sigma(p+1)}, \dots, t_{\sigma(p+q)}) \cdot \\ &\quad f(t_{\pi(1)}, \dots, t_{\pi(p)}) g(t_{\pi(p+1)}, \dots, t_{\pi(p+q)}) dt_1 \dots dt_{p+q}. \end{aligned}$$

Without loss of generality, we may suppose that $p \leq q$. Suppose now that $(\sigma(1), \dots, \sigma(p))$ and $(\pi(1), \dots, \pi(p))$ has $k \geq 0$ elements in common. If we use the block notations, then

$$\begin{aligned} (t_{\sigma(1)}, \dots, t_{\sigma(p)}) &= (A_k, \tilde{A}) \\ (t_{\sigma(p+1)}, \dots, t_{\sigma(p+q)}) &= B \\ (t_{\pi(1)}, \dots, t_{\pi(p)}) &= (A_k, \tilde{C}) \\ (t_{\pi(p+1)}, \dots, t_{\pi(p+q)}) &= D \end{aligned}$$

where A_k is the subblock containing elements common to $(t_{\pi(1)}, \dots, t_{\pi(p)})$ and $(t_{\sigma(1)}, \dots, t_{\sigma(p)})$. Then we have

$$\lambda_{\sigma, \pi} = \int_{[0,1]^{p+q}} f(A_k, \tilde{A})g(B) \cdot f(A_k, \tilde{C})g(D) dt_1 \dots dt_{p+q}.$$

Note that $A_k \cup \tilde{A} \cup B = A_k \cup \tilde{C} \cup D = \{t_1, \dots, t_{p+q}\}$, $\tilde{A} \cap \tilde{C} = \emptyset$. Hence we have $\tilde{A} \cup B = \tilde{C} \cup D$. Since $\tilde{A} \cap \tilde{C} = \emptyset$, we have $\tilde{C} \subset B$ and $\tilde{A} \subset D$. From the fact that (\tilde{A}, B) and (\tilde{C}, D) are the partitions of the same set, we have $D \setminus \tilde{A} = B \setminus \tilde{C}$. Hence we can write, with the obvious notations:

$$\begin{aligned} \lambda_{\sigma, \pi} &= \\ &= \int_{[0,1]^{p+q}} f(A_k, \tilde{A})g(\tilde{C}, B \setminus \tilde{C}) \cdot f(A_k, \tilde{C})g(\tilde{A}, D \setminus \tilde{A}) dt_1 \dots dt_{p+q} \\ &= \int_{[0,1]^{p+q}} f(A_k, \tilde{A})g(\tilde{C}, B \setminus \tilde{C}) f(A_k, \tilde{C})g(\tilde{A}, B \setminus \tilde{C}) dA_k d\tilde{A} d\tilde{C} d(B \setminus \tilde{C}) \\ &= \int_{[0,1]^{q-p+2k}} (f \otimes_{p-k} g)(A_k, B \setminus \tilde{C}) (f \otimes_{p-k} g)(A_k, B \setminus \tilde{C}) \cdot dA_k d(B \setminus \tilde{C}) \\ &= \|f \otimes_{p-k} g\|_{L^2([0,1]^{q-p+2k})}^2 \end{aligned}$$

where we have used the relation $D \setminus \tilde{A} = B \setminus \tilde{C}$ in the second line of the above equalities. Note that for $k = p$ we have $\lambda_{\sigma, \pi} = \|f \otimes g\|_{L^2}^2$. Hence we have

$$\begin{aligned} &E[I_p^2(f)I_q^2(g)] \\ &= p! \|f\|^2 \cdot q! \|g\|^2 \\ &\geq (p+q)! \left[\frac{1}{((p+q)!)^2} \left[\sum_{\sigma, \pi} \lambda_{\sigma, \pi}(k=p) + \sum_{\sigma, \pi} \lambda_{\sigma, \pi}(k \neq p) \right] \right]. \end{aligned}$$

The number of $\lambda_{\sigma, \pi}$ with $(k = p)$ is exactly $\binom{p+q}{p} (p!)^2 (q!)^2$, hence we have

$$p!q! \|f\|^2 \|g\|^2 \geq p!q! \|f \otimes g\|^2 + \sum_{k=0}^{p-1} c_k \|f \otimes_{p-k} g\|_{L^2([0,1]^{q-p+2k})}^2$$

with $c_k > 0$. For this relation to hold we should have

$$\|f \otimes_{p-k} g\| = 0, \quad k = 0, \dots, p-1$$

in particular for $k = p-1$, we have

$$\|f \otimes_1 g\| = 0.$$

(\Leftarrow): From the Proposition 1, we see that it is sufficient to prove

$$((I + \mathcal{L})^{-1} e^{i\alpha F} \nabla F, \nabla I_q(g)) = 0 \text{ a.s.}$$

with $F = I_p(f)$, under the hypothesis $f \otimes_1 g = 0$ a.s.: Let us write $e^{i\alpha I_p(f)} = \sum_k I_k(h_k)$, then

$$\begin{aligned} e^{i\alpha I_p(f)} \nabla I_p(f) &= p \cdot \sum_k I_k(h_k) \cdot I_{p-1}(f) \\ &= p \cdot \sum_k \sum_{r=0}^{k \wedge (p-1)} \alpha_{p,k,r} I_{p-1+k-2r}(h_k \otimes_r f). \end{aligned}$$

Hence

$$(I + \mathcal{L})^{-1} e^{i\alpha F} \nabla F = p \cdot \sum_k \sum_{r=0}^{k \wedge (p-1)} (1 + p + k - 1 - 2r)^{-1} I_{p-1+k-2r}(h_k \otimes_p f).$$

When we take the scalar product with $\nabla I_q(g)$, we will have terms of the type:

$$\begin{aligned} (I_{p-1+k-2r}(h_k \otimes_r f), I_{q-1}(g))_H &= \\ &= \sum_{i=1}^{\infty} I_{p-1+k-2r}(h_k \otimes_r f(e_i)) I_{q-1}(g(e_i)). \end{aligned}$$

If we use the multiplication formula to calculate each term, we find the terms as

$$\begin{aligned} &\sum_{i=1}^{\infty} \int (h_k \otimes_r f(e_i))(t_1, \dots, t_{p+k-2r-1}) g(e_i)(t_1, \dots, t_{q-1}) dt_1 dt_2 \dots \\ &= \int \int_{\theta=0}^1 (h_k \otimes_r f(\theta))(t_1, \dots, t_{p+k-2r-1}) g(\theta, t_1, \dots, t_{q-1}) d\theta dt_1 \dots \end{aligned}$$

From the hypothesis we have

$$\int_0^1 f(\theta, t_1 \dots) g(\theta, s_1 \dots) d\theta = 0 \quad \text{a.s.},$$

hence the Fubini theorem completes the proof. QED

Corollary 1: Let f and g be symmetric L^2 -kernels respectively on $[0, 1]^p$ and $[0, 1]^q$. Let

$$S_f = \text{span}\{f \otimes_{p-1} h : h \in L^2([0, 1])^{p-1}\}$$

and

$$S_g = \text{span}\{g \otimes_{q-1} k : k \in L^2([0, 1])^{q-1}\}.$$

Then the following are equivalent:

- i) $I_p(f)$ and $I_q(g)$ are independent
- ii) $S_f \perp S_g$ in H
- iii) The Gaussian-generated σ -fields $\sigma\{I_1(k); k \in S_f\}$ and $\sigma\{I_1(l); l \in S_g\}$ are independent.

Proof: (i \Rightarrow ii): (i) implies that $f \otimes_1 g = 0$ a.s. If $a \in S_f$, $b \in S_g$ then $a = f \otimes_{p-1} h$ and $b = g \otimes_{q-1} k$ (rather finite sums of these kind of vectors). Then

$$\begin{aligned} (a, b) &= (f \otimes_{p-1} h, g \otimes_{q-1} k) = (f \otimes_1 g, h \otimes k)_{(L^2)^{\otimes p+q-2}} \quad (\text{Fubini}) \\ &= 0. \end{aligned}$$

(ii \Rightarrow i) If $(f \otimes_1 g, h \otimes k) = 0 \forall h \in L^2([0, 1])^{p-1}, k \in L^2([0, 1])^{q-1}$, then $f \otimes_1 g = 0$ a.s. since finite combinations of $h \otimes k$ are dense in $L^2([0, 1])^{p+q-2}$.

(ii \Leftrightarrow iii) Is obvious. QED

Proposition: Suppose that $I_p(f)$ is independent of $I_q(g)$ and $I_p(f)$ is independent of $I_r(h)$. Then $I_p(f)$ is independent of $\{I_q(g), I_r(h)\}$.

Proof: We have $f \otimes_1 g = f \otimes_1 h = 0$ a.s. This implies the independence of $I_p(f)$ and $\{I_q(g), I_r(h)\}$ from the calculations similar to those of the proof of sufficiency of the theorem. QED

In a similar way we have

Proposition: Let $\{I_{p_\alpha}(f_\alpha); \alpha \in J\}$ and $\{I_{q_\beta}(g_\beta); \beta \in K\}$ be two arbitrary families of multiple Wiener integrals. The two families are independent if and only if $I_{p_\alpha}(f_\alpha)$ is independent of $I_{q_\beta}(g_\beta)$ for all $(\alpha, \beta) \in J \times K$.

Corollary: If $I_p(f)$ and $I_q(g)$ are independent, so are also $I_p(f)(w + \tilde{h})$ and $I_q(g)(w + \tilde{k})$ for any $\tilde{h}, \tilde{k} \in H$.

Proof: Let us denote, respectively, by h and k the Lebesgue densities of \tilde{h} and \tilde{k} . We have then

$$I_p(f)(w + \tilde{h}) = \sum_{i=0}^p \binom{p}{i} (I_{p-i}(f), h^{\otimes i})_{H^{\otimes i}}.$$

Let us define $f[h^{\otimes i}] \in L^2[0, 1]^{p-i}$ by $I_{p-i}(f[h^{\otimes i}]) = (I_{p-i}(f), h^{\otimes i})$. If $f \otimes_1 g = 0$ then it is easy to see that

$$f[h^{\otimes i}] \otimes_1 g[k^{\otimes j}] = 0,$$

hence the corollary follows from Theorem 1.

QED

From the corollary it follows

Corollary: $I_p(f)$ and $I_q(g)$ are independent if and only if the germ σ -fields

$$\sigma\{I_p(f), \nabla I_p(f), \dots, \nabla^{p-1} I_p(f)\}$$

and

$$\sigma\{I_q(g), \dots, \nabla^{q-1} I_q(g)\}$$

are independent.

Corollary: Let $X, Y \in L^2(\mu)$, $Y = \sum_0^\infty I_n(g_n)$. If

$$\nabla X \otimes_1 g_n = 0 \quad \text{a.s. } \forall n,$$

then X and Y are independent.

Proof: This follows from Prop. 1

QED

Corollary: In particular, if $\tilde{h} \in H$, then $\nabla_{\tilde{h}}\varphi = 0$ a.s. implies that φ and $I_1(h) = \delta\tilde{h}$ are independent.

Chapter VIII

Moment inequalities for Wiener functionals

In several applications, as limit theorems, large deviations, etc., it is important to control the (exponential) moments of Wiener functionals by those of their derivatives. In this chapter we will give some results on this subject.

1 Exponential tightness

First we will show the following result which is a consequence of the Doob inequality:

Theorem 1: Let $\varphi \in D_{p,1}$ for some $p > 1$. Suppose that $\nabla\varphi \in L^\infty(\mu, H)$. Then we have

$$\mu\{|\varphi| > c\} \leq 2 \exp - \frac{(c - E[\varphi])^2}{2\|\nabla\varphi\|_{L^\infty(\mu, H)}^2} \quad \text{for any } c \geq 0.$$

Proof: Suppose that $E[\varphi] = 0$. Let $(e_i) \subset H$ be a complete, orthonormal basis of H . Define $V_n = \sigma\{\delta e_1, \dots, \delta e_n\}$ and let $\varphi_n = E[P_{1/n}\varphi|\nabla_n]$, where P_t denotes the Ornstein-Uhlenbeck semigroup on W . Then, from Doob's Lemma,

$$\varphi_n = f_n(\delta e_1, \dots, \delta e_n).$$

Note that, since $f_n \in \bigcap_{p,k} W_{p,k}(\mathbb{R}^n, \mu_n) \Rightarrow f_n$ is C^∞ on \mathbb{R}^n from the Sobolev injection theorem. Let $(B_t; t \in [0, 1])$ be an \mathbb{R}^n -valued Brownian motion. Then

$$\begin{aligned} \mu\{|\varphi_n| > c\} &= \mathbb{P}\{|f_n(B_1)| > c\} \\ &\leq \mathbb{P}\left\{\sup_{t \in [0,1]} |E[f_n(B_1)|\mathcal{B}_t]| > c\right\} \\ &= \mathbb{P}\left\{\sup_{t \in [0,1]} |Q_{1-t}f_n(B_t)| > c\right\}, \end{aligned}$$

where \mathbb{P} is the canonical Wiener measure on $C([0, 1], \mathbb{R}^n)$ and Q_t is the heat kernel associated to (B_t) , i.e.

$$Q_t(x, A) = \mathbb{P}\{B_t + x \in A\}.$$

From the Ito formula, we have

$$Q_{1-t}f_n(B_t) = Q_1f_n(B_0) + \int_0^t (DQ_{1-s}f_n(B_s), dB_s).$$

By definition

$$\begin{aligned} Q_1f_n(B_0) &= Q_1f_n(0) = \int f_n(y) \cdot Q_1(0, dy) = \\ &= \int f_n(y) e^{-\frac{|y|^2}{2}} \frac{dy}{(2\pi)^{n/2}} \\ &= E[E[P_{1/n}\varphi|V_n]] \\ &= E[P_{1/n}\varphi] \\ &= E[\varphi] \\ &= 0. \end{aligned}$$

Moreover we have $DQ_t f = Q_t Df$, hence

$$Q_{1-t}f_n(B_t) = \int_0^t (Q_{1-s}Df_n(B_s), dB_s) = M_t^n.$$

The Doob-Meyer process $\langle M^n, M^n \rangle_t$ of the martingale M^n can be controlled as

$$\begin{aligned} \langle M^n, M^n \rangle_t &= \int_0^t |DQ_{1-s}f_n(B_s)|^2 ds \leq \\ &\leq \int_0^t \|Df_n\|_{C_b}^2 ds = t \|\nabla f_n\|_{C_b}^2 = t \|\nabla f_n\|_{L^\infty(\mu_n)}^2 \leq \\ &\leq t \|\nabla \varphi\|_{L^\infty(\mu, H)}^2. \end{aligned}$$

Hence from the exponential Doob inequality, we obtain

$$P\left\{ \sup_{t \in [0,1]} |Q_{1-t}f_n(B_t)| > c \right\} \leq 2 \exp -\frac{c^2}{2\|\nabla \varphi\|_{L^\infty}^2}.$$

Hence

$$\mu\{|\varphi_n| > c\} \leq 2 \exp \frac{-c^2}{2\|\nabla \varphi\|_{L^\infty(\mu, H)}^2}.$$

Since $\varphi_n \rightarrow \varphi$ in probability the proof is completed. QED

Corollary: Under the hypothesis of the theorem, for any $\lambda < \frac{1}{2\|\nabla \varphi\|_{L^\infty}^2}$, we have

$$E[\exp \lambda |\varphi|^2] < \infty.$$

In particular, for any $\lambda < \frac{1}{2}$, we have

$$E[\exp \lambda \|w\|_W^2] < \infty \quad (\text{Fernique's lemma}).$$

Proof: The first part follows from the fact that, for $F > 0$ a.s.,

$$E[F] = \int_0^\infty P\{F > t\} dt.$$

The second part follows from the fact that $\|\nabla \|w\|_H\| \leq 1$. QED

We will see another application of this result later.

2 Coupling inequalities

We begin with the following elementary lemma (cf. [8]):

Lemma: Let X be a Gaussian r.v. on \mathbb{R}^d . Then for any convex function U on \mathbb{R} and C^1 -function $V : \mathbb{R}^d \rightarrow \mathbb{R}$, we have the following inequality:

$$E[U(V(X) - V(Y))] \leq E\left[U\left(\frac{\pi}{2}(V'(X), Y)_{\mathbb{R}^d}\right)\right],$$

where Y is an independent copy of X and E is the expectation with respect to the product measure.

Proof: Let $X_\theta = X \sin \theta + Y \cos \theta$. Then

$$\begin{aligned} V(X) - V(Y) &= \int_0^{\pi/2} \frac{d}{d\theta} V(X_\theta) d\theta \\ &= \int_0^{\pi/2} (V'(X_\theta), X'_\theta)_{\mathbb{R}^d} d\theta \\ &= \frac{\pi}{2} \int_0^{\pi/2} (V'(X_\theta), X'_\theta)_{\mathbb{R}^d} d\tilde{\theta} \end{aligned}$$

where $d\tilde{\theta} = \frac{d\theta}{\pi/2}$. Since U is convex, we have

$$U(V(X) - V(Y)) \leq \int_0^{\pi/2} U\left(\frac{\pi}{2}(V'(X_\theta), X'_\theta)\right) d\tilde{\theta}.$$

Moreover X_θ and X'_θ are two independent Gaussian random variables with the same law as the one of X . Hence

$$\begin{aligned} E[U(V(X) - V(Y))] &\leq \int_0^{\pi/2} E\left[U\left(\frac{\pi}{2}(V'(X), Y)\right)\right] d\tilde{\theta} \\ &= E\left[U\left(\frac{\pi}{2}(V'(X), Y)\right)\right]. \end{aligned} \quad \text{QED}$$

Now we will extend this result to the Wiener space:

Theorem 2: Suppose that $\varphi \in D_{p,1}$, for some $p > 1$ and U is a lower bounded, convex function (hence lower semi-continuous) on \mathbb{R} . We have

$$E[U(\varphi(w) - \varphi(z))] \leq E\left[U\left(\frac{\pi}{2}I_1(\nabla\varphi(w))(z)\right)\right]$$

where E is taken with respect to $\mu(dw) \times \mu(dz)$ on $W \times W$ and on the classical Wiener space, we have

$$I_1(\nabla\varphi(w))(z) = \int_0^1 \frac{d}{dt} \nabla\varphi(w, t) dz_t.$$

Proof: Suppose first that

$$\varphi = f(\delta h_1(w), \dots, \delta h_n(w))$$

with f smooth on \mathbb{R}^n , $h_i \in H$, $(h_i, h_j) = \delta_{ij}$. We have

$$\begin{aligned} I_1(\nabla\varphi(w))(z) &= I_1\left(\sum_{i=1}^n \partial_i f(\delta h_1(w), \dots, \delta h_n(w)) h_i\right) \\ &= \sum_{i=1}^n \partial_i f(\delta h_1(w), \dots, \delta h_n(w)) I_1(h_i)(z) \\ &= (f'(X), Y)_{\mathbb{R}^n} \end{aligned}$$

where $X = (\delta h_1(w), \dots, \delta h_n(w))$ and $Y = (\delta h_1(z), \dots, \delta h_n(z))$. Hence the inequality is trivially true in this case.

For general φ , let (h_i) be a complete, orthonormal basis in H , $V_n = \sigma\{\delta h_1, \dots, \delta h_n\}$ and let

$$\varphi_n = E[P_{1/n}\varphi|V_n],$$

where $P_{1/n}$ is the Ornstein-Uhlenbeck semigroup on W .

We have then

$$E[U(\varphi_n(w) - \varphi_n(z))] \leq E\left[U\left(\frac{\pi}{2}I_1(\nabla\varphi_n(w))(z)\right)\right].$$

Let π_n be the orthogonal projection from H onto $\text{span}\{h_1, \dots, h_n\}$. We have

$$\begin{aligned} I_1(\nabla\varphi_n(w))(z) &= I_1(\nabla_w E_w[P_{1/n}\varphi|V_n])(z) \\ &= I_1(E_w[e^{-1/n}P_{1/n}\pi_n\nabla\varphi|V_n])(z) \\ &= I_1(\pi_n E_w[e^{-1/n}P_{1/n}\nabla\varphi|V_n])(z) \\ &= E_z[I_1^z(E_w[e^{-1/n}P_{1/n}^w\nabla\varphi|V_n])|\tilde{V}_n] \end{aligned}$$

where \tilde{V}_n is the copy of V_n on the second Wiener space. Then

$$\begin{aligned} &E\left[U\left(\frac{\pi}{2}I_1(\nabla\varphi_n(w))(z)\right)\right] \\ &\leq E\left[U\left(\frac{\pi}{2}I_1(E_w[e^{-1/n}P_{1/n}\nabla\varphi|V_n])(z)\right)\right] \\ &= E\left[U\left(\frac{\pi}{2}e^{-1/n}E_w[I_1(P_{1/n}\nabla\varphi(w))(z)|V_n]\right)\right] \\ &\leq E\left[U\left(\frac{\pi}{2}e^{-1/n}I_1(P_{1/n}\nabla\varphi(w))(z)\right)\right] \\ &= E\left[U\left(\frac{\pi}{2}e^{-1/n}P_{1/n}^w I_1(\nabla\varphi(w))(z)\right)\right] \\ &\leq E\left[U\left(\frac{\pi}{2}e^{-1/n}I_1(\nabla\varphi(w))(z)\right)\right] \\ &= E\left[U\left(\frac{\pi}{2}P_{1/n}^{(z)} I_1(\nabla\varphi(w))(z)\right)\right] \\ &\leq E\left[U\left(\frac{\pi}{2}I_1(\nabla\varphi(w))(z)\right)\right]. \end{aligned}$$

Now Fatou's lemma completes the proof. QED

Let us give some consequences of this result:

Theorem 3: The following Poincaré inequalities are valid:

- i) $E[\exp(\varphi - E[\varphi])] \leq E\left[\exp\frac{\pi^2}{8}|\nabla\varphi|_H^2\right],$
- ii) $E[|\varphi - E[\varphi]|] \leq \frac{\pi}{2}E[|\nabla\varphi|_H].$
- iii) $E[|\varphi - E[\varphi]|^{2k}] \leq \left(\frac{\pi}{2}\right)^{2k} \frac{(2k)!}{2^{k!}} E[|\nabla\varphi|_H^{2k}], k \in \mathbb{N}.$

Remark: Let us note that the result of (ii) can not be obtained with the classical methods, such as the Ito-Clark representation theorem, since the optional projection is not a continuous map in L^1 -setting. Moreover, using the Hölder inequality and the Stirling formula, we deduce the following set of inequalities:

$$\|\varphi - E[\varphi]\|_p \leq p \frac{\pi}{2} \|\nabla\varphi\|_{L^p(\mu, H)},$$

for any $p \geq 1$. To compare this result with those already known, let us recall that using first the Ito-Clark formula, then the Burkholder-Davis-Gundy inequality combined with the convexity inequalities for the dual projections and some duality techniques, we obtain, only for $p > 1$ the inequality

$$\|\varphi - E[\varphi]\|_p \leq K p^{3/2} \|\nabla\varphi\|_{L^p(\mu, H)},$$

where K is some positive constant.

Proof: Replacing the function U of Theorem 2 by the exponential function, we have

$$\begin{aligned} E[\exp(\varphi - E[\varphi])] &\leq E_w \times E_z[\exp(\varphi(w) - \varphi(z))] \leq \\ &\leq E_w \left[E_z \left[\exp \frac{\pi}{2} I_1(\nabla\varphi(w))(z) \right] \right] \\ &= E \left[\exp \frac{\pi}{2} |\nabla\varphi(w)|_H^2 \right]. \end{aligned}$$

(ii) and (iii) are similar with $U(x) = |x|^k$, $k \in \mathbb{N}$.

QED

Theorem 4: Let $\varphi \in D_{p,2}$ for some $p > 1$ and that $\nabla|\nabla\varphi|_H \in L^\infty(\mu, H)$ (in particular, this is satisfied if $\nabla^2\varphi \in L^\infty(\mu, H \tilde{\otimes}_2 H)$). Then there exists some $\lambda > 0$ such that

$$E[\exp \lambda |\varphi|] < \infty.$$

Proof: From Theorem 3, (i), we know that $E[\exp \lambda|\varphi - E[\varphi]|] \leq 2E\left[\exp \frac{\lambda^2\pi^2}{8}|\nabla\varphi|^2\right]$. Hence it is sufficient to prove that

$$E[\exp \lambda^2|\nabla\varphi|^2] < \infty$$

for some $\lambda > 0$. However Theorem 1 applies since $\nabla|\nabla\varphi| \in L^\infty(\mu, H)$.
QED

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