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by

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ANDERSON'S BROWNIAN MOTION AND THE INFINITE DIMENSIONAL ORNSTEIN-UHLENBECK PROCESS

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Introduction

Anderson's construction [2] of Brownian motion as the standard part of a random walk with infinitesimal increments is one of the success stories of nonstandard analysis. Almost every subsequenct development in nonstandard probability theory is inspired – directly or indirectly – by Anderson's work. The purpose of this paper is to point out how another extremely important process in stochastic analysis – the infinite dimensional Ornstein-Uhlenbeck process – can be derived very easily from Anderson's construction.

To explain the basic idea, let us first recall what Anderson did. Choose an infinitely large integer $N \in {}^*\mathbf{N}$ and let $\Delta t = 1/N$. Think of

$$T = \{0, \Delta t, 2\Delta t, \ldots, 1 - \Delta t\}$$

as a hyperfinite timeline. Let Ω be the set of all internal functions $\omega : T \to \{1, -1\}$ and denote the internal, uniform probability measure on Ω by P (i.e. $P(A) = |A|/|\Omega|$ for all internal sets A). By $L(P)$ we shall mean the Loeb measure of P. Anderson's hyperfinite random walk $B : \Omega \times T \to {}^*\mathbf{R}$ is defined by

$$B(\omega, t) = \sum_{s<t} \omega(s)\sqrt{\Delta t}.$$

Anderson showed that for $L(P)$-a.a. ω, the internal function $B(\omega, \cdot)$ is S-continuous, and hence defines a continuous standard function $b(\omega, \cdot)$ ($= {}^\circ B(\omega, \cdot)$) from [0,1] to $\mathbf{R}$. Moreover, the standard process $b(\omega, t)$ is a Brownian motion on $(\Omega, L(P))$ (see, e.g. [1], [2], [4], [5], [6], or [10] for the details).

The infinite dimensional Ornstein-Uhlenbeck process u is a stochastic process taking values in the space $C([0, 1])$ of continuous functions. Intuitively, it looks like a continuous, random modification of Brownian paths which keeps the Wiener measure invariant. Using Anderson's construction we can make this intuition rigorous in the following way.

Pick an initial element ω_0 in Ω (and do it in such a way that $B(\omega_0, \cdot)$ is S-continuous). At time 0, toss an unfair coin for each $s \in T$ to decide whether you want to reverse the sign of s-th component $\omega_0(s)$ or not; the probability

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drop the superscript and write Θ for Θ^{ω_0}. We now define the *hyperfinite Ornstein-Uhlenbeck* process U $(= U^{\omega_0})$ by

$$U(\xi,t)(\cdot) = B(\Theta(\xi,t),\cdot)$$

In this section we shall study the basic properties of U as a nonstandard object. We begin with a simple observation.

1.1 Lemma. For all $s, t \in T$ and $\omega_0 \in \Omega$

$$Q\{\xi \in \Xi | \Theta^{\omega_0}(\xi,t)(s) = \omega_0(s)\} = [(1-\Delta t)^{t/\Delta t} + 1]/2 \approx (e^{-t}+1)/2$$

Proof: Let

$$p_t = Q\{\xi \in \Xi | \Theta^{\omega_0}(\xi,t)(s) = \omega_0(s)\}$$

Since the probability of switching the s-th coordinate between time t and time $t+\Delta t$ is $\Delta t/2$, we have

$$p_{t+\Delta t} = p_t(1-\Delta t/2) + (1-p_t)\Delta t/2$$

and hence

$$p_{t+\Delta t} = p_t(1-\Delta t) + \Delta t/2$$

Solving this difference equation with initial condition $p_0 = 1$, we get

$$p_t = [(1-\Delta t)^{t/\Delta t} + 1]/2 \approx (e^{-t}+1)/2$$

For each $s \in T$, let $\mathcal{F}_s$ be the internal algebra on Ξ generated by the random variables $\{\xi(t)(r) | t, r \in T, r < s\}$. Whenever we say that something is a *martingale* in the s-variable, we shall mean an internal martingale adapted to the filtration $\{\mathcal{F}_s\}$.

1.2 Lemma. Fix t and ω_0. Then

$$\begin{aligned} E_Q[U(\xi,t)(s+\Delta t) - U(\xi,t)(s) | \mathcal{F}_s] &= \omega_0(s)\sqrt{\Delta t}\,(1-\Delta t)^{t/\Delta t} \\ &= \omega_0(s)\sqrt{\Delta t}\,e^{-t} + o(\sqrt{\Delta t}) \end{aligned}$$

where $o(\sqrt{\Delta t})$ denotes a quantity which is infinitesimal compared to $\sqrt{\Delta t}$.

Proof: With p_t as in the preceding lemma, we have

$$\begin{aligned} &E_Q[U(\xi,t)(s+\Delta t) - U(\xi,t)(s) | \mathcal{F}_s] = \omega_0(s)\sqrt{\Delta t}\cdot p_t - \omega_0(s)\sqrt{\Delta t}\cdot(1-p_t) = \\ &\quad \omega_0(s)\sqrt{\Delta t}\,(2p_t - 1) = \omega_0(s)\sqrt{\Delta t}\,(1-\Delta t)^{t/\Delta t} = \omega_0(s)\sqrt{\Delta t}\,e^{-t} + o(\sqrt{\Delta t}) \end{aligned}$$

1.3 Lemma. Fix ω_0 and t. Then

$$U(\xi,t)(s) = (1-\Delta t)^{t/\Delta t}(U(\xi,0)(s) + K^{(\omega_0,t)}(\xi,s))$$

where $K^{(\omega_0,t)}$ is an S-continuous martingale such that the standard part of $K^{(\omega_0,t)}/\sqrt{e^{2t}-1}$ is a Brownian motion. Moreover, for all $s,s'\in T$

$$[K^{(\omega_0,t)}](\xi,s) - [K^{(\omega_0,t)}](\xi,s') \leq 4e^2|s-s'|$$

where $[K^{(\omega_0,t)}]$ denotes the quadratic variation of $K^{(\omega_0,t)}$.

Proof: The process defined by

$$\begin{aligned} K^{(\omega_0,t)}(\xi,s) &:= (1-\Delta t)^{-t/\Delta t}U(\xi,t)(s) - \sum_{r<s}\omega_0(r)\sqrt{\Delta t} = \\ &= (1-\Delta t)^{-t/\Delta t}U(\xi,t)(s) - U(\xi,0)(s) \end{aligned}$$

is a martingale by Lemma 1.2. Observe that since

$$|\Delta K^{(\omega_0,t)}(\xi,s)| \leq 2\sqrt{\Delta t}\,(1-\Delta t)^{-t/\Delta t} \leq 2e\sqrt{\Delta t}$$

we have

$$[K^{(\omega_0,t)}](\xi,s) - [K^{(\omega_0,t)}](\xi,s') \leq 4e^2|s-s'|$$

where $[K^{(\omega_0,t)}]$ is the quadratic variation of $K^{(\omega_0,t)}$. Thus $[K^{(\omega_0,t)}]$ is S-continuous, and so is $K^{(\omega_0,t)}$ according to Theorem 4.4.16 in [1].

Observe next that

$$\begin{aligned} E_Q(\Delta K^{(\omega_0,t)}(s)^2|\mathcal{F}_s) &= [\omega_0(s)\sqrt{\Delta t}\,((1-\Delta t)^{-t/\Delta t}-1)]^2 p_t + \\ &\quad +[\omega_0(s)\sqrt{\Delta t}\,((1-\Delta t)^{-t/\Delta t}+1)]^2(1-p_t) \\ &= \Delta t\Big\{[(1-\Delta t)^{-t/\Delta t}-1]^2(1+(1-\Delta t)^{t/\Delta t})/2 + \\ &\quad +[(1-\Delta t)^{-t/\Delta t}+1]^2(1-(1-\Delta t)^{t/\Delta t})/2\Big\} \\ &= \Delta t((1-\Delta t)^{-2t/\Delta t}-1) = \Delta t(e^{2t}-1) + o(\Delta t) \end{aligned}$$

Hence the compensator process $\langle K^{(\omega_0,t)}\rangle$ satisfies

$$\langle K^{(\omega_0,t)}(s)\rangle := \sum E_Q(\Delta K^{(\omega_0,t)}(r)^2|\mathcal{F}_r) \approx s((1-\Delta t)^{-2t/\Delta t}-1) \approx s(e^{2t}-1)$$

and since $\langle K^{(\omega_0,t)}(s)/\sqrt{e^{2t}-1}\rangle \approx s$, we see from (the proof of) Theorem 4.4.18 in [1] that the standard part of $K^{(\omega_0,t)}/\sqrt{e^{2t}-1}$ is a Brownian motion.

So far we have let s vary for fixed t. If we reverse the situation, we first obtain the following result.

1.4 Lemma. Fix ω_0 and s. Then

$$E_Q(U(\xi,t+\Delta t)(s)-U(\xi,t)(s)|U(\xi,t)(s)=x)=-x\Delta t$$

Proof: Let $\omega=\Theta^{\omega_0}(\xi,t)$. Since $B(\omega,s)=U(\xi,t)(s)=x$, we must have

$$|\{r\in T|r<s \text{ and } \omega(r)=1\}=\frac{s}{\Delta t}+\frac{x}{2\sqrt{\Delta t}}$$
$$|\{r\in T|r<s \text{ and } \omega(r)=-1\}=\frac{s}{\Delta t}-\frac{x}{2\sqrt{\Delta t}}$$

When we move from t to $t+\Delta t$, the expected number of switches among components belonging to the first set is $(s/\Delta t+x/2\sqrt{\Delta t})\cdot\Delta t/2=s/2+x\sqrt{\Delta t}/4$, and the expected number of switches among components belonging to the second set is $(s/\Delta t-x/2\sqrt{\Delta t})\cdot\Delta t/2=s/2-x\sqrt{\Delta t}/4$. Since each switch of the first kind changes U by $-2\sqrt{\Delta t}$ and each switch of the second kind changes it by $2\sqrt{\Delta t}$, the expected change is $-2\sqrt{\Delta t}\,(s/2+x\sqrt{\Delta t}/4)+2\sqrt{\Delta t}\,(s/2-x\sqrt{\Delta t}/4)=-x\Delta t$.

1.5 Lemma. Fix ω_0 and s, and assume that ω_0 is nearstandard. Then

$$U(\xi,t)(s)=(1-\Delta t)^{t/\Delta t}(U(\xi,0)(s)+\sum_{r=0}^{t-\Delta t}(1-\Delta t)^{-(r+\Delta t)/\Delta t}\Delta W(\xi,r))$$

where W is an S-continuous martingale whose standard part is $\sqrt{2s}$ times a Brownian motion. Moreover, if we let

$$N^{(\omega_0,s)}(\xi,t))=\sum_{r<t}(1-\Delta t)^{-(r+\Delta t)/\Delta t}\Delta W(\xi,r),$$

then

$$[N^{(\omega_0,s)}(\xi,t)]-[N^{(\omega_0,s)}(\xi,t)]\leq 9e^2|t-t'|$$

for all $t,t'\in T$.

Proof: By Lemma 1.4, the process

$$W(\xi,t)=U(\xi,t)(s)-U(\xi,0)(s)+\sum_{r<t}U(\xi,r)(s)\Delta t$$

is a martingale (by which we mean an internal martingale with respect to the obvious filtration $\mathcal{G}_t$). Rewriting this definition as

$$U(\xi,t)(s)=U(\xi,0)(s)-\sum_{r<t}U(\xi,r)(s)\Delta t+W(\xi,t)$$

we get a difference equation for U. Solving it, we see that

$$U(\xi,t)(s)=(1-\Delta t)^{t/\Delta t}U(\xi,0)(s)+\sum_{r<t}(1-\Delta t)^{(t-r-\Delta t)/\Delta t}\Delta W(\xi,r)$$
$$=(1-\Delta t)^{t/\Delta t}(U(\xi,0)(s)+\sum_{r<t}(1-\Delta t)^{-(r+\Delta t)/\Delta t}\Delta W(\xi,r))$$

Returning to W, we first observe that since the maximal value of $U(\xi, r)(s)$ is $s/\sqrt{\Delta t} \leq 1/\sqrt{\Delta t}$, we always have

$$|\Delta W(\xi, t)| \leq |\Delta U(\xi, t)(s)| + |U(\xi, t)(s)\Delta t| \leq 2\sqrt{\Delta t} + \sqrt{\Delta t} = 3\sqrt{\Delta t}$$

and hence

$$[W(\xi, t)] - [W(\xi, t')] \leq 9|t - t'|$$

for all $t, t' \in T$. By Theorem 4.2.16 and Proposition 4.4.3 in [1], this means that W is S-continuous and S-square integrable. It follows that if $U(\xi, 0)(s)$ is finite, then with probability one, $U(\xi, t)(s)$ remains finite for all t. From this we see that

$$E_Q(\Delta W(t)^2 | \mathcal{G}_t) \leq E_Q(|\Delta U(t)(s) - U(t)(s)\Delta t|^2 | \mathcal{G}_t) = (4\Delta t + o(\Delta t))s/2 = 2s\Delta t + o(\Delta t)$$

and hence

$$\langle W(t) \rangle \approx 2st$$

almost everywhere. By Theorem 4.4.18 in [1], the standard part of $W\sqrt{2s}$ is a Brownian motion.

Defining $N^{(\omega_0, s)}(\xi, t) = \sum_{r<t}(1 - \Delta t)^{-(r+\Delta t)/\Delta t}\Delta W(\xi, r)$, we finally observe that

$$[N^{(\omega_0,s)}(\xi, t)] - [N^{(\omega_0,s)}(\xi, t')] \leq e^2([W(\xi, t)] - [W(\xi, t')]) \leq 9e^2|t - t'|$$

Combining Lemmas 1.3 and 1.5, we get:

1.6 Proposition. For any nearstandard ω_0

$$U^{(\omega_0)}(\xi, t)(s) = (1 - \Delta t)^{t/\Delta t}(U(\xi, 0) + M^{(\omega_0)}(\xi, t, s))$$

where $M^{(\omega_0)}(\xi, t, s)$ is an S-continuous martingale in each of the variables s and t when the other is kept fixed. For each t, the standard part of $(\xi, s) \to M^{(\omega_0)}(\xi, t, s)$ is of the form $\sqrt{e^{2t} - 1}\, w(\xi, s)$ where w is a Brownian motion. For fixed s, the standard part of $(\xi, t) \to M^{(\omega_0)}(\xi, t, s)$ is of the form $\sqrt{2s} \int e^r db(\xi, r)$ where b is a Brownian motion.

Proof: Just observe that in the notation of Lemma 1.3, we have $M^{(\omega_0)}(\xi, t, s) = K^{(\omega_0, t)}(\xi, s)$, and in the notation of Lemma 1.5, $M^{(\omega_0)}(\xi, t, s) = N^{(\omega_0, s)}(\xi, t)$.

Remark. It may be helpful to rephrase the results of this section in standard terms. If we keep t fixed and vary s, then Lemma 1.3 tells us that the standard part $u(\xi, t)(s)$ of $U(\xi, t)(s)$ can be expressed as

$$u(\xi, t)(s) = e^{-t}(u(\xi, 0)(s) + \sqrt{e^{2t} - 1}\, w(s))$$

where $w(\cdot)$ is a Brownian motion (depending on t).

Proof: By the Burkholder-Davis-Gundy inequalities (see, e.g., page 126 of [1])

$$E_Q(|M(\xi,t,s)-M(\xi,t,s')|^p) \leq$$
$$\leq p\sqrt{12}\,E_Q([M](\xi,t,s)-[M](\xi,t,s'))^{p/2} \leq 2^p e^p p\sqrt{12}\,|s-s'|^{p/2}$$

where the last step uses that $[M](\xi,t,s)-[M](\xi,t,s') \leq 4e^2|s-s'|$ (recall Lemma 1.3 and that $M(\xi,t,s)=K^{(\omega_0,t)}(\xi,s)$).

We then do the same with s and t interchanged.

2.2 Lemma. Fix ω_0 and s. For each $p>1$,

$$E_Q(|M(\xi,t,s)-M(\xi,t',s)|^p) \leq K_p|t-t'|^{p/2}$$

where K_p is a constant depending only on p.

Proof: Just as above, but replacing Lemma 1.3 by Lemma 1.5.

2.3 Proposition. Assume that ω_0 is nearstandard. Then for almost all ξ, the process $(s,t)\to U(\xi,t)(s)$ is (jointly) S-continuous.

Proof: As already observed, it suffices to show that $M(\xi,t,s)$ is jointly S-continuous. Choose $p>4$, and note that by Lemmas 2.1 and 2.2

$$E_Q[|M(\xi,t,s)-M(\xi,t',s')|^p] \leq$$
$$\leq 2^p(E_Q[|M(\xi,t,s)-M(\xi,t',s)|^p]+E_Q[|M(\xi,t',s)-M(\xi,t',s')|^p]) \leq$$
$$\leq K\|(s,t)-(s',t')\|^{p/2}$$

for some constant K. The proposition follows from Kolmogorov's theorem.

We can now define the standard part of U as the process $u:\Xi\times[0,1]\to C([0,1])$ given by

$$u(\xi,t)(s)={}^{\circ}U(\xi,t')(s')$$

where $t'\approx t$ and $s'\approx s$. It follows immediately from Proposition 2.1 that u is continuous ($L(Q)$-a.e.).

3 The Brownian sheet representation

If m is the standard part of M, then clearly

$$u(\xi,t)(s)=e^{-t}(u(\xi,0)(s)+m(\xi,t,s))$$

If we fix t, then according to Proposition 1.6, the process $(\xi,s)\to m(\xi,t,s)$ is of the form $\sqrt{e^{2t}-1}\,w(\xi,s)$ for some Brownian motion w. Another way of expressing this relationship is to say that the process w' defined implicitly by $m(\xi,t,s) = w'(\xi, s(e^{2t}-1))$ is a Brownian motion. If we instead fix s, we know that $(\xi,t) \to m(\xi,t,s)$ is of the form $\sqrt{2s}\int e^r db(r)$, which means that it is a continuous martingale with quadratic variation

$$2s\int_0^t e^{2r}dr = s(e^{2t}-1)$$

Hence the process b' defined implicitly by $m(\xi,t,s) = b'(\xi, s(e^{2t}-1))$, is a Brownian motion.

With this observation in mind, it is natural to introduce a new random field $v(\xi,t,s)$ by

$$v(\xi, e^{2t}-1, s) = m(\xi,t,s)$$

or, put more explicitly,

$$v(\xi,t,s) = m(\xi, \ln(t+1)/2, s)$$

3.1 Proposition. v is a Brownian sheet with $v(0,s) = v(t,0) = 0$.

Proof: We have to show that v is a continuous, Gaussian field with covariance

$$E(v(t,s)v(t',s')) = \min(t,t')\cdot\min(s,s').$$

It is obvious that v is Gaussian, and the continuity was proved in the previous section. To compute the covariance, we may clearly assume that $t \leq t'$. There are two cases to consider; $s \leq s'$ and $s > s'$. Since they are easy and quite similar, we only treat the second one.

Observe that

$$\begin{aligned}&E(v(t,s)v(t',s')) =\\ &= E([v(t,s') + (v(t,s) - v(t,s')]\cdot[v(t,s') + (v(t',s') - v(t,s')]) =\\ &= E(v(t,s')^2)\end{aligned}$$

since v is a martingale in each variable, and time evolution in the t- and the s-direction are independent. But then

$$\begin{aligned}E(v(t,s)v(t',s')) &= E(v(t,s')^2) = E(m(\ln(t+1)/2, s')^2) =\\ &= t\cdot s' = \min(t,t')\cdot\min(s,s')\end{aligned}$$

3.2 Corollary. We have

$$u(\xi,t)(s) = e^{-t}(u(\xi,0)(s) + v(\xi, e^{2t}-1, s))$$

where v is a Brownian sheet with $v(0,s) = v(t,0) = 0$.

4 Infinite dimensional Ornstein-Uhlenbeck processes

Infinite dimensional Ornstein-Uhlenbeck processes were introduced by Malliavin [7] and plays a fundamental role in the Malliavin calculus. Put briefly, one might say that these processes and their infinitesimal generator – the Ornstein-Uhlenbeck operator – play the same part in infinite dimensional calculus as Brownian motion and the Laplace operator do in finite dimensions.

There are many ways of describing infinite dimensional Ornstein-Uhlenbeck processes. We could have taken the description in Corollary 3.2 as our definition (see Meyer [8]), but it is more conventional to use a characterization which says that an Ornstein-Uhlenbeck process u is a continuous, strong Markov process with values in $C([0,1])$ generating the semigroup

$$T_t f(x) = \int (e^{-t}x + \sqrt{1-e^{-2t}} \cdot y) dW(y) \tag{1}$$

where W is the Wiener measure on $C([0,1])$ (see, e.g., Watanabe [11]). We shall show that our process u satisfies these criteria.

4.1 Lemma. Assume that ω_0 is nearstandard with standard part x. If $f : C([0,1]) \to \mathbf{R}$ is square integrable with respect to the Wiener measure W, then

$$E_{L(Q)}[f(u^{(\omega_0)}(\xi,t)(\cdot)] = \int f(e^{-t}x(\cdot) + \sqrt{1-e^{2t}} \cdot y(\cdot)) dW(y)$$

Proof: Observe that if f is bounded and continuous, then

$$\begin{aligned} E_{L(Q)}[f(u^{(\omega_0)}(\xi,t)(\cdot)] &= {}^\circ E_Q[{}^*f(U^{(\omega_0)}(\xi,t)(\cdot)] = \\ &= {}^\circ E_Q[{}^*f((1-\Delta t)^{t/\Delta t}(B(\omega_0,\cdot) + M(\xi,t,\cdot))] = \\ &= \int f(e^{-t}x(\cdot) + \sqrt{1-e^{-2t}}\, y(\cdot)) dW(y) \end{aligned}$$

by nonstandard measure theory and Propositions 1.6 and 2.3. Using the Monotone Convergence Theorem, the result is easily extended to nonnegative, square integrable functions, and the general case follows by treating positive and negative parts separately.

4.2 Theorem. Assume ω_0 is nearstandard with standard part $x \in C([0,1])$. Then $u^{(\omega_0)}$ is an Ornstein-Uhlenbeck process starting at x.

Proof: We know from Proposition 2.3 that u is continuous, and the lemma takes care of (1). It only remains to check that u is a strong Markov process. Observe that for any two initial conditions ω_0 and ω_0', $U^{\omega_0}(\xi,t)(s) -$

$U^{\omega_0'}(\xi,t)(s) = U^{\omega_0}(\xi,0)(s) - U^{\omega_0'}(\xi,0)(s)$ for all t and s. Thus if the initial conditions are infinitely close, one process is just an infinitesimal translation of the other. From this the Markov property follows easily (for instance by an appeal to Theorem 5.4.17 in [1] or Theorem 6.8 in [5], but this is certainly an overkill).

5 The hyperfinite Ornstein-Uhlenbeck operator

Having completed our serious work, we may amuse ourselves by taking a look at some nonstandard consequences. Define the *hyperfinite Ornstein-Uhlenbeck operator* L to be the infinitesimal generator

$$LF(\omega) = E_Q(F(U^{(\omega)}(\xi,0) - F(U^{(\omega)}(\xi,\Delta t)))/\Delta t$$

of U. Observe that L acts on the space $L^2(\Omega,Q)$ of all internal functions $F:\Omega\to\mathbf{R}$ with the inner product $\langle F,G\rangle = \Sigma F(\omega)G(\omega)Q(\omega)$. For each subset A of Ω, let $\mathcal{X}_A \in L^2(\Omega)$ be the function

$$\mathcal{X}_A(\omega) = \begin{cases} \prod \omega(s) & \text{if } A \neq \varnothing \\ 1 & \text{if } A = \varnothing \end{cases}$$

Since the set $\{\mathcal{X}_A\}$ is orthonormal and has the right cardinality, it must be a basis for $L^2(\Omega)$, and hence any element $F \in L^2(\Omega)$ can be written uniquely as a sum $F = \sum_{A\subset\Omega} F(A)\mathcal{X}_A$ where $F(A) \in {}^*\mathbf{R}$ (this is often called the *Walsh expansion* of F).

5.1 Lemma. For all subsets A of Ω

$$L\mathcal{X}_A = \lambda_{|A|}\mathcal{X}_A$$

where

$$\lambda_m = \sum_{\substack{k\leq m\\ k \text{ odd}}} \binom{m}{k} (\Delta t/2)^{k-1}(1-\Delta t/2)^{m-k}$$

When A is finite, $\lambda_{|A|}$ is infinitely close to $|A|$, and when A is infinite, $\lambda_{|A|}$ is infinitesimal.

Proof:

$$L_{\mathcal{X}_A}(\omega) = E_Q[\mathcal{X}_A(U^{(\omega)}(\xi,0)) - \mathcal{X}_A(U^{(\omega)}(\xi,\Delta t))]/\Delta t =$$

$=$ (the probability that an odd number of components in A is switched) $\times$

$$\times\, 2\mathcal{X}_A(\omega)/\Delta t = \sum_{\substack{k\leq |A|\\ k \text{ odd}}} \binom{|A|}{k} (\Delta t/2)^{k}(1-\Delta t/2)^{|A|-k}\cdot 2\mathcal{X}_A(\omega)/\Delta t$$

$$= \lambda_{|A|}\mathcal{X}_A(\omega)$$

as finite dimensional Brownian motion is the standard part of a finite dimensional, nearest neighbor random walk), it adds some extra force to the idea that the Ornstein-Uhlenbeck process is really the infinite dimensional counterpart of finite dimensional Brownian motion.

Let us now turn to the second alternative construction. We still keep the discrete timeline T, but this time the state space Ω will be the space of all internal maps $\omega : T \to \mathbf{R}$, and the internal probability measure P on Ω will be the one making all increments $\Delta\omega(t) = \omega(t+\Delta t) - \omega(t)$ independent with distribution $N(0, \sqrt{\Delta t})$. Now let all the increments $\Delta\omega(t)$ perform independent, one-dimensional Ornstein-Uhlenbeck processes (scaled to keep the initial measure invariant). The standard part of this random motion will be an infinite dimensional Ornstein-Uhlenbeck process.

Again this alternative construction has some advantages and some disadvantages compared to the original one. It is less elementary, but it supports the intuitive idea that an infinite dimensional Ornstein-Uhlenbeck process is one where the infinitesimal increments perform independent, one-dimensional Ornstein-Uhlenbeck processes. It also lends itself more easily to the translation of finite dimensional calculus to an infinite dimensional setting. Finally, if one wants to study flows on Wiener space, the discrete setting is very restrictive and unnatural, while the continuous model offered by our alternative construction works quite nicely.

In my opinion, the three slightly different constructions above all deserve further study. Although none of them can claim to be the simplest approach to the infinite dimensional Ornstein-Uhlenbeck process (the one suggested by Corollary 3.2 is hard to beat), they all offer additional insight into the nature and structure of one of the most important processes in today's stochastic analysis.

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