The Pressure Equation in Anisotropic Medium

Jon Gjerde
Department of Mathematics
University of Oslo
Box 1053 Blindern, N-0316 Oslo
Norway

December 12, 1995

Abstract

In this paper we will prove existence and uniqueness theorems for the stochastic differential equations (both smooth and singular case)

$$\begin{aligned} \nabla_x \cdot \{\mathrm{Exp}\{\mathcal{W}_x^s\} \diamond \nabla_x u\} &= g(x) \quad x \in D \\ u(x) &= f(x) \quad x \in \partial D \end{aligned}$$

where D is a smooth domain, f, g are stochastic functions and $\mathrm{Exp}\{\mathcal{W}_x^s\}$ is a positive white noise matrix. We will show that these equations have solutions in the space $(\mathcal{S})^{-1}$ of generalized white noise distributions in a strong differentiation sense. The connection between the smooth and singular solution will also be studied.

Keywords: Generalized white noise distributions, Wick product, Hermite Transform.

§1 Introduction

We will in this paper apply white noise analysis to obtain existence and uniqueness theorems for some stochastic partial differential equations. Within the white noise analysis, there are several choices of possible solutions spaces, but we will only work in the space of generalized white noise distributions, known as the Kondratiev distribution space. This space is used in [V] to obtain Hilbert space methods for solving several classes of stochastic partial differential equations, including one-dimensional weak solutions of the equations we are going to solve. The one-dimensional pressure equation for fluid flow in a stochastic medium was first solved by Holden et al. ([HLØUZ3]). Note that this solution is actually given by an explicit formula, a task which seems impossible in the multi-dimensional case. Other interesting stochastic partial differential equations, all which are possible to solve explicitly, are

- The transport equation in a stochastic medium ([GjHØUZ]).
- The Dirichlet equation ([Gj2]).
- The Burgers equation ([HLØUZ2]).

- The Schrödinger equation ([HLØUZ]).

We will use the following scheme to solve our equations:

- Take the Hermite transform as developed in [LØU]. This transforms our original equation into a complex valued function on an infinite-dimensional ellipsoid.
- Instead of developing, if possible at all, complex existence and uniqueness theorems suitable for this theory, we will solve the transformed equation only for those elements in the ellipsoid which have real components. This can be done by modifying the well known method of continuity (as used in [F] to solve elliptic differential equation).
- This method will show that if the coefficients and input data are smooth in the parameter, then this is so also for the parameter in the solution.
- It is now possible to use well known maximum principles to obtain that the solution is real analytic in the parameter.
- From this, we may obtain that the solution is bounded analytic on the infinite-dimensional ellipsoid, and then conclude, by using the inverse Hermite transform, that our solution exists as a unique generalized white noise distribution.

Note that this scheme is a slight generalization from that used in [Gj3] to solve several parabolic stochastic differential equations.

For a physical interpretation of our equation, please read [HLØUZ3] and [Ø4].

§2 Preliminaries on multidimensional white noise

We will now give a short introduction of definitions and results from multidimensional Wick calculus, taken mostly from [Gj], [HLØUZ3], [HKPS] and [KLS].

In the following we will fix the parameter dimension n and space dimension m.

Let

$$\mathcal{N} := \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n)$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^n$, and

$$\mathcal{N}^* := (\prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n))^* \approx \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n)$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

Let $\mathcal{B} := \mathcal{B}(\mathcal{N}^*)$ denote the Borel σ-algebra on $\mathcal{N}^*$ equipped with the weak star topology and set

$$\mathcal{H} := \bigoplus_{i=1}^{m} \mathcal{L}^2(\mathbb{R}^n)$$

where $\oplus$ denotes orthogonal sum.

Since $\mathcal{N}$ is a countably Hilbert nuclear space (cf. eg. [Gj]) we get, using Minlos' theorem, a unique probability measure ν on $(\mathcal{N}^*, \mathcal{B})$ such that

$$\int_{\mathcal{N}^*} e^{i\langle\omega,\phi\rangle}\, d\nu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{\mathcal{H}}} \quad \forall\phi \in \mathcal{N}$$

where $\|\phi\|^2_{\mathcal{H}} = \sum_{i=1}^{m} \|\phi_i\|^2_{\mathcal{L}^2(\mathbb{R}^n)}$.

Note that if $m = 1$ then ν is usually denoted by μ.

THEOREM 2.1 [Gj] We have the following

1. $\otimes_{i=1}^{m} \mathcal{B}(\mathcal{S}'(\mathbb{R}^n)) = \mathcal{B}(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n))$
2. $\nu = \times_{i=1}^{m} \mu$

DEFINITION 2.2 [Gj] The triple

$$(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n), \mathcal{B}, \nu)$$

is called the (m-dimensional) (n-parameter) **white noise probability space.**

For $k = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$ let

$$h_k(x) := (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}})$$

be the Hermite polynomials and

$$\xi_k(x) := \pi^{-\frac{1}{4}}((k-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{k-1}(\sqrt{2}x) \ ; \ k \geq 1$$

the Hermite functions.

It is well known that the family $\{\tilde{e}_\alpha\} \subset \mathcal{S}(\mathbb{R}^n)$ of tensor products

$$\tilde{e}_\alpha := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_n}$$

forms an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^n)$.

Give the family of all multi-indecies $\zeta = (\zeta_1, \ldots, \zeta_n)$ a fixed ordering

$$(\zeta^{(1)}, \zeta^{(2)}, \ldots, \zeta^{(k)}, \ldots) \text{ where } \zeta^{(k)} = (\zeta_1^{(k)}, \ldots, \zeta_n^{(k)})$$

and define $\tilde{e}_k := \tilde{e}_{\zeta^{(k)}}$.

Let $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis of $\mathcal{H}$ we get from the collection

$$\{(\overbrace{0, \ldots, 0}^{i-1}, \tilde{e}_j, \overbrace{0, \ldots, 0}^{m-i}) \in \mathcal{H} \ \ 1 \leq i \leq m, 1 \leq j < \infty\}$$

and let $\gamma : \mathbb{N} \to \mathbb{N}$ be a function such that

$$e_k = (0, \ldots, 0, \tilde{e}_{\zeta(\gamma(k))}, 0, \ldots, 0).$$

Finally, let $(\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(k)}, \ldots)$ with $\beta^{(k)} = (\beta_1^{(k)}, \ldots, \beta_n^{(k)})$ be a sequence such that $\beta^{(k)} = \zeta^{(\gamma(k))}$.

If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index of non-negative integers we put

$$H_\alpha(\omega) := \prod_{i=1}^{k} h_{\alpha_i}(\langle \omega, e_i \rangle).$$

From theorem 2.1 in [HLØUZ] we know that the collection

$$\{H_\alpha(\cdot); \alpha \in \mathbb{N}_0^k; k = 0, 1, \ldots\}$$

forms an orthogonal basis for $\mathcal{L}^2(\mathcal{N}^*, \mathcal{B}, \nu)$ with $\|H_\alpha\|_{\mathcal{L}^2(\nu)} = \alpha!$ where $\alpha! = \prod_{i=1}^{k} \alpha_i!$.

This implies that any $f \in \mathcal{L}^2(\nu)$ has the unique representation

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

where $c_\alpha \in \mathbb{R}$ for each multi-index α and

$$\|f\|^2_{\mathcal{L}^2(\nu)} = \sum_\alpha \alpha! c_\alpha^2.$$

DEFINITION 2.3 [Gj] The m-dimensional **white noise map** is a map

$$W : \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n) \times \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R}^m$$

given by

$$W^{(i)}(\phi, \omega) := \omega_i(\phi_i) \quad 1 \le i \le m$$

PROPOSITION 2.4 [Gj] The m-dimensional white noise map W satisfies the following

1. $\{W^{(i)}(\phi, \cdot)\}_{i=1}^{m}$ is a family of independent normal random variables.
2. $W^{(i)}(\phi, \cdot) \in \mathcal{L}^2(\nu)$ for $1 \le i \le m$.

DEFINITION 2.5 [HLØUZ3] Let $0 \le \rho \le 1$.

- Let $(\mathcal{S}_n^m)^\rho$, the space of **generalized white noise test functions**, consist of all

  $$f = \sum_\alpha c_\alpha H_\alpha \in \mathcal{L}^2(\nu)$$

  such that

  $$\|f\|^2_{\rho,k} := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbf{N})^{\alpha k} < \infty \quad \forall k \in \mathbb{N}$$

- Let $(\mathcal{S}_n^m)^{-\rho}$, the space of **generalized white noise distributions**, consist of all formal expansions

$$F = \sum_\alpha b_\alpha H_\alpha$$

such that

$$\sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho} (2N)^{-\alpha q} < \infty \text{ for some } q \in \mathbb{N}$$

where

$$(2N)^\alpha := \prod_{i=1}^{k} (2^n \beta_1^{(i)} \cdots \beta_n^{(i)})^{\alpha_i} \text{ if } \alpha = (\alpha_1, \ldots, \alpha_k).$$

We know that $(\mathcal{S}_n^m)^{-\rho}$ is the dual of $(\mathcal{S}_n^m)^{\rho}$ (when the later space has the topology given by the seminorms $\|\cdot\|_{\rho,k}$) and if $F = \sum b_\alpha H_\alpha \in (\mathcal{S}_n^m)^{-\rho}$ and $f = \sum c_\alpha H_\alpha \in (\mathcal{S}_n^m)^{\rho}$ then

$$\langle F, f\rangle = \sum_\alpha b_\alpha c_\alpha \alpha!.$$

It is obvious that we have the inclusions

$$(\mathcal{S}_n^m)^1 \subset (\mathcal{S}_n^m)^\rho \subset (\mathcal{S}_n^m)^{-\rho} \subset (\mathcal{S}_n^m)^{-1} \quad \rho \in [0,1]$$

and in the remaining of this paper we will consider the larger space $(\mathcal{S}_n^m)^{-1}$.

DEFINITION 2.6 [HLØUZ3] The Wick product of two elements in $(\mathcal{S}_n^m)^{-1}$ given by

$$F = \sum_\alpha a_\alpha H_\alpha \ , \ G = \sum_\beta b_\beta H_\beta$$

is defined by

$$F \diamond G = \sum_\gamma c_\gamma H_\gamma$$

where

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$$

LEMMA 2.7 [HLØUZ3] We have the following

1. $F, G \in (\mathcal{S}_n^m)^{-1} \Rightarrow F \diamond G \in (\mathcal{S}_n^m)^{-1}$
2. $f, g \in (\mathcal{S}_n^m)^{1} \Rightarrow f \diamond g \in (\mathcal{S}_n^m)^{1}$

DEFINITION 2.8 [HLØUZ3] Let $F = \sum b_\alpha H_\alpha$ be given. Then the Hermite transform of F,denoted by $\mathcal{H}F$, is defined to be (whenever convergent)

$$\mathcal{H}F := \sum_\alpha b_\alpha z^\alpha$$

where $z = (z_1, z_2, \cdots)$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$ if $\alpha = (\alpha_1, \ldots, \alpha_k)$.

LEMMA 2.9 [HLØUZ3] If $F, G \in (\mathcal{S}_n^m)^{-1}$ then

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)$$

for all z such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exists.

LEMMA 2.10 [HLØUZ3] Suppose $g(z_1, z_2, \cdots)$ is a bounded analytic function on $\mathbf{B}_q(\delta)$ for some $\delta > 0, q < \infty$ where

$$\mathbf{B}_q(\delta) := \{\zeta = (\zeta_1, \zeta_2, \cdots) \in \mathbb{C}_0^{\mathbb{N}}; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2\mathbf{N})^{\alpha q} < \delta^2\}.$$

Then there exists $X \in (\mathcal{S}_n^m)^{-1}$ such that $\mathcal{H}X = g$.

LEMMA 2.11 [HLØUZ3] Suppose $X \in (\mathcal{S}_n^m)^{-1}$ and that f is an analytic function in a neighborhood of $\mathcal{H}X(0)$ in $\mathbb{C}$. Then there exists $Y \in (\mathcal{S}_n^m)^{-1}$ such that $\mathcal{H}Y = f \circ \mathcal{H}X$.

LEMMA 2.12 [HLØUZ3] Let $\{X_n\}_{n=1}^\infty, X$ be given elements in $(\mathcal{S}_n^m)^{-1}$.
Then the following are equivalent

1. $X_n \to X$ in $(\mathcal{S}_n^m)^{-1}$.
2. $\exists(\delta > 0, q \in \mathbb{N}, M > 0)$ such that $\mathcal{H}X_n(z) \to \mathcal{H}X(z)$ as $n \to \infty$ and $|\mathcal{H}X_n(z)| \leq M$ for all $z \in \mathbf{B}_q(\delta)$.

EXAMPLE 2.13 Define the x-shift of ϕ, denoted by ϕ_x, by $\phi_x(y) := \phi(y - x)$.Then

$$\mathrm{Exp}\{W_{\phi_x}^{(i)}\} \in (\mathcal{S}_n^m)^{-1} \quad 1 \leq i \leq m, \forall x \in \mathbb{R}^n$$

which is an immediate consequence of proposition 2.4 and lemma 2.11.

EXAMPLE 2.14 Let a symmetric $u \times u$-matrix $\mathcal{W}^s(\phi, \cdot)$ be given by

$$(\mathcal{W}^s)_{ij}(\phi, \cdot) := W^{(\hat{\sigma}(i,j))}(\phi, \cdot) \in (\mathcal{S}_n^{\frac{u(u+1)}{2}})^{-1}$$

where

$$\hat{\sigma}(i,j) := \begin{cases} \sigma(j - u + i(u - \frac{i-1}{2})) & \text{when } i \leq j \\ \hat{\sigma}(j,i) & \text{when } i > j \end{cases}$$

and σ is an arbitrary element in the permutation group of $\frac{u(u+1)}{2}$ elements. We are now able, using lemma 2.11, to construct the white noise exponential matrix, with components in $(\mathcal{S}_n^{\frac{u(u+1)}{2}})^{-1}$, as the matrix

$$\mathrm{Exp}\{\mathcal{W}^s\} := \sum_{k=0}^{\infty} \frac{1}{k!} (\mathcal{W}^s)^{\diamond k}$$

where the Wick-exponents are in ordinary matrix multiplication sense.

§3 Existence and uniqueness of the pressure equation with smooth noise

We will in this section apply a generalized version of the method of continuity as used in [F, GT] to solve deterministic elliptic differential equations of second order.

Suppose that $D \in \mathbb{R}^n$ is an open bounded set. We then use the notation $(0 < \gamma \leq 1)$

$$H_\gamma^D(v) := \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|v(x) - v(y)|}{|x-y|^\gamma}$$

and define the space $C^\gamma(D)$ of Hölder continuous functions on D (exponent γ) as all functions $u : D \mapsto \mathbb{R}$ with finite norm

$$\|u\|_\gamma := \|u\|_{L^\infty(D)} + H_\gamma^D(u) < \infty$$

($L^\infty(D)$ being the space of bounded measurable functions on D with essential supremum norm). Similarly, we define $C^{2+\gamma}(D)$ as all functions $u \in C^2(D)$ (the twice continuously differentiable functions on D) with finite norm

$$\|u\|_{2+\gamma} := \|u\|_{L^\infty(D)} + \sum_{1 \leq i \leq n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^\infty(D)} + \sum_{1 \leq i,j \leq n} \left\| \frac{\partial^2 u}{\partial x_i x_j} \right\|_{L^\infty(D)} + \sum_{1 \leq i,j \leq n} H_\gamma^D\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$$

Let now $\Theta_k^{\delta,q} = \mathbf{B}_q(\delta) \cap \mathbb{R}^k$.

We then define $C_{\Theta_k^{\delta,q}}^{\gamma,1}(D)$ as all functions $u : D \times \Theta_k^{\delta,q} \to \mathbb{R}$ with $D \ni x \mapsto u(x,\cdot) \in C^1(\Theta_k^{\delta,q})$ and finite norm

$$\|u\|_{\gamma,1}^{\delta,q,k} := \sup_{\lambda \in \Theta_k^{\delta,q}} \|u(\cdot,\lambda)\|_\gamma + \sup_{\lambda \in \Theta_k^{\delta,q}} \sum_{1 \leq i \leq k} \left\| \frac{\partial u}{\partial \lambda_i}(\cdot,\lambda) \right\|_\gamma < \infty$$

Finaly, we define $C_{\Theta_k^{\delta,q}}^{2+\gamma,1}(D)$ as all functions $u : D \times \Theta_k^{\delta,q} \to \mathbb{R}$ with $D \ni x \mapsto u(x,\cdot) \in C^1(\Theta_k^{\delta,q})$, $\Theta_k^{\delta,q} \ni \lambda \mapsto u(\cdot,\lambda) \in C^2(D)$, $\Theta_k^{\delta,q} \ni \lambda \mapsto \frac{\partial}{\partial \lambda_i} u(\cdot,\lambda) \in C^2(D)$ $(1 \leq i \leq k)$ and finite norm

$$\|u\|_{2+\gamma,1}^{\delta,q,k} := \sup_{\lambda \in \Theta_k^{\delta,q}} \|u(\cdot,\lambda)\|_{2+\gamma} + \sup_{\lambda \in \Theta_k^{\delta,q}} \sum_{1 \leq i \leq k} \left\| \frac{\partial u}{\partial \lambda_i}(\cdot,\lambda) \right\|_{2+\gamma} < \infty$$

LEMMA 3.1 $C_{\Theta_k^{\delta,q}}^{\gamma,1}$, $C_{\Theta_k^{\delta,q}}^{2+\gamma,1}$ are Banach spaces $\forall k \in \mathbb{N}, \delta > 0, q \in \mathbb{N}$.

PROOF:

We will show the lemma for $C_{\Theta_k^{\delta,q}}^{\gamma,1}$, $C_{\Theta_k^{\delta,q}}^{2+\gamma,1}$ follows similarly. Let $\{u_m\}_{m=1}^\infty$ be a Cauchy sequence in $C_{\Theta_k^{\delta,q}}^{\gamma,1}$, i.e., given $\epsilon > 0$ there exists $N > 0$ such that $\|u_n - u_m\|_{\gamma,1}^{\delta,q,k} \leq \epsilon$ whenever $n, m \geq N$. In particular,

$$\max\{|u_n(x,\lambda) - u_m(x,\lambda)|, |\frac{\partial u_n}{\partial \lambda_i}(x,\lambda) - \frac{\partial u_m}{\partial \lambda_i}(x,\lambda)|\} \leq \epsilon \quad \forall x \in D, \lambda \in \Theta_k^{\delta,q} \tag{1}$$

and

$$\frac{|u_n(x,\lambda)+u_m(y,\lambda)-u_n(y,\lambda)-u_m(x,\lambda)|}{|x-y|^\gamma} \le \epsilon \quad \forall (x \ne y) \in D, \lambda \in \Theta_k^{\delta,q} \tag{2}$$

and

$$\frac{|\frac{\partial u_n}{\partial \lambda_i}(x,\lambda)+\frac{\partial u_m}{\partial \lambda_i}(y,\lambda)-\frac{\partial u_n}{\partial \lambda_i}(y,\lambda)-\frac{\partial u_m}{\partial \lambda_i}(x,\lambda)|}{|x-y|^\gamma} \le \epsilon \quad \forall (x \ne y) \in D, \lambda \in \Theta_k^{\delta,q}, 1 \le i \le k \tag{3}$$

whenever $n, m \geq N$. From (1) it follows that there exists functions u and v_i such that $u_n \to u$ and $\frac{\partial u_n}{\partial \lambda_i} \to v_i$ pointwise in $x \in D, \lambda \in \Theta_k^{\delta,q}$. By passing to the limit in (1), the convergence is seen to be uniformly in $\lambda \in \Theta_k^{\delta,q}$ and $x \in D$. A classical result then says that u is λ_i-differentiable with $\frac{\partial u}{\partial \lambda_i} = v_i$. By passing to the limit in (2) and (3) also, we obtain, that $u \in C^{\gamma,1}_{\Theta_k^{\delta,q}}$ and $u_n \to u$ in the $\|\cdot\|^{\delta,q,k}_{\gamma,1}$-norm. ■

The following theorem is well known and will be used frequently:

THEOREM 3.2 [F, Theorem 18, page 86] Suppose that L is an elliptic operator with Hölder continuous coefficients (exponent γ) in $\bar{D}$, where D is an open bounded set with $\partial D \in C^{2+\gamma}$(as defined in [GT, page 94]). If $f \in C^{2+\gamma}(\bar{D}), g \in C^\gamma(D)$, then there exists a unique solution $u \in C^{2+\gamma}(D)$ of the problem

$$\begin{aligned} Lu &= g(x) \quad x \in D \\ u(x) &= f(x) \quad x \in \partial D \end{aligned}$$

Moreover, there exists a constant $K > 0$, only dependent on L, D and γ such that

$$\|u\|_{2+\gamma} \le K(\|f\|_{2+\gamma} + \|g\|_\gamma)$$

We are now ready to state the main result of this section:

THEOREM 3.3 Let D be an open, bounded domain in $\mathbb{R}^n$ with $\partial D \in C^{2+\gamma}$ $(0 < \gamma < 1)$. Assume further that we are given functions $g \in C^\gamma(D)$ and $f \in C^{2+\gamma}(\bar{D})$. Then the stochastic pressure equation in anisotropic medium with smooth noise

$$\nabla_x \cdot \{\mathrm{Exp}\{\mathcal{W}^s_{\phi_x}\} \diamond \nabla_x u\} = g(x) \quad x \in D \tag{4}$$

$$u(x) = f(x) \quad x \in \partial D \tag{5}$$

has a unique solution $D \ni x \mapsto u(x) \in (\mathcal{S}_n^{\frac{n(n+1)}{2}})^{-1}$.

REMARK 3.4 (Strong differentiation)

The derivatives in (4) are taken with respect to x in $(\mathcal{S}_n^{\frac{n(n+1)}{2}})^{-1}$. By this we mean that the limit in $(\mathcal{S}_n^{\frac{n(n+1)}{2}})^{-1}$,

$$\frac{\partial u}{\partial x_k}(x) := \lim_{\epsilon \to 0} \frac{u(x+\epsilon e_k) - u(x)}{\epsilon}$$

where e_k is the k'th unit vector in $\mathbb{R}^n$, exists together with the other derivatives. This is, because of lemma 2.12, equivalent with the existence of $\delta > 0, q \in \mathbb{N}$ such that

$$\lim_{\epsilon\to 0} \frac{\mathcal{H}u(x+\epsilon e_k, z) - \mathcal{H}u(x,z)}{\epsilon} = \frac{\partial}{\partial x_k}\mathcal{H}u(x,z)$$

pointwise, uniformly bounded, whenever $z \in \mathbf{B}_q(\delta)$.

PROOF:

We will find $q \in \mathbb{N}, \delta > 0$ and a function $D \ni x \mapsto \tilde{u}(x,\cdot) = \{\mathcal{H}u(x)\}(\cdot) \in A_b(\mathbf{B}_q(\delta))$, the space of all bounded analytic functions on $\mathbf{B}_q(\delta)$, which solves the equation

$$\nabla_x \cdot \{e^{\tilde{\mathcal{W}}^s_{\phi_x}} \nabla_x \tilde{u}\} = g(x) \quad x \in D \tag{6}$$
$$\tilde{u}(x,z) = f(x) \quad x \in \partial D \tag{7}$$

when $\mathbf{B}_q(\delta)$. The proof consists of several lemmas:

LEMMA 3.5 Let $q \in \mathbb{N}$ and $\delta > 0$ be arbitrary. Then

1. $|\tilde{W}^{(i)}_{\phi_x}(\lambda)| \le \delta \|\phi\|_{\mathcal{H}}$ whenever $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}^{\mathbb{N}}_0$.
2. $\frac{\partial}{\partial \lambda_l} \tilde{W}^{(i)}_{\phi_x} = (\phi^{(i)}_x, e^{(i)}_l)_{\mathcal{L}^2(\mathbb{R}^n)}$ if $\lambda = (\lambda_1, \cdots, \lambda_k)$ with $l \le k$.

PROOF:

We have

$$W^{(i)}_{\phi_x}(\omega) = \sum_{k=1}^{\infty} (\phi^{(i)}_x, e^{(i)}_k)_{\mathcal{L}^2(\mathbb{R}^n)} \langle \omega, e_k \rangle$$

which gives the second result and

$$\begin{aligned}|\tilde{W}^{(i)}_{\phi_x}(\lambda)|^2 &= |\sum_{k=1}^{\infty} (\phi^{(i)}_x, e^{(i)}_k)\lambda_k|^2 \le \sum_{k=1}^{\infty} (\phi^{(i)}_x, e^{(i)}_k)^2 \sum_{k=1}^{\infty} |\lambda_k|^2 \\ &\le \|\phi\|^2_{\mathcal{H}} \sum_{\alpha \ne 0} |\lambda^{\alpha}|^2 (2\mathbf{N})^{\alpha q} \le \delta^2 \|\phi\|^2_{\mathcal{H}}\end{aligned}$$

which gives the first result. ■

LEMMA 3.6 Let $\delta > 0, q \in \mathbb{N}$ and $\gamma \in (0,1)$ be given. Then there exists a constant $C > 0$ such that

$$\sup_{x\in D} \max_{i,j,k} \{\|\tilde{W}^{(k)}_{(\frac{\partial\phi}{\partial y_i})_x}\|_{\gamma}, \|\frac{\partial}{\partial \lambda_j}\tilde{W}^{(k)}_{\phi_x}\|_{\gamma}, \|\frac{\partial}{\partial \lambda_j}\tilde{W}^{(k)}_{(\frac{\partial\phi}{\partial y_i})_x}\|_{\gamma}\} \le C$$

PROOF:

We know from lemma 3.5 that

$$|\tilde{W}^{(i)}_{\phi_x}(\lambda) - \tilde{W}^{(i)}_{\phi_{x^0}}(\lambda)| \le 2\delta\|\phi\|_{\mathcal{H}} \\ \le 2\delta\|\phi\|_{\mathcal{H}}|x - x^0|^\gamma$$

whenever $|x - x^0| \ge 1$. Assume now that $|x - x^0| < 1$. By the mean-value theorem we get

$$|\tilde{W}^{(i)}_{\phi_x}(\lambda) - \tilde{W}^{(i)}_{\phi_{x^0}}(\lambda)| \le (\sum_{l=1}^{n} \sup_{x,\lambda} |\tilde{W}^{(i)}_{(\frac{\partial\phi}{\partial y_l})(x)}(\lambda)|)|x - x^0| \\ \le (\sum_{l=1}^{n} \sup_{x,\lambda} |\tilde{W}^{(i)}_{(\frac{\partial\phi}{\partial y_l})_x}(\lambda)|)|x - x^0|^\gamma$$

and the first term is now estimated by using the boundedness of lemma 3.5.

The second and third terms are estimated similarly using lemma 3.5. ∎

LEMMA 3.7 Let $\delta > 0$ and $q \in \mathbb{N}$ be given. Then there exists a constant $\rho_0 > 0$ such that

$$\sum_{1\le i,j\le n} (e^{\tilde{W}^s_{\phi_x}})_{ij}\xi_i\xi_j \ge \rho_0|\xi|^2$$

whenever $\xi \in \mathbb{R}^n, x \in D$ and $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}^N_0$.

PROOF:

We have

$$\sum_{1\le i,j\le n} (e^{\tilde{W}^s_{\phi_x}})_{ij}\xi_i\xi_j = |\xi|^2 e^{\tilde{W}_{\psi(x,\xi)}}$$

where

$$\psi = \frac{1}{|\xi|^2} \sum_{1\le i,j\le n} \xi_i\xi_j\psi_{i,j}(x)$$

and $\psi_{i,j}$ is given in [Gj, page 27] with $\|\psi_{i,j}(x)\|_{\mathcal{H}} \le \|\phi\|_{\mathcal{H}}$. It then follows that we may choose $\rho_0 := e^{-\|\phi\|_{\mathcal{H}}\delta}$ which concludes the proof. ∎

LEMMA 3.8 Let $\delta > 0, \gamma \in (0,1)$ and $q \in \mathbb{N}$ be given. Then there exists constants $A_1 > 0, A_2 > 0$ such that

1. $\sup_{i,j} |(\partial^\alpha e^{\tilde{W}^s_{\phi_x}})_{i,j}|_\gamma \le A_1 A_2^{|\alpha|}$
2. $\sup_{i,j,h} |(\partial^\alpha \frac{\partial}{\partial x_h} e^{\tilde{W}^s_{\phi_x}})_{i,j}|_\gamma \le A_1 A_2^{|\alpha|}(1 + |\alpha|)$

for all $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}^N_0$.

PROOF:

We claim, because of lemma 3.6, that

$$A_1 := \sup_{x\in D} \max_{i,k}\{\|\tilde{W}^{(k)}_{(\frac{\partial\phi}{\partial y_i})_x}\|_\gamma, 1\} e^{n\|\tilde{W}^{(k)}_{\phi_x}\|_\gamma} \le \max\{C, 1\} e^{nC}$$

and

$$A_2 := n \sup_{x\in D} \max_{i,j,k}\{\|\frac{\partial}{\partial\lambda_j}\tilde{W}^{(k)}_{\phi_x}\|_\gamma, \|\frac{\partial}{\partial\lambda_j}\tilde{W}^{(k)}_{(\frac{\partial\phi}{\partial y_i})_x}\|_\gamma\} \le nC$$

are valid choices. We will in this proof use the consistent matrix-norm $\|\cdot\| := n\max_{1\le i,j\le n}\|(\cdot)_{ij}\|_\gamma$. Note that

$$e^{\tilde{\mathcal{W}}^s_{\phi_x}} = \sum_{k=0}^{\infty}\frac{1}{k!}(\tilde{\mathcal{W}}^s_{\phi_x})^k$$

where the convergence is in the $\|\cdot\|$-norm.

Claim 1:

Using the fact that $\partial^\alpha W_\phi = 0$ when $|\alpha| > 1$, we obtain that $(\partial^\alpha(\mathcal{W}^s_{\phi_x})^k)_{ij}$ consists of $n^{k-1}k(k-1)\cdots(k-|\alpha|+1)$ terms of type

$$\tilde{W}^{(\beta_1)}_{\phi_x}\cdots\tilde{W}^{(\beta_{k-|\alpha|})}_{\phi_x}\cdot\overbrace{\frac{\partial}{\partial\lambda_1}\tilde{W}^{(\beta_{k-|\alpha|+1})}_{\phi_x}}^{\alpha_1\text{-times}}\cdots\overbrace{\frac{\partial}{\partial\lambda_{l(\alpha)}}\tilde{W}^{(\beta_k)}_{\phi_x}}^{\alpha_{l(\alpha)}\text{-times}}$$

where $0 \le \beta_i \le \frac{n(n+1)}{2}$ $(1 \le i \le k)$ and $l(\alpha)$ is the length of α. From this we get the estimate

$$\begin{aligned}\|e^{\tilde{\mathcal{W}}^s_{\phi_x}}\| &\le \sum_{k=0}^{\infty}\frac{1}{k!}\|\partial^\alpha(\tilde{\mathcal{W}}^s_{\phi_x})^k\|\\ &\le \sum_{k=0}^{\infty}\frac{1}{k!}k(k-1)\cdots(k-|\alpha|+1)\|\tilde{\mathcal{W}}^s_{\phi_x}\|^{k-|\alpha|}A_2^{|\alpha|}\\ &\le A_1A_2^{|\alpha|}\end{aligned}$$

which finishes claim 1.

Claim 2:

As before, we obtain that $(\partial^\alpha\frac{\partial}{\partial x_h}(\tilde{\mathcal{W}}^s_{\phi_x})^k)_{ij}$ consists of $n^{k-1}k(k-1)\cdots(k-|\alpha|)$ terms of type

$$\tilde{W}^{(\beta_1)}_{(\frac{\partial\phi}{\partial y_h})_x}\tilde{W}^{(\beta_2)}_{\phi_x}\cdots\tilde{W}^{(\beta_{k-|\alpha|})}_{\phi_x}\cdot\overbrace{\frac{\partial}{\partial\lambda_1}\tilde{W}^{(\beta_{k-|\alpha|+1})}_{\phi_x}}^{\alpha_1\text{-times}}\cdots\overbrace{\frac{\partial}{\partial\lambda_{l(\alpha)}}\tilde{W}^{(\beta_k)}_{\phi_x}}^{\alpha_{l(\alpha)}\text{-times}}$$

and $\alpha_i n^{k-1}k(k-1)\cdots(k-|\alpha|+1)$ $(1 \le i \le l(\alpha))$ terms of type

$$\frac{\partial}{\partial\lambda_i}\{\tilde{W}^{(\beta_1)}_{(\frac{\partial\phi}{\partial y_h})_x}\}\tilde{W}^{(\beta_2)}_{\phi_x}\cdots\tilde{W}^{(\beta_{k-|\alpha|+1})}_{\phi_x}\cdot\overbrace{\frac{\partial}{\partial\lambda_1}\tilde{W}^{(\beta_{k-|\alpha|+2})}_{\phi_x}}^{\alpha_1\text{-times}}\cdots\overbrace{\frac{\partial}{\partial\lambda_i}\tilde{W}^{(\beta_{k-|\alpha|+1+i})}_{\phi_x}}^{(\alpha_i-1)\text{-times}}\cdots\overbrace{\frac{\partial}{\partial\lambda_{l(\alpha)}}\tilde{W}^{(\beta_k)}_{\phi_x}}^{\alpha_{l(\alpha)}\text{-times}}$$

Claim 2 now follows as in the proof of claim 1 above. ■

LEMMA 3.9 The equation given by (6) has a unique solution $\tilde{u}(x,\lambda)$ for each $\lambda \in \mathbf{B}_q(\delta)\cap\mathbb{R}^N_0$. Moreover, the function $\lambda \mapsto \tilde{u}(x,\lambda)$ is in $C^\infty(\Theta^{\delta,q}_k)$ $(\forall k\in\mathbb{N}, \delta > 0, q\in\mathbb{N})$ for each fixed $x \in D$.

PROOF:

Let

$$\mathcal{L}_t := t\mathcal{L} + (1-t)\Delta \quad t \in [0,1]$$

where $\mathcal{L}$ is the elliptic operator

$$\mathcal{L} := \sum_{i=1}^{n}\sum_{j=1}^{n}(e^{\tilde{\mathcal{W}}^s_{\phi_x}})_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}(e^{\tilde{\mathcal{W}}^s_{\phi_x}})_{ij}\frac{\partial}{\partial x_i}$$

we get from equation (6). Denote by $\Sigma_k^{\delta,q}$ the set of all values $t \in [0,1]$ for which the problem

$$\mathcal{L}_t u = g \quad x \in D, \lambda \in \Theta_k^{\delta,q}$$
$$u(x,\lambda) = 0 \quad x \in \partial D, \lambda \in \Theta_k^{\delta,q}$$

has a unique solution u in $C^{2+\gamma,1}_{\Theta_k^{\delta,q}}(D)$ for any g in $C^{\gamma,1}_{\Theta_k^{\delta,q}}(D)$. We will show that $1 \in \Sigma_k^{\delta,q}$.

Case 1: $(0 \in \Sigma_k^{\delta,q})$

We know from [KS, theorem 5.7.2] that

$$u(x,\lambda) = -\frac{1}{2}\hat{E}^x[\int_0^{\tau_D} g(b_s,\lambda)\, ds]$$

where $(b_s, \hat{P}^x)$ is a standard Brownian motion in $\mathbb{R}^n$, $\hat{E}^x$ denotes expectation with respect to $\hat{P}^x$ and $\tau_D = \inf\{s > 0; b_s \notin D\}$ solves

$$\Delta u = g \ ; \ u|_{\partial D} = 0$$

for each $\lambda \in \Theta_k^{\delta,q}$. In particular,

$$\|u(\cdot,\lambda)\|_{2+\gamma} \le K\|g(\cdot,\lambda)\|_\gamma \le K\|g\|_{\gamma,1}^{\delta,q,k}$$

for some constant $K > 0$ independent of $\lambda \in \Theta_k^{\delta,q}$. We see that

$$\frac{\partial u}{\partial \lambda_i}(x,\lambda) = -\frac{1}{2}\hat{E}^x[\int_0^{\tau_D}\frac{\partial g}{\partial \lambda_i}(b_s,\lambda)\, ds]$$

i.e, $\frac{\partial u}{\partial \lambda_i}$ solves

$$\Delta\frac{\partial u}{\partial \lambda_i} = \frac{\partial g}{\partial \lambda_i} \ ; \ \frac{\partial u}{\partial \lambda_i}\bigg|_{\partial D} = 0$$

and

$$\|\frac{\partial}{\partial \lambda_i}u(\cdot,\lambda)\|_{2+\gamma} \le K\|\frac{\partial g}{\partial \lambda_i}(\cdot,\lambda)\|_\gamma \le K\|g\|_{\gamma,1}^{\delta,q,k}$$

From this we obtain that $u \in C^{2+\gamma,1}_{\Theta_k^{\delta,q}}(D)$.

Case 2: $(0 \in \Sigma_k^{\delta,q} \Rightarrow 1 \in \Sigma_k^{\delta,q})$

Assume now that $t_0 \in \Sigma_k^{\delta,q}$. We then define a mapping $\mathcal{A}$ from $C^{2+\gamma,1}_{\Theta_k^{\delta,q}}(D)$ into itself given by $\mathcal{A}u = v$ where v is the solution of

$$\begin{aligned} \mathcal{L}_{t_0} v &= \mathcal{L}_{t_0} u - \mathcal{L}_t u + g \quad & x \in D \\ v(x,\lambda) &= 0 & x \in \partial D \end{aligned}$$

Note that the mapping $\mathcal{A}$ is well defined because of our assumption since

$$\begin{aligned} \|\mathcal{L}_s u\|_{\gamma,1}^{\delta,q,k} &\leq \sum_{i,j=1}^{n} s\|(e^{\tilde{\mathcal{W}}^s_{\Phi_x}})_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}\|_{\gamma,1}^{\delta,q,k} + (1-s)\|\Delta u\|_{\gamma,1}^{\delta,q,k} + s\sum_{i,j=1}^{n}\|\frac{\partial}{\partial x_j}(e^{\tilde{\mathcal{W}}^s_{\Phi_x}})_{ij}\frac{\partial u}{\partial x_i}\|_{\gamma,1}^{\delta,q,k} \\ &\leq \hat{K}\|u\|_{2+\gamma,1}^{\delta,q,k} < \infty \end{aligned}$$

where $\hat{K}$ is some constant, and we have used lemma 3.8 together with the fact that there exists a constant $H > 0$ ([GT, lemma 6.35]) such that

$$\|fg\|_{\gamma,1}^{\delta,q,k} \leq \|f\|_{\gamma,1}^{\delta,q,k}\|g\|_{\gamma,1}^{\delta,q,k} \; ; \; \|\frac{\partial u}{\partial x_i}\|_{\gamma,1}^{\delta,q,k} \leq H\|u\|_{2+\gamma,1}^{\delta,q,k} \; ; \; \|\frac{\partial^2}{\partial x_i \partial x_j}u\|_{\gamma,1}^{\delta,q,k} \leq H\|u\|_{2+\gamma,1}^{\delta,q,k}$$

We have

$$\begin{aligned} \|\mathcal{A}u_1 - \mathcal{A}u_2\|_{2+\gamma} &\leq K\|(t_0 - t)\mathcal{L}(u_1 - u_2) + (t - t_0)\Delta(u_1 - u_2)\|_\gamma \\ &\leq K_1|t - t_0|\|u_1 - u_2\|_{2+\gamma} \end{aligned} \tag{8}$$

where K_1 is independent of $\lambda, t_0, t, u_1, u_2$ and g. Similarly, we obtain using (8) and lemma 3.8 that

$$\begin{aligned} \|\frac{\partial(\mathcal{A}u_1 - \mathcal{A}u_2)}{\partial\lambda_i}\|_{2+\gamma} &\leq K\|\frac{\partial}{\partial\lambda_i}(\mathcal{L}_{t_0}(u_1 - u_2)) - \frac{\partial}{\partial\lambda_i}(\mathcal{L}_t(u_1 - u_2))\|_\gamma \\ &\quad + K\|(\frac{\partial}{\partial\lambda_i}\mathcal{L}_{t_0})(Au_1 - Au_2)\|_\gamma \\ &\leq K_2|t - t_0|\|u_1 - u_2\|_{2+\gamma} + K_2|t - t_0|\|\frac{\partial(u_1 - u_2)}{\partial\lambda_i}\|_{2+\gamma} \end{aligned}$$

where K_2 also is independent of $\lambda, t_0, t, u_1, u_2$ and g. It follows that

$$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_{2+\gamma,1}^{\delta,q,k} \leq (K_1 + kK_2)|t - t_0|\|u_1 - u_2\|_{2+\gamma,1}^{\delta,q,k}$$

By choosing $|t - t_0|$ small enough and using Banach's fixpoint theorem, we obtain a element $u \in C^{2+\gamma,1}_{\Theta_k^{\delta,q}}(D)$ such that $\mathcal{A}u = u$ or equivalent $\mathcal{L}_t u = g$ with $u(x) = 0$ on ∂D. We may now cover the interval $[0,1]$ by small intervals, since $0 \in \Sigma_k^{\delta,q}$, to obtain that $1 \in \Sigma_k^{\delta,q}$. ■

LEMMA 3.10 Let $q \in \mathbb{N}, \delta > 0$ and $\gamma \in (0,1)$ be given. Then there exists a constant $\bar{K} > 0$ such that

$$\sup_{\lambda \in \mathbf{B}_q(\delta)\cap\mathbb{R}_0^N} \|\partial^\alpha \tilde{u}(\cdot,\lambda)\|_{2+\gamma} \leq n^2\bar{K}A_1((3n^2\bar{K}A_1 + 3)A_2)^{|\alpha|}|\alpha|! \tag{9}$$

for all multi-indecies α.

PROOF:

Put $\bar{K} := \max\{1, K, H, K(\|g\|_{\gamma,1}^{\delta,q,k} + \|f\|_{2+\gamma,1}^{\delta,q,k})\}^3$. It then follows from the definition of $\bar{K}$ and theorem 3.2 that

$$\|\tilde{v}(\cdot,\lambda)\|_{2+\gamma} \leq K(\|g\|_\gamma + \|f\|_{2+\gamma}) \leq \bar{K}$$

from which it follows that formula (9) holds for $|\alpha| = 0$. By applying the operator $\frac{\partial}{\partial\lambda_i}$ $(i \in \mathbb{N})$ on equation (6), we obtain the equation

$$\nabla_x \cdot \{e^{\mathcal{W}^s_{\phi_x}} \nabla_x(\frac{\partial}{\partial\lambda_i}\tilde{u})\} = -\nabla_x \cdot \{\frac{\partial}{\partial\lambda_i}(e^{\mathcal{W}^s_{\phi_x}})\nabla_x\tilde{u}\}$$

which gives us the estimate

$$\|\tilde{u}(\cdot,\lambda)\|_{2+\gamma} \leq Kn^2(A_1A_2 + A_1A_2(1+1))HK(\|g\|_\gamma + \|f\|_{2+\gamma}) \leq 3n^2\bar{K}A_1A_2$$

as wanted. Let now α be given and assume formula (9) holds for all β with $|\beta| < |\alpha|$.

By applying the differential operator ∂^α on equation (6), we obtain

$$0 = \partial^\alpha(\nabla_x \cdot \{e^{\tilde{\mathcal{W}}^s_{\phi_x}}\nabla_x\tilde{u})\}) = \nabla_x \cdot \{e^{\tilde{\mathcal{W}}^s_{\phi_x}}\nabla_x\partial^\alpha\tilde{u}\} \tag{10}$$

$$+ \sum_{\substack{\beta\leq\alpha \\ \beta\neq 0}} \frac{\alpha!}{\beta!(\alpha-\beta)!}\nabla_x \cdot \{\partial^\beta(e^{\tilde{\mathcal{W}}^s_{\phi_x}})\nabla_x\partial^{\alpha-\beta}\tilde{u}\} \tag{11}$$

From our induction hypothesis, we obtain

$$\begin{aligned}
\|\partial^\alpha\tilde{u}\|_{2+\gamma} &\leq K\sum_{\substack{\beta\leq\alpha \\ \beta\neq 0}} \frac{\alpha!}{\beta!(\alpha-\beta)!}n^2A_1A_2^{|\beta|}(2+|\beta|)Hn^2\bar{K}A_1((3A_1n^2K+3)A_2)^{|\alpha-\beta|}|\alpha-\beta|! \\
&\leq \bar{K}\sum_{j=1}^{|\alpha|}\sum_{\substack{\beta\leq\alpha \\ |\beta|=j}} \frac{\alpha!}{\beta!(\alpha-\beta)!}n^2A_1A_2^j(2+j)n^2\bar{K}A_1((3A_1n^2\bar{K}+3)A_2)^{|\alpha|-j}(|\alpha|-j)! \\
&= \bar{K}\sum_{j=1}^{|\alpha|}\binom{|\alpha|}{j}n^2A_1A_2^j(2+j)n^2\bar{K}A_1((3A_1n^2\bar{K}+3)A_2)^{|\alpha|-j}(|\alpha|-j)! \\
&= n^2\bar{K}A_1((3A_1n^2\bar{K}+3)A_2)^{|\alpha|}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{(2+j)}{j!}\frac{n^2\bar{K}A_1}{(3A_1n^2\bar{K}+3)^j}
\end{aligned}$$

and since

$$\begin{aligned}
\sum_{j=1}^{|\alpha|}\frac{(2+j)}{j!}\frac{n^2\bar{K}A_1}{(3A_1n^2\bar{K}+3)^j} &\leq 3n^2\bar{K}A_1\sum_{j=1}^{\infty}\frac{1}{(3A_1n^2\bar{K}+3)^j} \\
&= 3n^2\bar{K}A_1\frac{1}{3n^2A_1\bar{K}+2} \leq 1
\end{aligned}$$

this concludes the proof. ■

LEMMA 3.11 The function $\tilde{u}^*$ given by

$$\tilde{u}^*(x,z) := \sum_\alpha \frac{1}{\alpha!}[\partial^\alpha\tilde{u}(x,0)]z^\alpha \tag{12}$$

solves (6) and $D \ni x \mapsto \tilde{u}^*(x,\cdot) \in A_b(\mathbf{B}_q(\delta))$.

PROOF:

Suppose now that $z = (z_1, \cdots, z_l)$ $(l \in \mathbb{N})$ with $s := \sum_{i=1}^{l} |z_i| < r := ((3n^2\bar{K}A_1 + 3)A_2)^{-1}$. Then

$$
\begin{aligned}
|\tilde{u}^*(x,z)| &\le \sum_{\alpha\in\mathbb{Z}_0^l} \frac{1}{\alpha!} |[\partial^\alpha \tilde{u}(x,0)]z^\alpha| \le \sum_{\alpha\in\mathbb{Z}_0^l} \frac{1}{\alpha!} n^2\bar{K}A_1|\alpha|!r^{-|\alpha|}|z^\alpha| \\
&= n^2\bar{K}A_1 \sum_{j=0}^{\infty} \sum_{\substack{|\alpha|=j \\ \alpha\in\mathbb{Z}_0^l}} \frac{j!}{\alpha!} r^{-j}|z^\alpha| \\
&= n^2\bar{K}A_1 \sum_{j=0}^{\infty} r^{-j}s^j = \frac{n^2\bar{K}A_1 r}{r-s} < \infty
\end{aligned}
$$

where the last constant is independent of $l \in \mathbb{N}$. Note now that if $z \in \mathbf{B}_q(\delta)$ then

$$\sum_{i=1}^{l} |z_i| \le \sum_{\alpha} |z^\alpha| \le \delta \sum_{\alpha} (2N)^{-\frac{\alpha q}{2}}$$

where the last sum is finite by [V] when $q \ge 2(1 + \ln n)$. We may now choose $\delta > 0$ so small that the last sum is less than r.

It only remain to show that $\tilde{u}^*$ solves (6) for all $z \in \mathbf{B}_q(\delta)$. Note that we may differentiate termwise in both expressions because of the estimates of lemma 3.10. The claim now follows since we may write

$$(e^{\tilde{\mathcal{W}}^s_{\Phi_x}})_{ij} = \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha[(e^{\tilde{\mathcal{W}}^s_{\Phi_x}})_{ij}(0)]z^\alpha$$

and insert this expression and equation (12) into equation (6). By comparing terms, when $z = \lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^{\mathbb{N}}$, we see that $\tilde{u}^*$ solves (6). Both sides have analytic expansions, so the equality follows for all $z \in \mathbf{B}_q(\delta)$. ■

LEMMA 3.12 The strong derivatives in equation (4) exist.

PROOF:

Let $\delta > 0$ and $q \in \mathbb{N}$ be chosen as in lemma 3.11 and put

$$\tilde{u}_k^\epsilon(x,z) := \frac{\tilde{u}(x+\epsilon e_k, z) - \tilde{u}(x,z)}{\epsilon}$$

where e_k is the k'th unit vector in $\mathbb{R}^n$, then

$$\tilde{u}_k^\epsilon(x,z) = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial^\alpha\tilde{u}(x+\epsilon e_k, 0) - \partial^\alpha\tilde{u}(x,0)}{\epsilon}\right) z^\alpha.$$

Now, by using the mean-value theorem and lemma 3.10, we obtain the inequality

$$\left|\frac{\partial^\alpha\tilde{u}(x+\epsilon e_k, 0) - \partial^\alpha\tilde{u}(x,0)}{\epsilon}\right| \le n^2\bar{K}A_1((3n^2\bar{K}A_1+3)A_2)^{|\alpha|}|\alpha|! \quad \forall\epsilon > 0$$

so by using the dominated convergence theorem

$$\lim_{\epsilon\to 0} \tilde{u}_k^\epsilon(x,z) = \frac{\partial \tilde{u}}{\partial x_k}(x,z)$$

which is, because of remark 3.4, what we wanted to prove. The other derivatives are proven to exist in a similar manner. ■

The proof of theorem 3.3 is now completed. ■

§4 Existence and uniqueness of the pressure equation with singular noise

The m-dimensional singular white noise is defined as

$$W_x(\omega) := (W_x^{(1)}(\omega), \cdots, W_x^{(m)}(\omega))$$

where

$$W_x^{(i)}(\omega) := \sum_{k=1}^{\infty} e_k^{(i)}(x) H_{\epsilon(k)}(\omega) \quad (1 \le i \le m)$$

and

$$\epsilon(k) := (\overbrace{0, \cdots, 0}^{k-1}, 1) \quad \text{is a multi-index.}$$

We now from [HP, Chap. 21] that there exists a constant B such that $\|\xi_k\|_\infty \le B \ \forall k \in \mathbb{N}$. From this it follows that

$$|\tilde{W}_x^{(i)}(\lambda)| = |\sum_{k=1}^{\infty} e_k^{(i)}(x)\lambda_k| \le B^n \sum_{k=1}^{\infty} |\lambda_k| \le B^n \delta \sum_{\alpha\neq 0} (2N)^{-\frac{\alpha q}{2}}$$

which is finite when $q \ge 2(1+\ln n)$, proving that $W_x^{(i)} \in (\mathcal{S}_n^m)^{-1}$ $(1 \le i \le m)$. Note that it is also possible to define the symmetric singular exponential matrix $\operatorname{Exp}\{\mathcal{W}_x^s\}$ as an obvious analoge to example 2.14.

THEOREM 4.1 Let D be an open, bounded domain in $\mathbb{R}^n$ with $\partial D \in C^{2+\gamma}$ $(0 < \gamma < 1)$. Assume further that we are given functions $g \in C^\gamma(D)$ and $f \in C^{2+\gamma}(\bar{D})$. Then the stochastic pressure equation in anisotropic medium with singular noise

$$\nabla_x \cdot \{\operatorname{Exp}\{\mathcal{W}_x^s\} \diamond \nabla_x u\} = g(x) \quad x \in D \tag{13}$$

$$u(x) = f(x) \quad x \in \partial D \tag{14}$$

has a unique solution $D \ni x \mapsto u(x) \in (\mathcal{S}_n^{\frac{n(n+1)}{2}})^{-1}$.

PROOF:

We will find $q \in \mathbb{N}, \delta > 0$ and a function $D \ni x \mapsto \tilde{u}(x,z) \in A_b(\mathbf{B}_q(\delta))$ which solves the equation

$$\nabla_x \cdot \{e^{\tilde{\mathcal{W}}_x^s} \nabla_x \tilde{u}\} = g(x) \quad x \in D \tag{15}$$

$$\tilde{u}(x,z) = f(x) \quad x \in \partial D \tag{16}$$

when $\mathbf{B}_q(\delta)$. The proof consist of several lemmas and is organized similar to the proof of theorem 3.3. The lemmas which only need minor changes will be omitted.

LEMMA 4.2 Let $\gamma \in (0,1)$ be given. Then there exists $\delta > 0, q \in \mathbb{N}, C > 0$ such that

1. $\sup_{x\in D} \max_k \|\tilde{\mathcal{W}}_x^{(k)}\|_\gamma \leq C$
2. $\sup_{x\in D} \max_{i,k} \|\frac{\partial}{\partial x_i}\tilde{\mathcal{W}}_x^{(k)}\|_\gamma \leq C$
3. $\sup_{x\in D} \max_{i,k}\{\|\frac{\partial}{\partial \lambda_j}\tilde{\mathcal{W}}_x^{(k)}\|_\gamma, \|\frac{\partial^2}{\partial \lambda_j \partial x_i}\tilde{\mathcal{W}}_x^{(k)}\|_\gamma\} \leq C\mathbf{N}^{\epsilon(j)}$

for all $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^{\mathbf{N}}$.

PROOF:

Note that (see [HP], [HKPS]) the Hermite functions satisfy $\sup_{x\in\mathbb{R}}\{|\xi_k'(x)|, |\xi_k''(x)|\} \leq Bk$ for some constant $B > 0$. Since

$$|\frac{\partial}{\partial x_i}\tilde{\mathcal{W}}_x^{(k)}(\lambda)| = |\sum_{h=1}^{\infty} \frac{\partial}{\partial x_i} e_h^{(k)}(x)\lambda_h| \leq B^n \sum_{h=1}^{\infty} \mathbf{N}^{\epsilon(h)}|\lambda_h| \leq B^n\delta \sum_{\alpha\neq 0} (2\mathbf{N})^{-\frac{\alpha(2-q)}{2}}$$

and similary results hold for the other terms involved, the result follows as in the proof of lemma 3.6. ■

LEMMA 4.3 There exists a constant $\rho_0 > 0$ and $\delta > 0, q \in \mathbb{N}$ such that

$$\sum_{1\leq i,j\leq n} (e^{\tilde{\mathcal{W}}_x^s})_{ij}\xi_i\xi_j \geq \rho_0|\xi|^2$$

whenever $\xi \in \mathbb{R}^n, x \in D$ and $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^{\mathbf{N}}$.

PROOF:

We have

$$\sum_{1\leq i,j\leq n} (e^{\tilde{\mathcal{W}}_x^s})_{ij}\xi_i\xi_j = |\xi|^2 e^{\frac{\xi^T}{|\xi|}\tilde{\mathcal{W}}_x\frac{\xi}{|\xi|}}$$

and we may now choose $\rho_0 = e^{-C\delta}$ where $C, \delta > 0, q \in \mathbb{N}$ is as in lemma 4.2. ■

LEMMA 4.4 Let $\gamma \in (0,1)$ be given. Then there exists constants $A_1 > 0, A_2 > 0$ and $\delta > 0, q \in \mathbb{N}$ such that

1. $\sup_{i,j} |(\partial^\alpha e^{\tilde{\mathcal{W}}_x^s})_{i,j}|_\gamma \leq A_1 A_2^{|\alpha|}\mathbf{N}^\alpha$
2. $\sup_{i,j,h} |(\partial^\alpha \frac{\partial}{\partial x_h} e^{\tilde{\mathcal{W}}_x^s})_{i,j}|_\gamma \leq A_1 A_2^{|\alpha|}(1+|\alpha|)\mathbf{N}^\alpha$

for all $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^N$.

PROOF:

Let $\delta > 0, q \in \mathbb{N}$ be as in lemma 4.2. The proof now follows as in the proof of lemma 3.8 together with lemma 4.2. In particular,

$$A_1 := \max\{C, 1\}e^{nC} \quad A_2 := nC$$

are valid choices. ■

LEMMA 4.5 Let $\gamma \in (0, 1)$ be given. Then there exists a constant $\bar{K} > 0$ and $\delta > 0, q \in \mathbb{N}$ such that

$$\sup_{\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^N} \|\partial^\alpha \tilde{u}(\cdot, \lambda)\|_{2+\gamma} \leq n^2 \bar{K} A_1((3n^2 \bar{K} A_1 + 3)A_2)^{|\alpha|} \mathbf{N}^\alpha |\alpha|! \tag{17}$$

for all multi-indecies α.

PROOF: This goes exactly as in the proof of lemma 3.10 if we use $\delta > 0, q \in \mathbb{N}$ from lemma 4.2. ■

LEMMA 4.6 There exist $\delta > 0, q \in \mathbb{N}$ such that the function $\tilde{u}^*$ given by

$$\tilde{u}^*(x, z) := \sum_\alpha \frac{1}{\alpha!}[\partial^\alpha \tilde{u}(x, 0)] z^\alpha \tag{18}$$

solves (15) and $D \ni x \mapsto \tilde{u}^*(x, \cdot) \in A_b(\mathbf{B}_q(\delta))$.

PROOF: Let $\hat{\delta} > 0, \hat{q} \in \mathbb{N}$ be as in lemma 4.2. We will from now on assume that $0 < \delta < \hat{\delta}, q \geq \hat{q}$. Suppose now that $z = (z_1, \cdots, z_l)$ $(l \in \mathbb{N})$ with $s := \sum_{i=1}^l |z_i| < r := ((3n^2 \bar{K} A_1 + 3)A_2)^{-1}$. Then

$$\begin{aligned} |\tilde{u}^*(x, z)| &\leq \sum_{\alpha \in \mathbb{Z}_0^l} \frac{1}{\alpha!} |[\partial^\alpha \tilde{u}(x, 0)] z^\alpha| \leq \sum_{\alpha \in \mathbb{Z}_0^l} \frac{1}{\alpha!} n^2 \bar{K} A_1 |\alpha|! r^{-|\alpha|} \mathbf{N}^\alpha |z^\alpha| \\ &= n^2 \bar{K} A_1 \sum_{j=0}^\infty \sum_{\substack{|\alpha|=j \\ \alpha \in \mathbb{Z}_0^l}} \frac{j!}{\alpha!} r^{-j} |(\mathbf{N}z)^\alpha| \end{aligned}$$

where $(\mathbf{N}z)^\alpha = (\mathbf{N}^{\epsilon(1)} z_1, \cdots, \mathbf{N}^{\epsilon(l)} z_l)^\alpha$. Note now that if $z \in \mathbf{B}_q(\delta)$ then

$$\sum_{i=1}^l |(\mathbf{N}z)_i| \leq \sum_\alpha \delta (2\mathbf{N})^{\epsilon_i (1 - \frac{q}{2})} \leq \delta \sum_\alpha (2\mathbf{N})^{\alpha(1 - \frac{q}{2})} < \infty$$

when $q \geq 4 + 2 \ln n$, from which we may choose $\delta > 0$ so small that the last sum is less than r.

■

The proof of theorem 4.1 is now completed. ■

We will now show that there is a nice connection between the solutions of theorem 3.3 and 4.1.

EXAMPLE 4.7 (Connection beween smooth and singular white noise)

Let $\psi \in \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n)$ with $\int \psi^{(i)}\,dx = 1$ and $\|\psi^{(i)}\|_{\mathcal{L}^1(\mathbb{R}^n)} \leq K \ (1 \leq i \leq m)$ be given. Define $\psi_k(x) := k^n \psi(kx)$, then $\psi_k^{(i)}(x) \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^n)$ by [R, proposition 2.2.3]. We have

$$\tilde{W}^{(i)}_{(\psi_k)_x}(z) = \sum_{l=1}^{\infty} ((\psi_k^{(i)})_x, e_l^{(i)})_{\mathcal{L}^2(\mathbb{R}^n)} z_l \to \sum_{l=1}^{\infty} e_l(x) z_l = \tilde{W}_x(z) \quad (1 \leq i \leq m)$$

and

$$\begin{aligned} |\tilde{W}^{(i)}_{(\psi_k)_x}(z)| &\leq \sum_{l=1}^{\infty} |((\psi_k)_x^{(i)}, e_l^{(i)})_{\mathcal{L}^2(\mathbb{R}^n)}||z_l| \\ &\leq KB^n \delta \sum_{\alpha} (2\mathbf{N})^{-\frac{\alpha q}{2}} \end{aligned} \tag{19}$$

so it follows from lemma 2.12 that $W^{(i)}_{(\psi_k)_x} \to W_x^{(i)}$ in $(\mathcal{S}_n^m)^{-1} \ (1 \leq i \leq m)$.

THEOREM 4.8 Let u_k be the smooth solutions given from theorem 3.3 with $\phi = \psi_k$ and u the singular solution given from theorem 4.1. Then $u_k \to u$ in $(\mathcal{S}_n^{\frac{n(n+1)}{2}})^{-1}$.

PROOF:

Note that we are able to show a version of lemma 4.2 (x replaced by ψ_k) which holds uniformly in k. We then obtain, as in the proof of lemma 4.5, that there exist constants B_1, B_2 such that

$$\|\partial^\alpha \tilde{u}_k(\cdot, 0)\|_{2+\gamma} \leq B_1 B_2^{|\alpha|} \mathbf{N}^\alpha |\alpha|! \quad (\forall k \in \mathbb{N}) \tag{20}$$

Note that $\tilde{u}_k(x,0) = \tilde{u}(x,0)$. Let now $\partial^\alpha \tilde{u}_{k_i}(x,0)$ be an arbitrary subsequence of $\partial^\alpha \tilde{u}_k(x,0)$. Then, because of inequality (20), we may use the Ascoli-Arzela theorem to obtain a function $\tilde{v}(\alpha, x) \in C^{2+\gamma}(D)$ and a subsequence $\partial^\alpha \tilde{u}_{k_{i_j}}(x,0)$ which converges to $\tilde{v}(\alpha,x)$ in $\|\cdot\|_{2+\gamma}$-norm. By using induction and uniqueness we may go to the limit in (10) (with $\lambda = 0$) to obtain that $\partial^\alpha \tilde{u}_k(x,0) \to \tilde{v}(\alpha,x) = \partial^\alpha \tilde{u}(x,0)$. The conclusion now follows as in the proof of lemma 3.12. ∎

REMARK 4.9 With only minor changes, we may show that both theorem 3.3 and theorem 4.1 are valid when $\bar{D} \ni x \mapsto f(x) \in (\mathcal{S}^{\frac{n(n+1)}{2}})^{-1}$ and $D \ni x \mapsto g(x) \in (\mathcal{S}^{\frac{n(n+1)}{2}})^{-1}$ are functions such that

$$\sup_{\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^{\mathbf{N}}} \|\partial^\alpha \tilde{g}(\cdot, \lambda)\|_\gamma \leq M_1 M_2^{|\alpha|} (\mathbf{N}^q)^\alpha |\alpha|! \quad \forall \alpha \in \mathbb{N}_0^k; k = 0, 1, \ldots$$

and

$$\sup_{\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^{\mathbf{N}}} \|\partial^\alpha \tilde{f}(\cdot, \lambda)\|_{2+\gamma} \leq M_1 M_2^{|\alpha|} (\mathbf{N}^q)^\alpha |\alpha|! \quad \forall \alpha \in \mathbb{N}_0^k; k = 0, 1, \ldots$$

for some constants $M_1 > 0, M_2 > 0$ and $q > 0$.

Acknowledgments: The author would like to thank B. Øksendal for useful conversations.

This work is supported by Vista, a research cooperation between The Norwegian Academy of Science and Letters and Den Norske Stats Oljeselskap A.S (Statoil).

References

[F] A. Friedman: *Partial differential equations of parabolic type.* Prentice-hall, Englewood Cliffs, N. J. 1964.

[GHLØUZ] H. Gjessing, H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe, T. -S. Zhang: *The Wick Product.* In H. Niemi et al. (eds): Frontiers in pure and applied probability. TVP press, Moscow 1, 1992, pp. 29-67.

[Gj] J. Gjerde: *Multidimensional white noise with applications:* Cand. Scient. Thesis, University of Oslo, 1993.

[Gj2] J. Gjerde: *Two classes of stochastic Dirichlet equations which admit explicit solution formulas.* Preprint, University of Oslo 1995.

[Gj3] J. Gjerde: *Existence and uniqueness theorems for some stochastic parabolic partial differential equations.* Preprint, University of Oslo, 1995.

[GjHØUZ] J. Gjerde, H. Holden, B. Øksendal, J. Ubøe and T. -S. Zhang: *An equation modelling transport of a substance in a stochastic medium.* In E. Bolthausen (ed): Stochastic analysis, random fields and applications. Proc. for Ascona Conference 1993. Birkhäuser Verlag.

[GS] I. M. Gelfand and G. E. Shilov: *Generalized Functions ,Vol.2 : Spaces of fundamental and generalized functions.* Academic Press 1968 (English translation).

[GT] D. Gilbarg and N. Trudinger: *Elliptic partial differential equations of second order.* 2nd ed., Springer-Verlag, New York, 1983.

[GV] I. M. Gelfand and N. Y. Vilenkin: *Generalized Functions ,Vol.4: Application of Harmonic Analysis.* Academic Press 1964 (English translation).

[H] T. Hida: *Brownian Motion.* Springer Verlag 1980.

[HKPS] T. Hida, H. -H. Kuo, J. Potthoff and L. Streit: *White Noise Analysis.* Kluwer 1993.

[HLØUZ] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T. -S. Zhang: *Stochastic boundary value problems. A white noise functional approach.* Prob. Th. Rel. Fields. 95, 1993.

[HLØUZ2] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T. -S. Zhang: *The Burgers equation with a noise force.* Comm. PDE. 19, 1994.

[HLØUZ3] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T. -S. Zhang: *The pressure equation for fluid flow in a stochastic medium.* Manuscript, University of Oslo, Norway, 1994.

[HP] E. Hille and R. S. Phillips: *Functional analysis and semigroups.* Amer. Math. Soc. Colloq. Publ. 31. 1957.

[I] K. Itô: *Multiple Wiener integral.* J. Math. Soc. Japan 3, pp. 157-169, 1951.

[J] F. John: *Partial differential equations.* Fourth edition. Springer Verlag. 1982.

[KLS] Y. G. Kondratiev, P. Leukert and L. Streit: *Wick calculus in Gaussian analysis.* Manuscript. University of Bielefeld 1994.

[KS] I. Karatzas and S. E. Shreve: *Brownian motion and stochastic calculus.* Springer Verlag 1988

[LSU] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'cevac: *Linear and quasilinear equations of parabolic type.* Translations of Math. Monographs, AMS, Rhode Island, 1968.

[LØU] T. Lindstrøm, B. Øksendal and J. Ubøe: *Stochastic differential equations involving positive noise.* In M. Barlow and N. Bingham (editors): Stochastic Analysis. Cambridge Univ. Press, pp. 261-303, 1991.

[LØU2] T. Lindstøm, B. Øksendal and J. Ubøe: *Wick multiplication and Ito-Skorohod stochastic differential equations.* In S. Albeverio et al (editors): Ideas and Methods in Mathematical Analysis,Stochastics and Applications. Cambridge Univ. Press, pp.183-206, 1992.

[LØU3] T. Lindstrøm, B. Øksendal and J. Ubøe: *Stochastic modelling of fluid flow in porous media.* In S. Chen and J. Yong (eds): Control Theory, Stochastic analysis and applications. World Scientific, 1991.

[M] R. A. Minlos: *Generalized random processes and their extension to a measure.* Selected translations in mathematical statistics and probability. No. 3, Am. Math. Soc., Providence, Rhode Island, 1963.

[R] J. Rauch: *Partial differential equations.* Springer Verlag, 1991.

[V] G. Våge: *Existence and uniqueness of solution for stochastic partial differential equations.* Manuscript, University of Trondheim (NTH), Oslo, 1994.

[Z] T. -S. Zhang: *Characterizations of white noise test functions and Hida distributions.* In Stochastics and stochastics reports, Vol. 41, pp. 71–87.

[Ø] B. Øksendal:*Stochastic Differential Equations (third edition).* Springer Verlag 1992.

[Ø2] B. Øksendal: *A stochastic approach to moving boundary problems.* To appear in M.Pinsky (editor): Diffusion Processes and Related Problems in Analysis.Birkhäuser, 1990

[Ø3] B. Øksendal: *Stochastic partial differential equations ; A mathematical connection between macrocosmos and microcosmos.* In A. I. Cardoro et al. (eds): Stochastic Analysis and applications in Physics. NATO ASI Series. Vol C449, Kluwer Academic Publishers, pp. 283–305, 1994.

[Ø4] B. Øksendal: *Stochastic partial differential equations and applications to hydrodynamics.* Preprint, University of Oslo 1994.

[ØZ] B. Øksendal and T. -S. Zhang: *The stochastic Volterra equation.* Preprint, University of Oslo 1992.