On the Classification of C*-algebras of Real Rank Zero, III: The Infinite Case

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Preface

Most of the mathematical work underlying this paper was done when GAE visited AK in Sapporo in the spring of 1993, and when O. B., G.A.E. and AK visited DEE in Swansea in the summer of 1993. The visit of GAE to the University of Hokkaido was supported by a Canada-Japan Bilateral Scientific Exchange Fellowship awarded by the Natural Sciences and Engineering Research Council of Canada and the Japan Society for the Promotion of Science.

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One of us, O.B. was meant to prepare the manuscript, but due to other distractions, the writing went rather slowly. As a result, original results of the work have gradually been subsumed by subsequent developments [LP], [ElR], [Kir], [Phi]. We have already now separated out from this work the technical homotopy tools in [BEEK] (which were also used in [ElR]). Since some of the techniques and observations of this sequel may also have independent interest (especially those of Section 5 regarding a standard form of unitaries in Cuntz-circle algebras). We now publish this second part of the work as a review paper.

The final preparation of the manuscript took place when DEE visited AK in Sapporo in the spring of 95, supported by the Japan Society for the Promotion of Science and the Royal Society under the Bilateral Programme between the Isaac Newton Research Institute for Mathematical Sciences, Cambridge and the Research Institute for Mathematical Sciences, Kyoto University.

1. Statement of the Theorem

Recently, in [BKRS], a remarkable non-commutative Rokhlin property was established for the quasifree shift on the Fermion algebra. This led to the result that the Cuntz algebra $\mathcal{O}_2$ is isomorphic to its tensor product with the Fermion algebra (i.e., the UHF C*-algebra M_{2^∞}). Going substantially beyond this, Rørdam gave in [Rør1] a K-theoretical classification of inductive limits of arbitrary sequences of finite direct sums of matrix algebras over the Cuntz algebras $\mathcal{O}_n$, n finite and even. The invariant used was, in the stable case, exactly the invariant used in [Ell1] to classify AF algebras—namely, the set of Murray-von Neumann equivalence classes of projections, together with addition, in general only partially defined. (In [Ell1], this invariant was referred to as the (abstract) dimension range. It is of course also the invariant used by Murray and von Neumann in their classification of factors into types.)

The C*-algebras considered by Rørdam are of real rank zero, (by [Zha]), and have K_1 equal to zero (by [Cun2]). In this paper we shall extend Rørdam's classification to a class of algebras of real rank zero for which the K_1-group may be non-zero. (This step is analogous to the generalization in [Ell2] from matrix algebras over $\mathbb{C}$ to matrix algebras over $C(\mathbb{T})$, i.e., from AF algebras to real rank zero inductive limits of finite direct sums of matrix algebras over $C(\mathbb{T})$.)

We will only consider simple inductive limits which may be unital or not, and the invariant we shall use is simply K_*, together with the K_0-class of 1 if the algebra has a unit 1.

Theorem 1.1 *Let A and B be simple C*-algebras obtained as the inductive limits of sequences of direct sums of matrix algebras over algebras of the form*

$\mathcal{O}_n \otimes C(\mathbb{T})$ with $n = 2, 4, 6, \ldots$. Assume that A and B are either both unital or both nonunital and that all the maps between the summands $\mathcal{O}_n \otimes C(\mathbb{T})$ are either injective or zero. Let

$$\phi_* : K_*A \to K_*(B)$$

be an isomorphism of graded groups, and assume that

$$\phi_*([1_A]_0) = [1_B]_0$$

if A and B are unital.

It follows that ϕ_* arises from an isomorphism of C^*-algebras

$$\phi : A \to B$$

Remark 1.2 The hypotheses immediately imply that A and B are simple, σ-unital, infinite C*-algebras, and therefore of real rank zero, [Zha].

Remark 1.3

The restriction of n to even integers stems from the fact that the Rokhlin property only has been proved for the shift on M_{n^∞} when n is even. Enlarging the class of building blocks can be achieved by other means [Rør2], and very recently the Rokhling property has been established for general n, [Kis1], [Kis2].

Remark 1.4

It would be desirable to extend Theorem 1.1 to nonsimple inductive limits of this type, but even when A, B still are of real rank zero, we must then at least throw the ideal structure into the invariant, for example by using $K_*(A)$ together with the graded dimension range $D_*(A)$ of equivalence classes of partial unitaries as an invariant. But a recent result of Gong, [Gon], shows that this is not a complete invariant for C*-algebras of real rank zero and stable rank one which are inductive limits of finite direct sums of algebras of the form $M_n \otimes C(X)$ where X are 2-dimensional finite CW-complexes. His methods also show that $K_*(A), D_*(A)$ is not a complete invariant in our context, when A, B are of real rank zero.

In the hope that $D_*(A)$ will be part of the invariant in a more general situation, we describe it in more detail. We will also describe the elementary theory of our algebras for general real rank zero non-simple inductive limits in sections 2–3. $D_*(A)$ is the set of partial unitaries (the normal partial isometries) in the C*-algebra A modulo the relation of homotopy, with addition defined on classes with orthogonal representatives:

$$d_*(u_1 + u_2) = d_*(u_1) + d_*(u_2)$$

where u_1 and u_2 are partial unitaries with $u_1 u_2 = 0$ and $d_*(u)$ denotes the equivalence class of u (which we may perhaps call the graded dimension of u).

$D_*(A)$ is a local abelian semigroup with unit the class of 0. If u is a partial unitary in A, then the Murray-von Neumann equivalence class of the projection u^*u depends only on $d_*(u)$; let us denote it by $d_0(u)$. Similarly, the K_1-class of u depends only on $d_*(u)$; let us denote it by $d_1(u)$. Note that the set of classes $d_0(u)$, with addition defined as for d_*, is what was called the (abstract) dimension range of A, DA, in [Ell1]. Let us denote this now by D_0A, and refer to it also as the even part of the graded dimension range. In a somewhat similar way, let us denote the set of K_1-classes $d_1(u)$ by D_1A. The map

$$D_*A \to D_0A \oplus D_1A$$

$$d_*u \mapsto (d_0u, d_1u)$$

is additive. For the algebras that we shall consider, this map is injective.

Remark 1.5

It follows from Theorem 1.1 that if A_α denotes the irrational rotation C^*-algebras associated to the rotation α, then $\mathcal{O}_n \otimes A_\alpha \cong \mathcal{O}_n \otimes A_\beta$, if n is an even integer and α, β are any irrational numbers. This follows since A_α, A_β are inductive limits of sequences of direct sums of matrix algebras on $C(\mathbb{T})$ by [EE], and both tensor products have the same K-theory by the reasoning in Section 3.

In Sections 7 and 8 these techniques are extended to include $\mathcal{O}_n \otimes \mathcal{O}_m$ as building blocks. Consequently :

Theorem 1.6 *If n, m are positive even integers, and $k-1$ is the greatest common divisor of $n-1$ and $m-1$, then*

$$\mathcal{O}_n \otimes \mathcal{O}_m \cong \mathcal{O}_k \otimes \mathcal{O}_k \cong \mathcal{O}_k \otimes (C(\mathbb{T}) \rtimes G_k)$$

where G_k is the subgroup of $\mathbb{T}$ generated by $\{1/k^r; r \in \mathbb{N}\}$, acting on $\mathbb{T}$ by translations. Thus if $n-1$, $m-1$ are relatively prime, then

$$\mathcal{O}_n \otimes \mathcal{O}_m \cong \mathcal{O}_2$$

and

$$\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$$

for any simple unital C^*-algebra A in the class covered by Theorem 1.1.

Remark 1.7

A version of Theorem 1.1, with some more restrictions on the embeddings, has been proved in [GP]. The very recent complete classification of simple purely infinite separable σ-unital, nuclear C^*-algebras satisfying the universal coefficient theorem by Kirchberg and Phillips, [Kir], [Phi], has Theorems 1.1 and 1.6 as immediate corollaries.

2. Elementary Theory of the Inductive Limits

In this section we will characterize general inductive limits of finite direct sums of algebras of the form $M_n \otimes \mathcal{O}_k \otimes C(\mathbb{T})$, with $n = 1, 2, 3, ...$ and $k = 0, 2, 3, 4, ...$ with $\mathcal{O}_0 = \mathbb{C}$. The results will be more general than those which are needed in the context of the simple C*-algebras in Theorem 1.1.

We will characterize the inductive limits which are real rank zero by a property of small spectral variation, and we will characterize the ideal structure of these inductive limits. As usual, if p is a projection and u is a partial unitary in a C*-algebra A, then $[p]_0, [u]_1$ denote their canonical images in $K_0(A), K_1(A)$, respectively.

Recall from [Cun2] that a simple C*-algebra is called purely infinite if all nonzero hereditary sub-C*-algebras contain an infinite projection. If A is a unital purely infinite C*-algebra, we will in the next lemmas make use of the following known properties of A:

1. A has real rank zero, [Zha],

2. $K_0(A) = \{[p]_0 | p \text{ is a nonzero projection in } A\}$, [Cun2],

2. a. Any two nonzero projections in A has the same K_0 class if and only if they are Murray-von Neumann equivalent [Cun2],

3. $K_1(A) = U(A)/U_0(A)$, [Cun2],

4. Any element in $U_0(A)$ can be approximated by a unitary with finite spectrum, [Lin1],

5. Any nonzero projection in A contains a subprojection which is Murray-von Neumann equivalent to 1. [Consequence of 2 and 2a]

   5 will be used in the following forms:

5. a. For any $k_0 \in K_0(A)$ and any nonzero projection $p \in A$, there exists a nonzero projection $q \in A$ such that $q \leq p$ and $[q]_0 = k_0$,

5. b. For any $k_1 \in K_1(A)$ and any nonzero projection $p \in A$, there exists a unitary $u \in A$ such that $(1-p)u = u(1-p) = 1-p$ and $[u]_1 = k_1$.

   A useful consequence is

6. Given any $\varepsilon > 0$ and any two unitaries $u, v \in U(A)$ with $[u]_1 = [v]_1$ and Spec u, Spec v both ε-dense in $\mathbb{T}$, then there exists a $w \in U_0(A)$ such that
$$||u - wvw^*|| < 2\varepsilon$$
   (of course, when $[u]_1 \neq [1]_1$, then Spec $u =$ Spec $v = \mathbb{T}$ and the condition on the ε-density of Spec u and Spec v is automatic)[Ell3].

We are going to apply these results to the algebras $M_k \otimes \mathcal{O}_n$, where $k = 1, 2, \ldots$ and $n = 2, 3, 4, \ldots$. In the rest of this section we will consider C*-inductive limits $A = \varinjlim A_n$ where the C*-algebras A_n has the form
$$A_n = \bigoplus_{j=1}^{r_n} A_{n,j}$$
and
$$A_{n,j} = M_{[n,j]} \otimes \mathcal{O}_{\{n,j\}} \otimes C(\mathbb{T})$$
where $[n, j] \in \mathbb{N}$ and $\{n, j\} \in \{0, 2, 3, 4, \ldots\}$. Let $u = u_{n,j}$ denote the canonical unitary $1 \otimes 1 \otimes (z \to z)$ in $A_{n,j}$. If x is any element in A_n, let $\phi_{m,j;n}(x)$ denote

its image in $A_{m,j}$ by the mapping $A_n \to A_m \to A_{m,j}$, when $m \geq n$. The notation $\phi_n(x) = \phi_{\infty;n}(x)$ will be used to denote the image of x in A. The morphisms $A_n \to A_m$ are not assumed to be injective nor unital throughout this section. Let $1_{n,j}$ denote the central projection in A_n corresponding to the summand $A_{n,j}$.

Lemma 2.1 *If p is a nonzero projection in $B = M_k \otimes \mathcal{O}_n \otimes C(\mathbb{T})$, then*

$$pBp \cong M_i \otimes \mathcal{O}_n \otimes C(\mathbb{T})$$

where i is the class of $p(t)$ in $K_0(M_k \otimes \mathcal{O}_n)$ for any $t \in \mathbb{T}$.
(Thus $i \in \{1, ..., n-1\}$ when $n \geq 2$.)

Proof:

Let $q \in M_k \otimes \mathcal{O}_n$ be a fixed projection such that $[q]_0 = [p(t)]_0$ for all $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. For definiteness let $q = p(0)$. We may construct a norm continuous family u_t of unitaries such that $u_0 = 1$ and

$$u_t q u_t^* = p(t)$$

[BEEK]. Since $p(1) = q$ it follows that $u_1 \in \{q\}'$. Since $K_1(\mathcal{O}_n) = 0$ and $q(M_k \otimes \mathcal{O}_n)q \simeq M_i \otimes \mathcal{O}_n$, [Cun2], it follows from property 3, above, that qu_1q may be deformed in the unitary group of $q(M_k \otimes \mathcal{O}_n)q$ to the identity q of this algebra. It follows easily that we may modify $t \to u_t$ so that $u_1 = 1$. But then $u \in B$, and

$$\begin{aligned} pBp &\cong u^*pBpu = u^*puu^*Buu^*pu \\ &= qBq \cong M_i \otimes \mathcal{O}_n \otimes C(\mathbb{T}) \end{aligned}$$

where q also denotes the constant function $\mathbb{T} \ni t \to q$.

Let $u = u_{n,k}$. By a slight abuse of language we will use the term spectrum of $\phi(u) = \phi_{m,j;n}(u_{n,k})$ in $A_{m,j}$ to mean the spectrum of $\phi(u)$ inside the unital algebra $\phi(1_{n,k})A_{m,j}\phi(1_{n,k})$. By Lemma 2.1 this algebra, if nonzero, has the form $M_i \otimes \mathcal{O}_{\{m,j\}} \otimes C(\mathbb{T})$, and the term spectrum of $\phi(u)(t)$ will refer to the spectrum of the unitary operator valued function $\phi(u)$ at $t \in \mathbb{T}$. (The total spectrum of $\phi(u)$, respectively $\phi(u)(t)$, will be the spectrum as just defined, a closed subset of $\mathbb{T}$, in addition to 0 if $\phi(1_{n,k})$ is not the unit of $A_{m,j}$.) We use the notation $Sp\phi(u)$ and $Sp(\phi(u)(t))$ for the spectra, defined as above.

If U, V are subsets of a metric space, dist (U, V) denotes the usual Hausdorff distance of U and V:

$$dist(U,V) = \max\{\sup_{x\in U}\inf_{y\in V} dist(x,y), \sup_{y\in V}\inf_{x\in U} dist(y,x)\}$$

In our present setting, the property that A has real rank zero is characterized by a property of small spectral variation, just as in the case where the fibre of $A_{n,j}$ is $M_{[n,k]}$ rather than $M_{[n,k]} \otimes \mathcal{O}_{\{n,k\}}$, [BBEK], [BDK], [DNNP], [BE], [Su]. Before stating this property in Lemma 2.3 we need a well known general Lemma.

Lemma 2.2 *Let $(A_n, \phi_{m,n} : A_n \to A_m)$ be an inductive system of C^*-algebras with inductive limit A. Let $\phi_n = \phi_{\infty;n} : A_n \to A$ be the canonical morphism.*

1. *If $x, y \in A_n$, then*
$$\lim_{m\to\infty} ||\phi_{m,n}(x) - \phi_{m,n}(y)|| = ||\phi_n(x) - \phi_n(y)||$$
2. *If $p \in A$ is a projection and $\varepsilon > 0$, there exists a $n \in \mathbb{N}$ and a projection $q \in A_n$ with*
$$||\phi_n(q) - p|| < \varepsilon.$$
3. *If $p_1, ..., p_r \in A$ are mutually orthogonal projections and $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ and mutually orthogonal projections $q_1, ..., q_r \in A_n$ such that*
$$||\phi_n(q_i) - p_i|| < \varepsilon, \qquad i = 1, ..., r$$

Proof:

1. is an immediate consequence of the definition the inductive limit A by means of equivalence classes of sequences.
2. The projection p can be approximated arbitrarily well by an element of the form $\phi_m(x)$, where $x = x^* \in A_m$. Replacing x by $\phi_{m,n}(x)$ for a suitable large m, we may make $||x^2 - x||$ small, by (1). Modifying x by applying a function which is constant equal to 0 near 0, and constant equal to 1 near 1, we obtain q.

3. By the same reasoning as in (2) we may find elements $x_i = x_i^* \in A_n$ for some n such that $\phi_n(x_i)$ approximates p_i, and such that $||x_i^2 - x_i||$ are small and $||x_i x_j||$ are small for $i \neq j$. Using standard spectral-theoretic techniques as in [Gli],[Bra] one modifies the x_i's to obtain the q_i's.

Now, we return to our more specific inductive limits again. First note that the cases $\{n, j\} \geq 2$ and the cases $\{n, j\} = 0$ have to be treated slightly differently. The case where $\{n, j\} = 0$ for all (n, j) has been treated in detail in [BDK],[BE],[Su]. One way of summarizing the result in [Su], for the case that

$$A_{n,j} = M_{[n,j]} \otimes C(\mathbb{T})$$

is as follows: Let

$$\phi_{m,j;n,i} : M_{[n,i]} \otimes C(\mathbb{T}) \to M_{[m,j]} \otimes C(\mathbb{T})$$

be the partial morphisms arising from $\phi_{m;n}$. Each point ω in the right $\mathbb{T}$ (let us call it $\mathbb{T}_{m,j}$) corresponds to an irreducible representation π of $C(\mathbb{T}, M_{[m,j]})$ (by evaluation at ω), and $\pi \circ \phi_{m,j;n,i}$ decomposes into a finite number k of irreducible representations of $C(\mathbb{T}, M_{[n,i]})$ corresponding to (not necessarily distinct) points $\phi_1(\omega), ..., \phi_k(\omega)$ in $\mathbb{T}_{n,i}$. Here $0 \leq k \leq [m, j]/[n, i]$. Since $\mathbb{T}$ is connected, the number k is independent of $\omega \in \Omega_{m,j}$ [DNNP]. Let $S^k\mathbb{T}$ denote the k-fold symmetric product of $\mathbb{T}$; that is, $S^k\mathbb{T}$ is the Cartesian product $\mathbb{T}^k$ modulo the canonical action of the symmetric group S^k on this product, equipped with the quotient topology. Then $\phi_1(\omega), ..., \phi_k(\omega)$ defines a unique point $L_{n,i;m,j}(\omega)$ in $S^k\mathbb{T}$. The map $L_{n,i;m,j} : \mathbb{T}_{m,j} \to S^k\mathbb{T}_{n,i}$ is well defined and continuous [DNNP]. If d is the usual metric on $\mathbb{T}$, the topology on $S^k\mathbb{T}$ is defined from the metric

$$d'(S^k(\omega_1, ..., \omega_k), S^k(\phi_1, ..., \phi_k)) = \inf_{\sigma \in S^k} (\max_{1 \leq l \leq k} d(\omega_l, \phi_{\sigma(l)}))$$

Define the spectral variation of $\phi_{m,j;n,i}$ as

$$SV(\phi_{m,j;n,i}) = \sup_{s,t \in \mathbb{T}} d(L_{n,i;m,j}(s), L_{n,i;m,j}(t)),$$

and the spectral variation of $\phi_{m,n}$ as

$$SV(\phi_{m,n}) = \max_{ij} SV(\phi_{m,j;n,i}).$$

One may now prove that A has real rank zero if and only if

$$\lim_{m\to\infty} SV(\phi_{m,n}) = 0 \tag{2.1}$$

for each $n \in \mathbb{N}$, [Su].

Note that $L_{n,i;m,j}(s)$ is nothing but the set of eigenvalues of $\phi_{m,j;n,i}(u_{n,i})$, counted with multiplicity (divided by $[n,i]$). In the case that $\{m,j\} \geq 2$ we will now establish a very similar criterion to the one above, but simpler since we do not need to consider the multiplicity of the eigenvalues (roughly because this is infinite anyhow, due to the infiniteness of the fibers $\mathcal{O}_{\{m,j\}}$). The following Lemma 2.3 can be amalmagated with Su's criterion above to obtain that A has real rank zero if and only if any of the canonical unitaries $u_{n,i}$ has small spectral variation over the circles when mapped far out, where small spectral variation means (2.1) over the summands where the fibres are matrix algebras and condition 2 of the lemma over the other summands. The following lemma applies to the ideal generated by $A_{n,i}$ with $\{n,i\} \geq 2$.

Lemma 2.3 *Let $A = \varinjlim A_n$. Assume that $\{n,i\} \geq 2$ for all (n,i), but the connecting morphisms $\phi_{m,n}$ do not need to be unital. The following conditions are equivalent*

1. *A has real rank zero.*
2. *For any (n,k) and any $\epsilon > 0$, there exists an $m > n$ such that, with $u = u_{n,k}$,*

$$\max_i \sup_{s,t\in\mathbb{T}} dist(Sp(\phi_{m,i;n}(u)(t)), Sp(\phi_{m,i;n}(u)(s))) < \epsilon.$$

3. *For any n, any $x = x^* \in A_n$ and any $\epsilon > 0$, there exists an $m > n$ such that*

$$\max_i \sup_{s,t} dist(Sp(\phi_{m,i;n}(x)(t), Sp(\phi_{m,i;n}(x)(s)) < \epsilon.$$

Proof:

We prove $1 \Leftrightarrow 3 \Leftrightarrow 2$.

$1 \Rightarrow 3$: If A has real rank zero, and $x = x^* \in A_n$, then $\phi_n(x)$ may be approximated arbitrarily well by a finite linear combination $\sum_{k=1}^{r} \lambda_k p_k$, where $\lambda_k \in \mathbb{R}$ and $p_1, ..., p_k$ is a family of mutually orthogonal projections in A. By

Lemma 2.2 (3) we may find a $m \geq n$ and mutually orthogonal projections $q_1, ..., q_r$ in A_m such that $\phi_m(q_k)$ is close to p_k for $k = 1, ..., r$. But by 2.2 (1),

$$\lim_{k\to\infty} ||\phi_{k;n}(x) - \sum_{i=1}^{r} \lambda_i \phi_{k;m}(q_i)|| = ||\phi_n(x) - \sum_{i=1}^{r} \lambda_i \phi_m(q_i)||$$

so the element $\phi_{m,n}(x)$ can be approximated arbitrarily well by a self-adjoint element of finite spectrum in some A_m. Since the projections in $A_m = \bigoplus_j A_{m,j}$ are direct sums of projections, and a projection in

$$A_{m,j} = M_{[m,j]} \otimes \mathcal{O}_{\{m,j\}} \otimes C(\mathbb{T})$$

is a continuous function from $\mathbb{T}$ into projections of $M_{[m,j]} \otimes \mathcal{O}_{\{m,j\}}$, and therefore either nonzero over all points of $\mathbb{T}$ or zero over all points, it follows that the spectrum of $\phi_{m,j;n}(x)(t)$ is approximately constant in t, which is 3.

$3 \Rightarrow 1$. Assume 3. To show that A has real rank zero it suffices to show that for all n and all $x = x^* \in A_n$, there is an $m > n$ such that $\phi_{m,j;n}(x)$ can approximated by an element of finite spectrum in $A_{m,j}$ for all j. But using condition 3 it thus suffices to show that for any $\epsilon > 0$ there is a $\delta > 0$ such that for any self-adjoint valued continuous function x from $\mathbb{T}$ into $M_k \otimes \mathcal{O}_n$ if norm less than 1 such that $Sp(x(t))$ is approximately constant in t within δ there is a finite set $p_1, ..., p_r$ of projections in $M_k \otimes \mathcal{O}_n \otimes C(\mathbb{T})$ and real numbers $\lambda_1, ..., \lambda_n \in [-1, 1]$ such that

$$||x - \sum_{k=1}^{r} \lambda_k p_k|| < \epsilon.$$

This follows from the properties 1–6 listed in the beginning of Section 2. In fact it is only the implication $1 \Rightarrow 3$ which will be used later as A will be simple, and thus real rank zero is an overall assumption.

$2 \Leftrightarrow 3$. Note that the kernel of the map

$$\phi_{m,i;n,k} : A_{n,k} \to A_{m,i}(t)$$

consists of exactly those functions from $\mathbb{T}$ into $M_{[n,k]} \otimes \mathcal{O}_{\{n,k\}}$ which are zero on $Sp(\phi_{m,i;n,k}(u)(t)) \subseteq \mathbb{T}$. Thus, if $x = x^* \in A_{n,k}$, $Sp(\phi_{m,i;n,k}(x)(t))$, is the closure of the union of the spectra of $x(s)$ over $s \in Sp(\phi_{m,i;n,k}(u)(t))$. Since $Sp(x(s))$ is a continuous function of s, it is then clear that $2 \Rightarrow 3$, and the converse implication follows by letting x run through a sufficiently fine partition of the identity of $\mathbb{T}$.

We will now summarize Lemma 2.3 and Su's result.

Definition 2.4 *Let* $A = \varinjlim A_n$*, where*

$$A_n = \bigoplus_{i=1}^{r_n} A_{n,i} \text{ and } A_{n,i} = M_{[n,j]} \otimes \mathcal{O}_{\{n,j\}} \otimes C(\mathbb{T})$$

where $[n,j] \in \mathbb{N}$ *and* $\{n,j\} \in \{0,2,3,4,...\}$ *and the connecting homomorphims are unital. We say that* A has small spectral variation *if for any* (n,k) *and any* $\epsilon > 0$*, there exists an* $m > n$ *such that, with* $u = u_{n,k}$*,*

$$\sup_{s,t\in\mathbb{T}} dist(Sp(\phi_{m,i;n,k}(u)(t)), Sp(\phi_{m,i;n,k}(u)(s))) < \epsilon \text{ if } \{m,i\} \geq 2\,,$$

and

$$SV(\phi_{m,i;n,k}) < \epsilon, \text{ if } \{m,i\} = 0\,.$$

Corollary 2.5 *Let* A *be as in Definition 2.4. Then* A *has real rank zero if and only if* A *has small spectral variation.*

Proof:

This follows from Lemma 2.3 and [Su], together with the fact that there exists no nonzero homomorphism from $M_i \otimes \mathcal{O}_n \otimes C(\mathbb{T})$ into $M_m \otimes C(\mathbb{T})$; see the following remark.

Remark 2.6

Let $A = \varinjlim A_n$ be as before, and assume that A has real rank zero. It follows that the ideals in A have real rank zero, and hence has approximative identities consisting of projections, [BP]. On the other hand, if I is an ideal in A, and $I_n = \{x \in A_n | \phi_n(x) \in I\}$, then I_n is an ideal in A_n, and $I = \overline{\bigcup_n \phi_n(I_n)}$. (In fact, I identifies with the inductive limit of $I_1 \xrightarrow{\phi_{2,1}} I_2 \xrightarrow{\phi_{3,2}} I_3 \longrightarrow ....$) Since I is generated by its projections, each I_n is a subsum of the direct sum $\bigoplus_k A_{n,k}$, because if $p \in A_{n,k}$ is a projection, and $p(t) \neq 0$ for some $t \in \mathbb{T}$, then $p(t) \neq 0$ for all $t \in \mathbb{T}$. Thus we may associate to A a Bratteli diagram $\mathcal{D}(A)$ where the vertices are the indices (n,k), and there is an edge between (n,k) and $(n+1,i)$ if and only if $\phi_{n+1,i;n,k} \neq 0$. There is then the same $1-1$ correspondence between ideals in A and certain subsets of $\mathcal{D}(A)$ as for AF-algebras, [Bra]. This will be useful in identifying D_* with (K_* & ideal structure) in the next section.

One particular ideal in $A = \varinjlim A_n$ has special interest. This is the maximal infinite ideal I_∞. To define this, note that the only homomorphism from $M_i \otimes \mathcal{O}_n \otimes C(\mathbb{T})$ into $M_k \otimes \mathcal{O}_0 \otimes C(\mathbb{T}) = M_k \otimes C(\mathbb{T})$ when $n \geq 2$ is zero since 1 is infinite in the first algebra, while the latter algebra has no infinite projections. Thus the ideals

$$I_{n,\infty} = \bigoplus_{j;\{n,j\}\geq 2} A_{n,j}$$

in A_n are mapped into each other by $\phi_{n+1;n}$, and their inductive limit is I_∞. I_∞ has real rank zero, and all its projections are infinite. On the other hand A/I_∞ is a unital real rank zero inductive limit of circle algebras, and has been completely analyzed in [Ell2].

The subset $D_*(I_\infty)$ of $D_*(A)$ corresponding to I_∞ consists of all those $x \in D_*(A)$ such that nx is defined for all $n \in \mathbb{N}$ and $nx = x$ for some $n \in \mathbb{N}$.

We end this section with some remarks on simple inductive limits, which is what the statement of Theorem 1.1 is about. In this case the maximal infinite ideal I_∞ has to be equal to A (unless $I_\infty = 0$ which is the case already analyzed in [Ell2]), and hence we may throw away all the summands $A_{n,j}$ where $\{n,j\} = 0$ without changing A. Thus we assume from now that $\{n,j\} \geq 2$ for all n, j. We may also throw away all summands $A_{n,j}$ such that $\phi_n(A_{n,j}) = 0$ without changing A, even when A is non-simple. When A is simple, we may (assuming A_1 is chosen such that $\phi_1(A_1) \neq 0$) throw away all summands $A_{n,j}$ when $n \geq 2$ and $\phi_{n,j;n-1} = 0$ without changing A. In the AF-diagram this corresponds to throwing away all vertices from the second row onwards which are not hit by an edge from above. Starting from the second row and going down this results in that $\phi_{n,j;n-1} \neq 0$ for all $n \geq 2$ for the diagram consisting of the remaining vertices. The simplicity of A then manifests itself in the fact that for any (n,i) there is a $m > n$ such that $\phi_{m,j;n,i} \neq 0$ for any j.

We will say that an algebra of our type is AF-simple if the corresponding AF-diagram is that of a simple C*-algebra, i.e. if after throwing away all summands $A_{n,j}$ which are mapped into zero in A, the diagram has the property that if Λ is a set of vertices such that any descendant of any point in Λ is in Λ, and if all the decscendants of a vertex is in Λ, then the vertex itself is in Λ, then Λ is either the empty set or consists of all vertices. The simple C*-algebras of our type is then characterized as follows:

Proposition 2.7 *Consider C^*-inductive limits $A = \varinjlim A_n$ where*

$$A_n = \bigoplus_{j=1}^{r_n} M_{[n,j]} \otimes \mathcal{O}_{\{n,j\}} \otimes C(\mathbb{T})$$

where $[n,j] \in \mathbb{N}$ and $\{n,j\} \in \{2,3,4,...\}$. The following conditions are equivalent:

1. *A is simple.*

2. *A is AF-simple and has small spectral variation.*

Proof:
1 ⇒ 2: If A is simple, it is clear from the fact that

$$I = \overline{\bigcup_{n=1}^{\infty} (I \cap \phi_n(A_n))}$$

that A is AF simple [Bra]. That A has small spectral variation can be deduced from the fact that A is infinite and simple, and thus of real rank zero, and Lemma 2.3. One could also deduce this directly from the fact that an ideal in A is described by a certain sequence of open sets in the spectrum $\coprod_{k=1}^{r_n} \mathbb{T}$ of A_n. We leave the details to the reader.

2 ⇒ 1: Since A has real rank zero, any non-zeroclosed ideal I of A is generated by its projections. This implies that for any sufficiently large n there is a j such that $\phi_n(A_{n,j}) \subset I$. Then we obtain that I=A by using AF-simplicity in the same way as in the AF case [Bra].

3. Discussion of the Invariants

The graded dimension range $D_*(A)$ was introduced in section 1. In this section we will discuss this invariant in more detail in the context of section 2, and its relation with the more familiar invariant $K_*(A)$, [Bla].

We will first compute K_*. It is known that

$$K_*(\mathcal{O}_0) = K_*(\mathbb{C}) = (\mathbb{Z}, 0)$$

while

$$K_*(\mathcal{O}_n) = (\mathbb{Z}/(n-1)\mathbb{Z}, 0)$$

for $n = 2, 3, ...$, [Cun2]. Also

$$K_*(C(\mathbb{T})) = (\mathbb{Z}, \mathbb{Z}).$$

The graded group $K_*(\mathcal{O}_n \otimes \mathbb{C}(\mathbb{T}))$ can then be computed from the Künneth short exact sequence

$$0 \to K_*(A) \otimes K_*(B) \to K_*(A \otimes B) \to Tor_1^{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

[RS]. In our case, when $n = 2, 3, ...$,

$$K_*(\mathcal{O}_n) \otimes K_*(C(\mathbb{T})) = (\mathbb{Z}_{n-1}, 0) \otimes (\mathbb{Z}, \mathbb{Z}) = (\mathbb{Z}_{n-1}, \mathbb{Z}_{n-1})$$

and

$$Tor_1^{\mathbb{Z}}(K_*(\mathcal{O}_n), K_*(C(\mathbb{T}))) = Tor_1^{\mathbb{Z}}((\mathbb{Z}_{n-1}, 0), (\mathbb{Z}, \mathbb{Z})) = (0, 0)$$

as one can see by using the standard free resolution

$$0 \to \mathbb{Z} \overset{(n-1)\times}{\to} \mathbb{Z} \to \mathbb{Z}_{n-1} \to 0$$

of $\mathbb{Z}_{n-1}$. Hence, by the Künneth short exact sequence, $K_*(M_k \otimes \mathcal{O}_n \otimes C(\mathbb{T})) \cong (\mathbb{Z}_{n-1}, \mathbb{Z}_{n-1})$ for $n = 2, 3, ...$ Suitable generators are

$$K_0(1 \otimes 1 \otimes 1) = 1, \qquad K_1(1 \otimes 1 \otimes (z \to z)) = 1,$$

[Cun2]. Of course,

$$K_*(M_k \otimes \mathcal{O}_0 \otimes C(\mathbb{T})) \cong (\mathbb{Z}, \mathbb{Z})$$

with generators

$$K_0(p \otimes 1 \otimes 1) = 1,$$
$$K_1(p \otimes 1 \otimes (z \to z)) = 1,$$

where p is a one-dimensional projection in M_k.

Now, using the properties 1–5 of $M_k \otimes \mathcal{O}_n$ listed near the beginning of Section 2, it is not hard to prove that (see [Rør1]):

$$D_*(M_k \otimes \mathcal{O}_n) = \overline{0} \cup K_*(M_k \otimes \mathcal{O}_n) = \overline{0} \cup (\mathbb{Z}_{n-1}, 0)$$

for $n = 2, 3, \ldots$, where $\bar{0}$ is the Murray-von Neumann equivalence class of $0 \in M_k \otimes \mathcal{O}_n$, and addition in the semigroup $D_*(M_k \otimes \mathcal{O}_n)$ is given by the requirement that $\bar{0}$ is the zero element, and addition on the subset $K_*(M_k \otimes \mathcal{O}_n)$ is as earlier. Furthermore

$$D_*(M_k \otimes \mathcal{O}_n \otimes C(\mathbb{T})) = \bar{0} \cup K_*(M_k \otimes \mathcal{O}_n \otimes C(\mathbb{T})) = \bar{0} \cup (\mathbb{Z}_{n-1}, \mathbb{Z}_{n-1})$$

for $n = 2, 3, \ldots$ with addition defined analogously with the previous case. Suitable generators are

$$D_*(0) = \bar{0}, \qquad D_*(1 \otimes 1 \otimes (z \to z^m)) = (1, m), m \in \mathbb{Z}_{n-1}.$$

In the case $n = 0$:

$$D_*(M_k \otimes \mathcal{O}_0 \otimes C(\mathbb{T})) = \bar{0} \cup (\{1, \cdots, k\}, \mathbb{Z}).$$

This is a local abelian semigroup (with addition only locally defined). Suitable generators are

$$D_*(0) = \bar{0}, \qquad D_*(p \otimes 1 \otimes (z \to z^m)) = (1, m), m \in \mathbb{Z}$$

where p is a one-dimensional projection in M_k.

Put

$$D_{k,n} = D_*(M_k \otimes \mathcal{O}_n \otimes C(\mathbb{T})) = \begin{cases} \bar{0} \cup (\mathbb{Z}_{n-1}, \mathbb{Z}_{n-1}) & \text{when } n = 2, 3, \ldots \\ \bar{0} \cup (\{1, \cdots, k\}, \mathbb{Z}) & \text{when } n = 0. \end{cases}$$

It is clear by the usual spectral-theory argument that $D_*(A)$ is the inductive limits of the local semigroups

$$D_*(A_n) = \bigoplus_{j=1}^{r_n} D_*(A_{n,j}) = \bigoplus_{j=1}^{r_n} D_{[n,j],\{n,j\}}.$$

The maps

$$(\phi_{m,k;n,j})_* : D_*(A_{n,j}) \to D_*(A_{m,k})$$

are as follows: For $\Phi = \phi_{m,k;n,j}$, certainly

$$\Phi_*(\bar{0}) = \bar{0}$$

and if $\Phi = 0$, then

$$\Phi_*(d) = \bar{0}$$

for all $d \in D_{[n,j],\{n,j\}}$. If $\Phi \neq 0$ then $\Phi_* | K_*(A_{n,j})$ is the element in $Hom(K_*(A_{n,j}), K_*(A_{m,k}))$ induced by Φ. (If $\{n,j\} = 0$, replace $K_*(A_{n,j})$ by $(\{1,\dots,l\}, \mathbb{Z})$ for some l. Note that $\Phi_* = 0$ if $\{n,j\} \neq 0$ but $\{m,k\} = 0$).

Note that the AF-diagram of A, and thus (by Remark 2.) the ideal structure of A can be read off the inductive system

$$D_*(A_1) \to \dots,$$

i.e. the vertices are the points (n,k) and there is an edge between (n,k) and $(m+1,l)$ if and only if $(\phi_{n+1,l;n,k})_* \neq 0$. If we now define an ideal in $D_*(A)$ as a sub-semigroup I with the property that if $x+y \in I$, then $x, y \in I$, then it is easy to see that there is a canonical 1–1 correspondence between ideals in $D_*(A)$ and those subsets of $D_*(A)$ that corresponds to ideals in A.

Equivalently, we could define a pre-order on the local semi-group $D_*(A)$ by $x \leq z$ if there is a y with $x + y = z$, and then the ideals in $D_*(A)$ are the hereditary sub-semigroups of $D_*(A)$.

In order to formulate and prove a KK-intertwining argument, we need some information about

$$KK_*(A_n, A_m) = KK_*(\bigoplus_j A_{n,j}, \bigoplus_k A_{m,k}) = \bigoplus_{j,k} KK_*(A_{n,j}, A_{m,k}).$$

Put temporarily $A = M_j \otimes \mathcal{O}_n \otimes C(\mathbb{T})$, $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$. By the universal coefficent theorem, [RS], we have a short exact sequence

$$\begin{aligned} 0 \;\to\; & Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A,B) \\ \xrightarrow{\gamma}\; & Hom(K_*(A), K_*(B)) \to 0 \end{aligned}$$

where δ is of degree 1 and γ is of degree 0 as mappings of graded groups, and Hom means graded homomorphisms. If $n, m \geq 2$, then

$$K_*(A) = (\mathbb{Z}_{n-1}, \mathbb{Z}_{n-1}), \qquad K_*(B) = (\mathbb{Z}_{m-1}, \mathbb{Z}_{m-1}),$$

so one computes

$$Ext^1_{\mathbb{Z}}(K_*(A), K_*(B)) = (\mathbb{Z}_k, \mathbb{Z}_k)$$

where $k = gcd(n-1, m-1) \equiv$ the greatest common divisor of $n-1$ and $m-1$, and, furthermore

$$Hom(K_*(A), K_*(B)) = (\mathbb{Z}_k, \mathbb{Z}_k).$$

It follows from UCT that both $KK_0(A,B)$ and $KK_1(A,B)$ are extensions of $\mathbb{Z}_k$ by $\mathbb{Z}_k$, with $k = gcd(n-1, m-1)$, when $n, m \geq 2$. When $n = 0$, $m \geq 2$, then

$$K_*(A) = (\mathbb{Z}, \mathbb{Z}), \qquad K_*(B) = (\mathbb{Z}_{m-1}, \mathbb{Z}_{m-1}).$$

Now

$$Ext_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_{m-1}) = 0$$

since $\mathbb{Z}$ is free, and

$$Hom(\mathbb{Z}, \mathbb{Z}_{m-1}) = \mathbb{Z}_{m-1},$$

so

$$KK_*(A,B) = Hom(K_*(A), K_*(B)) = (\mathbb{Z}_{m-1}, \mathbb{Z}_{m-1})$$

when $n = 0$, $m \geq 2$. Similarly

$$Ext_1^{\mathbb{Z}}(\mathbb{Z}_{n-1}, \mathbb{Z}) = \mathbb{Z}/(n-1)\mathbb{Z} = \mathbb{Z}_{n-1},$$

$$Hom(\mathbb{Z}_{n-1}, \mathbb{Z}) = 0,$$

so

$$KK_*(A,B) = (\mathbb{Z}_{n-1}, \mathbb{Z}_{n-1})$$

when $n \geq 2$, $m = 0$. Finally

$$Ext_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0, \qquad Hom_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z},$$

so

$$KK_*(A,B) = Hom(K_*(A), K_*(B)) = (\mathbb{Z}, \mathbb{Z})$$

when $n = m = 0$.

4. The KK-intertwining Argument

In order to prove Theorem 1, we will adapt the KK-intertwining argument from [Ell2]. For this we need a uniqueness and a existence theorem. These are as follows.

Theorem 4.1 *(Uniqueness) For any sufficiently small $\epsilon > 0$ there exists a $\delta > 0$ with the following property:*

If ϕ, ψ are unital morphims from $A = M_i \otimes \mathcal{O}_n \otimes C(\mathbb{T})$ into $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$ where $i, k \in \mathbb{N}$, $n \in 2\mathbb{N}$, $m \in 2\mathbb{N}$, such that

$$Sp(\phi(v)(t)) \quad \text{and} \quad Sp(\psi(u)(s)) \quad \text{are } \delta\text{-dense in } \mathbb{T}$$

for all $t, s, \in \mathbb{T}$, where u is the canonical unitary in A, the following conditions are equivalent

1. $KK(\phi) = KK(\psi)$
2. *There exists a unitary $U \in B$ such that*

$$||\phi(s_j) - AdU \circ \psi(s_j)|| < \epsilon$$

for $j = 1, ..., n$, where s_j are the generators of $\mathcal{O}_n$,

$$||\phi(u) - AdU \circ \psi(u)|| < \epsilon,$$

and

$$||\phi(x) - AdU \circ \psi(x)|| < \epsilon||x||$$

for $x \in M_i$.

Theorem 4.2 *(Existence) Let $A = M_i \otimes \mathcal{O}_n \otimes C(\mathbb{T})$ and $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$ where $i, k \in \mathbb{N}$, $n \in 0 \cup 2\mathbb{N}$, $m \in 2\mathbb{N}$. Assume that*

$$m - 1 \left| \frac{k(n-1)}{gcd(i, n-1)} \right.$$

if $n \geq 2$. Let $g \in KK(A, B)$ be an element which induces the map

$$1 \in K_0(A) = \mathbb{Z}_{n-1} \mapsto s \in K_0(B) = \mathbb{Z}_{m-1}$$

where s is such that

$$(n-1)s = 0 \ mod(m-1), \qquad is = k \ mod(m-1).$$

It follows that there is a unital homomorphism ϕ of A into B such that $KK(\phi) = g$ and $Sp(\phi(u)(t)) = \mathbb{T}$ for any $t \in \mathbb{T}$.

Note that the number theoretic assumptions in Theorem 4.2 are there just to assume that there exists homomorphisms in $Hom(K_0(A), K_0(B))$ taking the class of the unit in $K_0(A)$ into the class of the unit in $K_0(B)$. These theorems will be proved, along with more concrete characterizations of the elements in $KK(A, B)$, in Section 6, see Theorem 6.8 and 6.9. It will be enough to consider the case $i = 1$ (cf. [Rør1], proof of Theorem 5.2).

For the KK-intertwining we will also need the fact, proved in [RS], that if $D = \varinjlim D_n$ and $K_*(C)$ is finitely generated, then $KK_*(C, D) = \varinjlim KK_*(C, D_n)$. Finally we will need that if A_n, B_m are finite direct sums of algebras of the form $M_k \otimes \mathcal{O}_i \otimes C(\mathbb{T})$, then $KK_*(A_n, B_m)$ is a finite abelian group. This was established in Section 3.

Proof of Theorem 1.1: We follow closely the KK-intertwining argument in [Ell2], Section 7. So assume that each of A and B is the inductive limit of a sequence of finite direct sums of basic building blocks

$$M_i \otimes \mathcal{O}_n \otimes C(\mathbb{T})$$

where $i \in \mathbb{N}$, $n \in 2\mathbb{N}$.

We first assume that A and B are unital. We may then assume that the unit in A, B is the image of the unit in A_1, B_1, respectively, and since A, B are simple it follows from Theorem 2 that we may choose the inductive sequences

$$A_1 \to A_2 \to \ldots \to A$$

$$B_1 \to B_2 \to \ldots \to B$$

so that the connecting maps $\phi_{n+1,j;n,i}$ are all nonzero and the spectral variation is arbitrarily small. These properties are preserved when going to subsequences.

Now, assume that there exists an isomorphism $\phi_* : K_*A \to K_*B$ such that $\phi_*([1_A]_0) = [1_B]_0$.

By Theorem 1.17 of [RS] (the universal coefficient theorem for KK) together with Proposition 7.3 of [RS], the isomorphism ϕ_* arises from an invertible element $\phi(A, B)$ of $KK(A, B)$, and so ϕ_*^{-1} arises from the inverse element $\phi(B, A)$ in $KK(B, A)$.

The canonical element $\phi(A_1, A) \in KK(A, A)$ coming from the morphism $A_1 \to A$, gives rise to an element $\phi(A_1, B) = \phi(A_1, A)\phi(A, B)$ in $KK(A_1, B)$.

But each K_*A_n or K_*B_n is of the form

$$\bigoplus_j (\mathbb{Z}_{\{n,j\}-1}, \mathbb{Z}_{\{n,j\}-1})$$

(the class of the identity being $\bigoplus_j([n,j],0))$ and hence, since these groups are finitely generated (even finite)

$$KK(A_1, B) = \varinjlim KK(A_1, B_n)$$

in the sense that if $\beta \in KK(A_1, B)$ there is a n and a $\gamma \in KK(A_1, B_n)$ such that $B = \gamma\phi(B_n, B)$. Pick such an n, relabel B_n to B_1, and name γ $\phi(A_1, B_1)$. We have obtained a commutative diagram of KK-elements which gives rise to a commutative diagram of K_*-maps:

$$\begin{array}{ccc}
K_*(A_1) & \longrightarrow & K_*(A) \\
\downarrow & & \uparrow\downarrow \\
K_*(B_1) & \longrightarrow & K_*(B)
\end{array}$$

Switching the role of the A's and B's in the above argument, we obtain for a sufficiently large n a commutative diagram of KK-elements

$$\begin{array}{cccc}
 & K_*(A_n) & \longrightarrow & K_*(A) \\
 & \nearrow \gamma_n & & \uparrow\downarrow \\
K_*(B_1) & & \longrightarrow & K_*(B)
\end{array}$$

Using commutativity of both diagrams we have

$$\phi(A_1, A_n)\phi(A_n, A) = \phi(A_1, A) = \phi(A_1, B_1)\phi(B_1, A) = \phi(A_1, B_1)\gamma_n\phi(A_n, A),$$

and so

$$(\phi(A_1, A_n) - \phi(A_1, B_1)\gamma_n)\phi(A_n, A) = 0.$$

But again using $KK(A_1, A) = \varinjlim KK(A_1, A_m)$ we see that

$$(\phi(A_1, A_n) - \phi(A_1, B_1)\gamma_n)\phi(A_n, A_m) = 0$$

for some $m \geq n$. Renaming A_m as A_2 and defining $\phi(B_1, A_2) = \gamma_n \phi(A_n, A_m)$ we obtain a commutative diagram of KK-elements

$$\begin{array}{ccccc}
K_*(A_1) & \longrightarrow & K_*(A_2) & \longrightarrow & K_*(A) \\
\downarrow & \nearrow & & & \uparrow\downarrow \\
K_*(B_1) & & \longrightarrow & & K_*(B).
\end{array}$$

Proceeding inductively in this manner, we obtain a commutative diagram of KK-elements

$$\begin{array}{ccccccccc}
K_*(A_1) & \rightarrow & K_*(A_2) & \rightarrow & K_*(A_3) & \rightarrow \dots \rightarrow & & & K_*(A) \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & & & \uparrow\downarrow \\
K_*(B_1) & \rightarrow & K_*(B_2) & \rightarrow & K_*(B_3) & \rightarrow \dots \rightarrow & & & K_*(B)
\end{array}$$

The argument proceeds as in [Ell2] using Existence Theorem 4.2 to lift the new KK-maps to the algebra level and then using the Uniqueness Theorem 4.1 to obtain approximate unitary equivalence at the algebra level of the triangles in the above diagram. In order to apply Theorem 4.1, one needs that the image of the canonical unitary u under a connecting map ϕ is such that $\mathrm{Sp}\,\phi(u)(t)$ is δ-dense for all $t \in \mathbb{T}$ (for arbitrarily small δ, by telescoping enough connecting maps if necessary). By injectivity the union of $\mathrm{Sp}\,\phi(u)(t)$, as t varies over $\mathbb{T}$ is dense in $\mathbb{T}$. Hence if the spectral variation of $\phi(u)$ is small with tolerance δ (which will follow by telescoping from the real rank-zero property) then $\mathrm{Sp}\,\phi(u)(t)$ itself must be δ-dense for each $t \in \mathbb{T}$.

5. Unitaries in $M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$

Here we show how to approximate unitaries with a spectral condition in a matrix algebra over $\mathcal{O}_m \otimes C(\mathbb{T})$, where $\mathcal{O}_m$ is a Cuntz algebra, by unitaries in a standard form. Consequently such unitaries with the same K-theory are approximately unitarily equivalent (cf. Lin's result [Lin1] for approximate unitary equivalence of unitaries in a simple purely infinite C^*-algebra, as mentioned in property 6 of Section 2).

Theorem 5.1 *There exists a constant $C > 0$ satisfying the following conditions: If $\delta > 0$, M is a positive integer, and U is a unitary in $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$ such that $M g_0 = k \bmod m-1$, $\delta > 1/M$, $[U] = l g_0 \bmod m-1$, and $\mathrm{Sp}(U(t))$ is δ-dense in $\mathbb{T}$ for all $t \in \mathbb{T}$, then there is a unitary $U_0 \in B$ such that*

$$\|U - U_0\| < C\delta$$
$$U_0 = u^l P_1 + \sum_{j=2}^{M} e^{2\pi i j/M} P_j$$

where $\{P_j : j = 1, \cdots, M\}$ are mutually orthogonal non-zero projections such that $[P_j] = g_0$. Here u is the standard unitary $t \to 1 \otimes 1 \otimes e^{2\pi i t}$, and g_0 is an arbitrary element in $\mathbb{Z}/(m-1)\mathbb{Z} = \{0, 1, \cdots, m-2\}$, viewed both as an element in $K_0(B)$ and in $K_1(B)$. We say that U_0 is of standard form of rank M.

Proof:

For a sufficiently small $\epsilon > 0$ there exists $\delta_1 > 0$ such that if $|t - s| < \delta_1$, then $\|U(s) - U(t)\| < \epsilon$. Choose a positive integer M such that $1/M < \delta_1$, and then by [Lin1] for each $k = 0, 1, 2, \cdots, M-1$, choose a unitary U_k with finite spectrum such that $\|U(k/M) - U_k\|$ is so small that $\|U_{k+1} - U_k\| < \epsilon$ for $k = 0, 1, 2, \cdots, M-1$ with $U_M = U_0$.

Now δ is a measure of the length of the gaps (if present) in the spectra of $U(t)$, hence of U_k. Choose an integer N so that $1/(2N+2) < \delta < 1/2N$; hence the spectral projections of U_k corresponding to spectral intervals of length greater than or equal to $1/2N$ are non-zero. With this N, we take $\epsilon > 0$ so small that homotopy arguments of [BEEK] can be applied, in particular those of Lemmata 2.2 and 2.9 (see also Proposition 3.1). More precisely, first define

$$U'_k = \sum_{l=0}^{2N-1} e^{\pi i l/N} p_{k,l}$$

where p_{kl} is the spectral projection of U_k for the interval $(\frac{2l-1}{4N}, \frac{2l+1}{4N}]$, $l = 0, 1, 2, \cdots, 2N-1$. We will connect U'_k with U'_{k+1} with a path of unitaries $v_k(t)$, $t \in [k/M, (k+1)/M]$, so that we always stay within a C/N neighbourhood of U'_k (or U_k) where C is a constant and the eigenvalues of $v_k(t)$ are as indicated in Figure 1.

Step 1. First move along the interval $[k/M,\ k/M + 1/5M]$ changing the eigenvalues $e^{\pi i 2r/N}$ and $e^{\pi i(2r+1)/N}$ (on $p_{k,2r}$ and $p_{k,2r+1}$ respectively) to $e^{\pi i(4r+1)/2N}$ (on $p_{k,2r} + p_{k,2r+1} \equiv P_{k,r}$) for $r = 0, 1, 2, \cdots, N-1$. So at $k/M + 1/5M$ we have

$$v_k(k/M + 1/5M) = \sum_{r=0}^{N-1} e^{\pi i(4r+1)/2N} P_{k,r}$$

Step 2. Next move back along the interval $[k/M + 4/5M,\ (k+1)/M]$ changing the eigenvalues $e^{\pi i(2r+1)/N}$ and $e^{\pi i 2(r+1)/N}$ (on $p_{k+1,2r+1}$ and $p_{k+1,2(r+1)}$ respectively) to $e^{\pi i(4r+3)/2N}$ (on $p_{k+1,2r+1} + p_{k+1,2(r+1)} = Q_{k,r}$) for $r = 0, 1, 2, \cdots, N-1$. So at $k/M + 4/5M$ we have

$$v_k(k/M + 4/5M) = \sum_{r=0}^{N-1} e^{\pi i(4r+3)/2N} Q_{k,r}\,.$$

Step 3. With this N and ϵ sufficiently small, we use the method of Proposition 3.1 of [BEEK] to bifurcate $P_{k,r}$ and $Q_{k,r}$ into

$$P_{k,r} = p^1_{k,r} + p^2_{k,r}, \qquad Q_{k,r} = q^1_{k,r} + q^2_{k,r}$$

with a unitary $U = e^{ih}$ (depending on k), with h small so that

$$\begin{aligned} U q^1_{k,r} U^* &= p^2_{k,r} \\ U q^2_{k,r} U^* &= p^1_{k,r+1}\,. \end{aligned}$$

Then along the interval $[k/M + k/5M,\ k/M + 2/5M]$ change the eigenvalue $e^{\pi i(4r+1)/2N}$ (on $P_{k,r} = p^1_{k,r} + p^2_{k,r}$) to $e^{\pi i 2r/N}$ and $e^{\pi i(2r+1)/N}$ (on $p^1_{k,r}$ and $p^2_{k,r}$ respectively) for $r = 0, 1, 2, \cdots, N-1$ so that at $k/M + 2/5M$ we have

$$v_k(k/M + 2/5M) = \sum_{r=0}^{N-1} (e^{\pi i 2r/N} p^1_{k,r} + e^{\pi i(2r+1)/N} p^2_{k,r})\,.$$

Similary working back on the interval $[k/M + 3/5M,\ k/M + 4/5M]$ change the eigenvalue $e^{\pi i(4r+3)/2N}$ (on $Q_{k,r} = q^1_{k,r} + q^2_{k,r}$) to $e^{\pi i(2r+1)N}$ and $e^{\pi i 2(r+1)/N}$ (on $q^1_{k,r}$ and $q^2_{k,r}$ respectively), for $r = 0, 1, 2, \cdots, N-1$. So at $k/M + 3/5M$ we have

$$v_k(k/M + 3/5M) = \sum_{r=0}^{N-1} (e^{\pi i(2r+1)/N} q^1_{k,r} + e^{\pi i 2(r+1)/N} q^2_{k,r})\,.$$

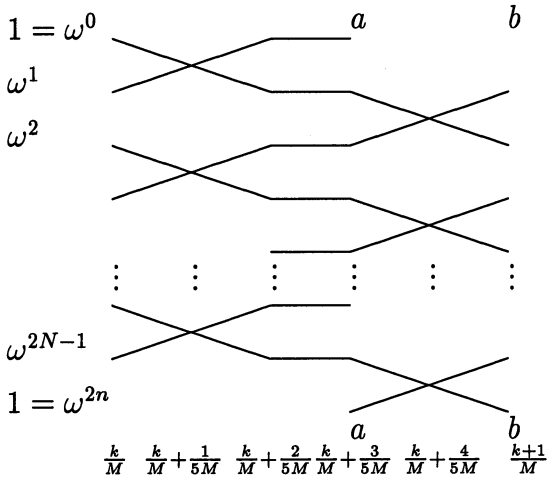


Figure 1:

Step 4. Finally by applying $\mathrm{Ad}(e^{ish})$, we can connect $v_k(k/M + 2/5M)$ and $v_k(k/M+3/5M)$ across the fifth and final interval $[k/M+2/5M,\ k/M+3/5M]$.

Putting the unitaries constructed on the intervals $[k/M,\ (k+1)/M]$ together we obtain a continuous path $v(t)$ of unitaries on the circle so that $\|v(t) - U(t)\| < C/N$. In each interval $[k/M, (k+1)/M]$, Figure 1 describes the eigenvalue functions where $\omega = e^{\pi i/N}$, and we have periodic boundary conditions in the vertical direction so that the two points labelled a (respectively b) are identified.

For each segment I between two crossings, there is a continuous function P_I of projections on I. (We consider $2N$ segments from each interval $[k/M+1/5M,\ k/M+4/5M]$ and $2N$ segments form each interval $[k/M+4/5M,\ (k+1)/M + 1/5M]$.) To each segment I assign an integer α_I so that $[P_I] = \alpha_I \bmod (m-1)$. At each crossing we have the weak conservation law $\alpha_1+\alpha_2 = \alpha_3 + \alpha_4 \bmod (m-1)$ as in Figure 2

We can take a choice of integer assigments to the segments so that we have a strict conservation law $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$ (but at this stage α_I may not necessarily be positive). We work through the crossings in the intervals $[k/2M,\ (k+1)/2M]$ in the order $k = 0, 1, 2, \cdots, 2M-1$. The only

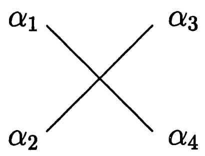


Figure 2:

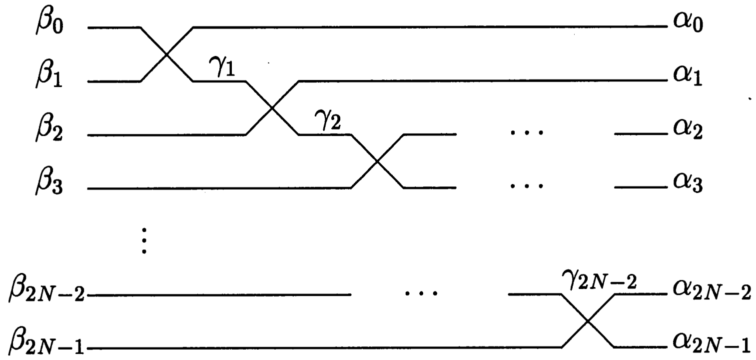


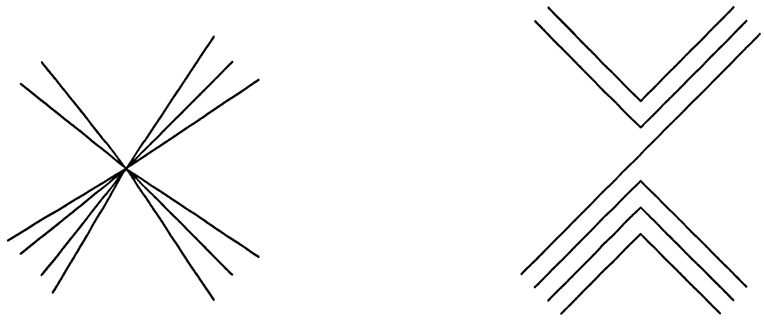
Figure 3:

problem may be that we may not necessarily have matched up the integers $\beta_0, \cdots, \beta_{2N-1}$ and $\alpha_0, \cdots, \alpha_{2N-1}$ from the last interval $[(2M-1)/2M, 1]$ and the first $[0, 1/2M]$, but at least $\beta_i = \alpha_i \bmod (m-1)$. To ensure periodic boundary conditions (in the horizontal direction) we introduce a further set of crossings as in Figure 3.

Here the eigen-projections remain the same; we are merely coalesing eigenvalues according to the procedure in Step 1 already used above. With our previous choices of α_i and β_i, we now solve for $\gamma_1, \cdots, \gamma_{2N-2}$ from

$$\gamma_i + \beta_{i+1} = \alpha_i + \gamma_{i+1} \quad i = 0, \cdots, 2N-1 \tag{5.1}$$

with $\gamma_0 = \beta_0$. At the final crossing, $\gamma_{2N-2} + \beta_{2N-1} = \alpha_{2N-2} + \alpha_{2N-1}$ will automatically follow from (5.1) and $\sum_{i=0}^{2N-1} \alpha_i = \sum_{i=0}^{2N-1} \beta_i$. By adding the same



(a) (b)

Figure 4:

large multiple of $m-1$ to each segment, we can ensure that each of α_I is positive with $[P_I] = \alpha_I \operatorname{mod}(m-1)$ and we still have the strict conservation law at each crossing. We now subdivide each projection P_I into α_I projections each of class 1, so that replacing each segment I by α_I lines, each crossing now looks as in Figure 4(a).

In order to remove the crossing and make the eigenvalues (for fixed t) distinct as in Figure 4(b) we need the following Lemma 5.2 to continuously match the incoming projections from the left with the outgoing projections on the right, and obtain a perturbation v' of v such that the estimate $\|v'(t) - U(t)\| < C/N$ persists.

Lemma 5.2 *Let $\{e_i\}$ and $\{f_i\}$ be orthogonal families of projections such that $\sum_{i=1}^{L} e_i = \sum_{i=1}^{L} f_i$, and $[e_i] = [f_i]$, $i = 1, 2, \cdots, L$. Then there exists an orthogonal family $\{g_i(t)\}$ of projection valued continuous functions on $[0,1]$ such that $g_i(0) = e_i$, $g_i(1) = f_i$ and $\sum_{i=1}^{L} g_i(t) = \sum_{i=1}^{L} g_i(0)$.*

We omit the easy proof of this.

We now write

$$v'(t) = \sum_{i=1}^{M} \lambda_i(t) e_i(t)$$

where for each t, $\{e_1(t), \cdots, e_M(t)\}$ is an orthogonal family of projections, $\sum_{i=1}^{M} e_i(t) = 1$, $[e_i(t)] = 1$, and the eigenvalues $(\lambda_1(t), \lambda_2(t), \cdots, \lambda_M(t))$ are distinct and lie in the same order on $\mathbb{T}$ for any t. Note that this M is different from the previous one and satisfies $M \geq 2N$ and $M = k \bmod (m-1)$. Actually we can make M as large as we wish just by making α_I large. As easily seen, by using this fact we can make $\{\lambda_1(t), \cdots, \lambda_M(t)\}$ arbitrarily dense in $\mathbb{T}$ for any $t \in \mathbb{T}$ keeping the estimate $\|v'(t) - U(t)\| < C/N$. (Just imagine the eigenvalues as functions on $\mathbb{T}$ form a graph on the 2-torus $\mathbb{T}^2$ as in Figure 1; the graph divides $\mathbb{T}^2$ into $2MN$ cells of size $1/M \times 1/N$; each cell can be handled just by the surrounding border lines.) Since $\{\lambda_1(0), \cdots, \lambda_M(0)\} = \{\lambda_1(1), \cdots, \lambda_M(1)\}$, there exists some b such that $\lambda_i(1) = \lambda_{i+b}(0)$ (where the indices are taken mod M). Then it follows that $e_i(1) = e_{i+b}(0)$ and $b = lg_0 = [u] \bmod (m-1)$. (But this fact will not be used later.)

Let N_0 be an integer such that $N_0 > 1/\delta$ and $N_0 g_0 = k \bmod (m-1)$. Having separated the M eigenvalues, we now group them back into N_0 subsets, where each subset will have many eigenvalues, all close to j/N_0, $j = 0, \cdots, N_0 - 1$. We choose $\epsilon > 0$ so that $\epsilon \ll 1/N_0$ and an open interval of length ϵ still contains at least m values in $\{\lambda_1(t), \cdots, \lambda_M(t)\}$ for any t.

Lemma 5.3 *There exist a subdivision of the unit interval,* $0 = t_0 < t_1 < t_2 < \cdots < t_{s-1} < t_s = 1$ *such that for each interval* (t_i, t_{i+1}) *there is a partition* $\{I_0^i, I_1^i, I_2^i, \cdots, I_{N_0-1}^i, I_{N_0}^i = I_0^i\}$ *of* $\{1, \cdots, M\}$ *such that* I_j^i *is an interval of the cyclically ordered set* $\{1, 2, \cdots, M\}$,

$$|\frac{j}{N_0} - \frac{1}{2\pi} \arg \lambda_a(t)| < \frac{1}{2N_0} + \epsilon \tag{5.2}$$

for $a \in I_j^i$, $t \in (t_i, t_{i+1})$, *and* $\#\,(I_j^i) = g_0 \bmod(m-1)$ *for* $i = 0, 1, 2, \cdots, s-1$.

Proof:

The (exponentials of the) intervals

$$(j/N_0 - (\frac{1}{2N_0} + \epsilon), j/N_0 + (\frac{1}{2N_0} + \epsilon)) \tag{5.3}$$

$j = 0, \cdots, N_{0-1}$ cover the circle and overlap by 2ϵ. Since there must be at least $2m$ points of $\{\frac{1}{\pi} \arg \lambda_a(t) : a = 1, 2, \cdots, M\}$ in each overlap, we can find a partition $I_0, \cdots, I_{N_0-1}$ so that (5.2) holds for $a \in I_j$ and $\#$

$(I_j) = g_0 \,\mathrm{mod}(m-1)$. Indeed if we begin with I_0, and then proceed with $I_1, \cdots I_{N_0-2}$, then $\#\ (I_{N_0-1}) = g_0 \,\mathrm{mod}(m-1)$ will automatically follow from $M = k\,\mathrm{mod}(n-1)$ and $N_0 g_0 = k\,\mathrm{mod}(m-1)$. Since the estimates persist in a neigbourhood of t we can apply a compactness argument.

Note that the segments of the intervals which do not overlap are of length $2(\frac{1}{2N_0} - \epsilon)$, so that here we unambiguously can decide which interval I_j^i to assign to such λ_a and we have $\#\ (I_j^i) \gg 1$. Also since the basic intervals (5.3) only overlap with their nearest neighbours, we must have

$$I_j^i \subset I_{j-1}^{i\pm 1} \cup I_j^{i\pm 1} \cup I_{j+1}^{i\pm 1}$$

We next arrange matters so that

$$\#((I_0^i \cup I_1^i) \cap (I_0^{i+1} \cup I_{N_0-1}^{i+1})) = g_0 \,\mathrm{mod}\,(m-1) \tag{5.4}$$

for $i = 0, 1, 2, \cdots, s-1$. We leave $\{I_j^0\}$ as they are and proceed inductively to modify $\{I_j^{i+1}\}$ if necessary so that (5.4) holds, having already settled on $\{I_j^0\}, \cdots, \{I_j^i\}$. If $r = \#((I_0^i \cap I_1^i) \cap (I_0^{i+1} \cup I_{N_0-1}^{i+1}))\,\mathrm{mod}(m-1)$, where $0 \le r < m-1$ we move $r - g_0$ elements cyclically from I_j^{i+1} to I_{j+1}^{i+1} for $j = 0, \cdots, N_0 - 1$. The cyclic rotation ensures that $\#\ (I_j^{i+1})$ remains the same (i.e. g_0 mod$(m-1)$). To find at most another m elements to rotate (since $|r - g_0| < m$), we have to increase the overlap of the intervals (5.3) by another ϵ on either side, so that we now have the modified estimates:

$$|j/N_0 - \frac{1}{2\pi}\arg(\lambda_a(t))| < \frac{1}{2N_0} + 2\epsilon$$
$$\text{for } a \in I_j^i,\ t \in (t_i, t_{i+1}),\ i = 0, 1, 2, \cdots, s-1.$$

Next we define projections p_j^r on (t_r, t_{r+1}) by

$$p_j^r(t) = \sum_{m \in I_j^r} e_m(t) \quad ; \quad t \in (t_r, t_{r+1}),$$

for $r = 0, 1, 2, \cdots, s-1, j = 0, 1, 2, \cdots, N_0 - 1$.

We define unitaries v_r on (t_r, t_{r+1}) as follows. If $l = 0$, we put

$$v_r(t) = \sum_{j=0}^{N_0-1} e^{2\pi i j/N_0} p_j^r(t).$$

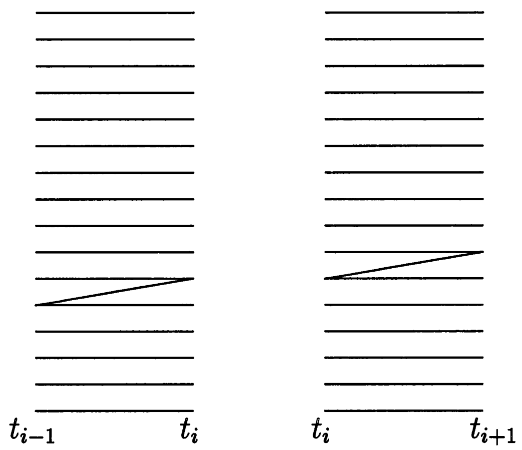


Figure 5:

If $l \neq 0$, we subdivide the partition $0 = t_0 < t_1 < \cdots < t_s = 1$ so that each $\{lt; t \in (t_r, t_{r+1})\} (\subset \mathbb{R}/\mathbb{Z})$ is contained in one of the intervals $[j/(N_0 - 1), (j+1)/(N_0 - 1)]$, $j = 0, \cdots, n_0 - 2$ and if $\{lt : t \in (t_r, t_{r+1})\}$ lies in the interval $[j_0/(N_0 - 1), (j_0 + 1)/(N_0 - 1)]$, we put

$$v_r(t) = \sum_{j=0}^{j_0} e^{2\pi i j/(N_0-1)} p_j^r(t) + e^{2\pi i l t} p_{j_0+1}^r(t) + \sum_{j=j_0+2}^{N_0-1} e^{2\pi i (j-1)/(N_0-1)} p_j^r(t).$$

Then for $t \in (t_r, t_{r+1})$:

$$\begin{aligned} \|v(t) - v_r(t)\| &< \begin{cases} 1/2N_0 + 2\epsilon & \text{if } l = 0 \\ 5/2N_0 + 2\epsilon & \text{if } l \neq 0 \end{cases} \\ &< 3\delta. \end{aligned}$$

So on each interval (t_i, t_{i+1}) we have the eigenvalue diagram of Figure 5, (drawn in this case for ℓ negative).

We next have to connect the two unitaries $w_0 = u_{i-1}(t_i)$ and $w_1 = v_i(t_i)$ (obtained from neighbouring intervals (t_{i-1}, t_i) and (t_i, t_{i+1}) for $i = 1, \cdots, s-1$) with a path of unitaries w_t whose eigenprojections move continuously from those of w_0 to those of w_1, with constant spectrum $Sp(w_t) = Sp(w_0)$. More precisely, we have to match up the eigenprojections with the same

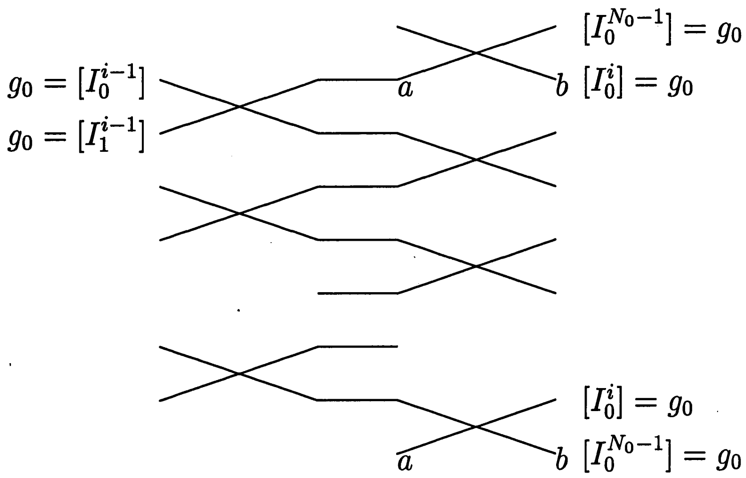


Figure 6:

constant eigenvalue or with the winding eigenvalue $t \to e^{2\pi ilt}$, i.e., all of $p_j^{i-1}(t_i)$ are matched with $p_j^i(t_i)$, possibly except that $p_{j_0}^{i-1}(t_i)$ is matched with $p_{j_0+1}^i(t_i)$ and $p_{j_0+1}^{i-1}(t_i)$ with $p_{j_0}^i(t_i)$ for some j_0. First we use the fact that $\{p_j^{i-1}(t_i),\, p_j^i(t_i);\ j = 0, \cdots, N_0-1\}$ is commutative and $p_j^{i-1}(t_i)p_j^i(t_i) \neq 0$ and connect w_0 and w_1 by finding a continuous path of projections from $p_j^{i-1}(t_i)$ to $p_j^i(t_i)$ for each j in such a way that the resulting path of unitaries remains in a small neighbourhood (of order $1/N_0$). Indeed to find such a path we can use Steps 1 to 4 and Figure 1 where the lines represent (singular) paths of projections rather than eigenvalues. (See Figure 6.)

However condition (5.4) ensures that the class of the projection on each segment is g_0 (and incidentally that $\#\,((I_r^{i-1} \cup I_{r+1}^{i-1}) \cap (I_r^i \cup I_{r-1}^i)) = g_0 \bmod (m-1)$ holds for each $r = 0, 1, \cdots$). This means we are in the situation described by Figure 4 and can separate the *paths* of projections to get Figure 7. Then, if this is not yet what we wanted, we rotate two projections $p_{j_0}^i(t_i)$ and $p_{j_0+1}^i(t_i)$ into the other keeping the sum constant (so that the resulting path of unitaries is close to constant). Finally we insert this path in the gap between w_0 and w_1.

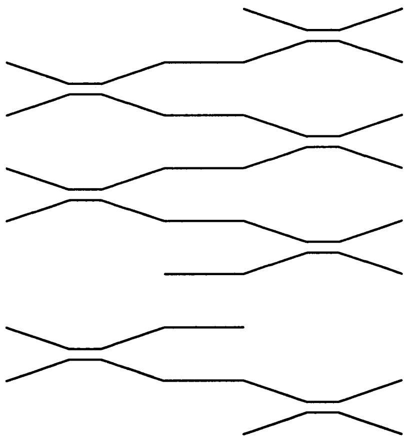


Figure 7:

Modifying the parameter t we will obtain a unitary valued continuous function v on $[0,1]$ such that if $l=0$,

$$v(t)=\sum_{j=0}^{N_0-1} e^{2\pi ij/N_0}p_j(t)$$

or if $l\neq 0$,

$$v(t)=\sum_{j=0}^{N_0-2} e^{2\pi ij/(N_0-1)}p_j(t)+e^{2\pi ilt}p_{N_0-1}(t)$$

and $\|U(t)-v(t)\|$ is of order δ.

We still have to connect $v_{s-1}(1)$ and $v_0(0)$, which again we will do according to Figure 6 (except for the rotation required). Note that by our condition (5.4) the class of each segment when we connected $v_{i-1}(t_i)$ with $v_i(t_i)$ in Figure 6 was g_0, and the contribution to K_1 from v in the construction on $[0,1]$ is lg_0 (when a closed path is made by connecting $v_{s-1}(1)$ and $v_0(0)$ in the way the eigenvalues are constant). However in connecting $v_{s-1}(1)$ to $v_0(0)$ as in Figure 6 we do not a priori know the class of the projections of the internal segments, although it is clear from the conservation law that there is some $\theta(\in K_0(\mathcal{O}_m))$ so that the classes of the projections on the segments are as labelled in Figure 8. The contribution to K_1 of this part is computed

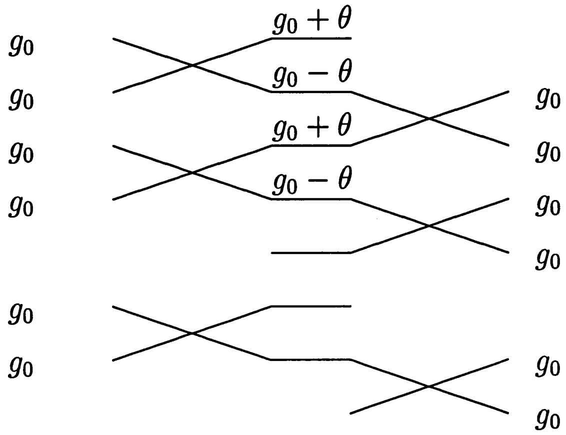


Figure 8:

to be θ (see [BEEK]). Having connected $v_{s-1}(1)$ to $v_0(0)$ according to Figure 8, we have the unitary v close to U, with $[v] = lg_0 + \theta$, but also $[v] = [U]$. So having chosen $[U] = lg_0$, we are forced to conclude that $\theta = 0$. Thus the class of each internal segment in Figure 8 is g_0 and we can proceed to separate the paths of projections as in Figure 7, and insert this connection between $v_{s-1}(1)$ and $v_0(0)$. Since v is close to U_0 of standard form with $M = N_0$, this completes the proof of Theorem 5.1.

6. $\mathcal{O}_n \otimes C(\mathbb{T}) \to M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$: Existence and uniqueness

Here we prove the basic existence and uniqueness results (Theorem 4.1 and Theorem 4.2) for lifting KK-maps between building blocks of the form a matrix algebra over $\mathcal{O}_n \otimes C(\mathbb{T})$, where $\mathcal{O}_n$ is a Cuntz algebra.

We recall from [BEEK], Theorem 4.1, the isospectral obstruction $F(\lambda, u)$ of a unitary operator u with respect to a unital endomorphism λ of a simple, purely infinite C^*-algebra A. Here $[u] = 0$, $\lambda(u) \approx u$ and $F(\lambda, u)$ is an element of $K_0(A)/(1 - \lambda_*)K_0(A)$. If v is a unitary in A, $vu \approx uv$, and λ is the inner automorphism $\mathrm{Ad}\, v$, then $\lambda_* = 1$, and the isospectral obstruction of u with respect to v is defined as:

$$\text{Isospec}(v,u) = F(\text{Ad}(v),u) \in K_0(A).$$

(The isospectral obstruction Isospec(u,v) can also be identified by [BEEK, Theorem 9.1] with the Bott element $B(v,u)$ of [Lor].)

The basic homotopy lemma of [BEEK, Theorem 8.1] shows that if $vu \approx uv$, (and $[u] = 0$ of course) and the isospectral obstruction Isospec(v,u) vanishes, then there is a rectifiable path u_t of universally bounded length of unitaries of A, connecting 1 to u and $vu_t \approx u_t v$ (uniformly in t). In Lemma 6.1 we deduce a version of this homotopy lemma for the building block $M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$.

If A is a purely infinite simple C^*-algebra and $B = A \otimes C(\mathbb{T})$, let $\pi_t = B \to A$ denote the evaluation map. If λ is an endomorphism of B given by a continuous family of endomorphisms λ_t of A, we write $F(\lambda,u)$ for $F(\lambda_t,u(t))$ if u is a unitary of B, with $[u(t)] = 0$, $\lambda(u) \approx u$. Note that $F(\lambda_t,u(t))$ is independent of t by homotopy invariance [BEEK, Theorem 4.1 (1)]. If v is a unitary of B, we define Isospec$(v,u) \equiv$ Isospec$(v(t),u(t))(= F(Ad\, v,u))$, as long as $vu \simeq uv$.

Lemma 6.1 *For any $\epsilon > 0$, there exists a $\delta > 0$ with the following properties: If A is a simple unital purely infinite C^*-algebra, and U and V are unitaries in $A \otimes C(\mathbb{T})$ with*

(i) $[V]_1 = 0$, $[U(t)]_1 = 0$,

(ii) $\|VU - UV\| < \delta$,

(iii) Isospec$(U,V) = 0$,

(iv) Sp$(U(t))$ *is δ-dense in $\mathbb{T}$ for all $t \in \mathbb{T}$,*

then there exists a rectifiable path V_t of unitaries in $A \otimes C(\mathbb{T})$ with

(a) $V_0 = 1$, $V_1 = V$,

(b) $\|V_t U - UV_t\| < \epsilon$,

(c) Length $(V_t) \leq L$,

where L is a universal constant (which can be estimated as $4(5\pi + 1)$).

Proof:

We denote by ι the embedding of $B = A \otimes C(\mathbb{T})$ into $A \otimes C[0,1]$. Then since $\mathrm{Sp}(\iota(U)(t))$ is δ-dense in $\mathbb{T}$ for any $t \in [0,1]$, $\iota(U)$ can be approximated by a unitary U_0 of $A \otimes C[0,1]$ with finite spectrum as in Theorem 5.1. Then as in [BEEK] Theorem 8.1, there is a path $\{V_t\}$ of unitaries such that

$$\begin{aligned} &V_0 = 1, \quad V_1 = V, \quad \|[\iota(U), V_t]\| \approx 0, \\ &\text{Length}\,(\{V_t\}) < L' \end{aligned}$$

where L' is a constant depending only on the algebra $A \otimes C[0,1]$. (By a similar calculation as in the proof of [BEEK] Theorem 8.1 we can estimate L' as $5\pi + 1$). By changing the parametrization and the path itself slightly if necessary we can assume that for any $s, t \in [0,1]$

$$V_s = 1 \qquad 0 \le s \le 1/2\,, \text{ and } \|V_s - V_t\| < 2L'|s-t|.$$

Define a path v by

$$v(t) = \begin{cases} V_{t+1/2}(0) & 0 \le t \le 1/2 \\ V_{\frac{3}{2}-t}(1) & 1/2 \le t \le 1 \end{cases}$$

Then since $v(0) = 1 = v(1)$ and v is continuous, v can regarded as a unitary of B, and satisfies $[v, u] \approx 0$ where u is a unitary of B with $u(t) = U(1)$. Since $[v] = 0$ (as v is the boundary of $V_s(t)$, $(s,t) \in [0,1] \times [0,1]$ and $[V_1] = 0 = [V_0]$ in $K_1(B)$), $F(\mathrm{Ad}\, v, u) = F(\mathrm{Ad}\, v(0), u(0)) = 0$, and u is constant, there is a path $\{v_t\}$ of unitaries of B such that

$$v_0 = 1, \quad v_1 = v, \quad [v_t, u] \approx 0, \text{ and Length}\,(\{v_t\}) < L''$$

where L'' is a constant. (The same estimate as for L' applies for L''). We can assume here for any $s, t \in [0,1]$ that $\|v_s - v_t\| < L''|s-t|$, and, by the construction, that $v_s(0) = 1$, $s \in [0,1]$ (We may just replace v_s by $v_s \cdot v_s(0)^*$ and L'' by $2L''$). Define

$$w(s,t) = \begin{cases} v_{2s}(t) & 0 \le s \le 1/2 \\ v_1(t + (1-2t)(s - \frac{1}{2})), & 1/2 \le s \le 1 \end{cases}$$

Then w is continuous on $[0,1] \times [0,1]$ and satisfies $w(0,t) = 1$, $w(1,t) = v_1(1/2) = V(0)$, and for any $s_1, s_2 \in [0,1]$

$$\|w(s_1, \cdot) - w(s_2, \cdot)\| \le 2\max(L', L'')|s_1 - s_2|$$

where $w(s,\cdot)$ is regarded as an element of $A \otimes C[0,1]$. Note that

$$w(s,0) = \begin{cases} v_{2s}(0) = 1 = V_s(0) & 0 \leq s \leq 1/2 \\ v(s-1/2) = V_s(0) & 1/2 \leq s \leq 1 \end{cases}$$

and so $w(s,0) = V_s(0)$, and that

$$w(s,1) = \begin{cases} v_{2s}(1) = 1 = V_s(1) & 0 \leq s \leq 1/2 \\ v(3/2-s) = V_s(1) & 1/2 \leq s \leq 1 \end{cases}$$

and so $w(s,1) = V_s(1)$. Since $w(s,0) = V_s(0)$ and $w(s,1) = V_s(1)$, we can connect $V_s(0)$ and $V_s(1)$ by using $w(s,\ \cdot)$. By squeezing the part of w into a small neighbourhood of $0 = 1$ and combining with $V_s(t)$ we will obtain a path V'_s of unitaries of B such that

$$V'_0 = 1,\ V'_1 = V, \quad \|[V'_t, U]\| \approx 0\,, \text{ and Length}\,(\{V'_t\}) < 2\max(L', L'').$$

This completes the proof.

We say that a homomorphism ψ of the free product $\mathcal{O}_n *_{\mathbb{C}} C(\mathbb{T})$ into a C^*-algebra B is of class $\delta > 0$ if

$$\|\lambda_\psi(\psi(u)) - \psi(u)\| < \delta$$

where u is the canonical unitary of $C(\mathbb{T})$ and λ_ψ is the endomorphism of $\psi(1)B\psi(1)$ defined by

$$\lambda_\psi(x) = \sum_{i=1}^{n} \psi(s_i)x\psi(s_i)^*.$$

Lemma 6.2 *If a unital homomorphism ϕ of $\mathcal{O}_n * C(\mathbb{T})$ into B is of class 1, then $(n-1)[\phi(u)] = 0$, where u is the canonical unitary of $C(\mathbb{T})$.*

Proof:

One knows that for any $i = 1, \cdots, n$

$$\phi(s_i)\phi(u)\phi(s_i)^* + 1 - \phi(s_i)\phi(s_i)^*$$

is equivalent to $\phi(u)$ (cf. [Rør1, Proposition 3.4], [Cun2, Lemma 1.1]). Hence

$$[\sum_{i=1}^{n} \phi(s_i)\phi(u)\phi(s_i)^*] = n[\phi(u)].$$

Since $\|\lambda_\phi(\phi(u)) - \phi(u)\| < 1$, one has $[\lambda_\phi(\phi(u))] = [\phi(u)]$, where $\lambda_\phi(x) = \sum \phi(s_i)x\phi(s_i)^*$. Thus $(n-1)[\phi(u)] = 0$. This completes the proof.

Let $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$. Let $G = \{g \in K_1(B) | (n-1)g = 0\}$. Since G is a subgroup of the cyclic group $K_1(B) = \mathbb{Z}/(m-1)\mathbb{Z}$, G has a generator g_0. We may sometimes identify $K_1(B)$ with $\{0, 1, \cdots, m-1\}$.

If ϕ is a unital homomorphism of $\mathcal{O}_n * C(\mathbb{T})$ into $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$, then $\lambda_\phi \in \mathrm{End}(B)$ satisfies that $\pi_t \circ \lambda_\phi = \lambda_{\pi_t \circ \phi} \circ \pi_t$, where $\lambda_{\pi_t \circ \phi} \in \mathrm{End}(M_k \otimes O_m)$. For a unitary w in B with $\lambda_\phi(w) \approx w$ we have already defined $F(\lambda_\phi, w) = F(\lambda_{\pi_t \circ \phi}, \pi_t(w))$, (note $[w(t)] = 0$ as $K_1(M_k \otimes O_m) = 0$).

Lemma 6.3 *For any $\epsilon > 0$, there exists $\delta > 0$ satisfying the following conditions: If ϕ, ψ are unital homomorphisms of $\mathcal{O}_n * C(\mathbb{T})$ into $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$ of class δ and satisfy*

(i) For any $t \in \mathbb{T}$, the spectra of $\phi(u)(t)$ and $\psi(u)(t)$ are δ-dense in $\mathbb{T}$,

(ii) $[\phi(u)] = [\psi(u)]$ in $K_1(B)$,

(iii) $F(\lambda_\phi, \phi(u)) = F(\lambda_\psi, \psi(u))$ in $K_0(B)/(n-1)K_0(B)$,

(iv) $[\sum_{i=1}^{n} \phi(s_i)\psi(s_i)^*] \in (n-1)K_1(B)$,

then there is a unitary U of B such that

(a) $\|\phi(S_i) - \mathrm{Ad}\, U(\psi(S_i))\| < \epsilon, i = 1, 2, \cdots, n$

(b) $\|\phi(u) - \mathrm{Ad}\, U(\psi(u))\| < \epsilon$.

Proof:

Let $\delta < \epsilon/C$ and $M > 1/\delta$ where C is in Theorem 5.1. We approximate $\phi(u)$ (resp. $\psi(u)$) by a unitary U_ϕ (resp. U_ψ) of standard form of rank M. Then there is a unitary w of B such that $wU_\psi w^* = U_\phi$, and we define ϕ' and ψ' by

$$\phi'(S_i) = \phi(S_i),$$

$$\phi'(u) = U_\phi,$$

$$\psi'(S_i) = w\psi(S_i)w^*,$$

$$\psi'(u) = U_\phi.$$

Note that $\psi'(s_i) = \mathrm{Ad}\, w \circ \psi(s_i)$ and

$$\|\psi'(u) - \mathrm{Ad}\, w \circ \psi(u)\| \le \|\psi(u) - U_\psi\| < \epsilon$$

and also that $\phi'(s_i) = \phi(s_i)$, $\|\phi'(u) - \phi(u)\| < \epsilon$. Hence it suffices to replace ϕ, ψ by ϕ', ψ'. (The condition (iii) holds for ϕ', ψ' for sufficiently small $\epsilon > 0$ because F is continuous.)

Now suppose that

$$\phi(u) = \psi(u) = u^l p_1 + \sum_{j=2}^{M} e^{2\pi i j/M} p_j,$$

where $[p_j] = g_0$. Since $(n-1)g_0 = 0$, there is a canonical set $\{T_i^j\}$ of n isometries of $p_j B p_j$:

$$\sum_{i=1}^{n} T_i^j T_i^{j*} = p_j.$$

By summing up over j, we obtain a canonical set $\{T_i\}$ of n isometries of B in the commutant of $\phi(u)$. Let

$$W_\phi = \sum_{i=1}^{n} \phi(s_i) T_i^*\,, \text{ and } W_\psi = \sum_{i=1}^{n} \psi(s_i) T_i^*.$$

If $[W_\psi] \equiv h \neq 0$, choose a subprojection q of p_1 such that $[q] = 1$ (where we identify $K_0(B)$ with $K_1(B)$). Then replace T_i by $T_i(u^h q + 1 - q)$, which is still in the commutant of $\phi(u)$, and the resulting W_ψ is trivial in $K_1(B)$. Since

$$W_\phi W_\psi^* = \sum \phi(s_i)\psi(s_i)^* = V$$

it follows that

$$[W_\phi] = [V] \in (n-1)K_1(B).$$

Let $g \in K_1(B)$ be such that $[W_\phi] = (n-1)g$.

Let U be a unitary of B such that U commutes with $\phi(u)$ and $[U] = g$ (the existence of such a unitary has been already demonstrated with $u^h q + 1 - q$) and let

$$\phi' = \operatorname{Ad} U \circ \phi.$$

Then

$$W_{\phi'} = \sum U\phi(s_i)U^* T_i^* = \sum U W_\phi T_i U^* T_i^* = U W_\phi \lambda(U^*)$$

where $\lambda(x) = \sum T_i x T_i^*$. Since $[\lambda(U^*)] = n[U^*]$, it follows that

$$[W_{\phi'}] = [W_\phi] - (n-1)g = 0.$$

Thus, since $F(\lambda_{\pi_t \circ \phi'}, \pi_t \circ \phi'(u)) = F(\lambda_{\pi_t \circ \phi}, \pi_t \circ \phi(u))$, by replacing ϕ by ϕ' we may assume that $[W_\phi] = 0$.

Since λ fixes $\phi(u)$ which has finite spectrum one obtains that

$$F(\lambda_{\pi_t \circ \phi}, \pi_t \circ \phi(u)) = F(\operatorname{Ad} W_\phi(t), \phi(u)(t)) + (n-1)K_0(B),$$

and a similar formula for ψ. Thus it follows that

$$F(\operatorname{Ad} W_\phi(t), \phi(u)(t)) - F(\operatorname{Ad} W_\psi(t), \psi(u)(t)) \in (n-1)K_0(B).$$

Let U be a unitary such that $U\phi(u)U^* \approx \phi(u)$ and define ϕ' by

$$\begin{aligned} \phi'(s_i) &= U\phi(s_i)U^* = U W_\phi \lambda(U^*) T_i \\ \phi'(u) &= \phi(u). \end{aligned}$$

Then $\phi' \approx \operatorname{Ad} U \circ \phi$ and

$$\begin{aligned} & F(\operatorname{Ad}(U W_\phi \lambda(U^*))(t), \phi'(u)(t)) \\ = \ & F(\operatorname{Ad} U(t), \phi(u)(t)) + F(\operatorname{Ad} W_\phi(t), \phi(u)(t)) + F(\operatorname{Ad} \lambda(U^*)(t), \phi(u)(t)). \end{aligned}$$

Since there is a continuous path from $\lambda(U^*)$ to U^{*n} which almost commute with $\phi(u)$, it follows that

$$F(\operatorname{Ad} \lambda(U^*)(t), \phi(u)(t)) = nF(\operatorname{Ad} U^*(t), \phi(u)(t)).$$

Hence

$$\begin{aligned} F(\operatorname{Ad}(U W_\phi \lambda(U^*))(t), \phi'(u)(t)) &= F(\operatorname{Ad} W_\phi(t), \phi(u)(t)) \\ &+ (n-1)F(\operatorname{Ad} U^*(t), \phi(u)(t)). \end{aligned}$$

Choosing U suitably (e.g., U can be a unitary which cyclically permutes subprojections of $p_2, \cdots, p_M$ as in [BEEK, Lemma 9.2]), and replacing ϕ by ϕ' we may assume that

$$F(\mathrm{Ad}\, W_\phi(t), \phi(u)(t)) = F(\mathrm{Ad}\, W_\psi(t), \psi(u)(t)).$$

We are now reduced to the following case: unital homomorphisms ϕ and ψ of class δ satisfying

$$\begin{aligned}
&\phi(u) = \psi(u) \quad \text{is of standard form of rank } M \\
&[W_\phi] = [W_\psi] = 0 \\
&F(\mathrm{Ad}\, W_\phi(t), \phi(u)(t)) = F(\mathrm{Ad}\, W_\psi(t), \psi(u)(t))
\end{aligned}$$

¿From the last condition it follows that $F(\mathrm{Ad}\, W_\phi(t)W_\psi(t)^*, \phi(u)(t)) = 0$. Since $[W_\phi W_\psi^*] = 0$, applying Lemma 6.1 one obtains that there is a path $\{V_t\}$ in B from 1 to $W_\phi W_\psi^*$ such that V_t nearly commutes with $\phi(u)$ and the length of $\{V_t\}$ is bounded by a universal constant depending only on B. Since $F(\mathrm{Ad}\, \lambda^l_{\pi_t \circ \psi}(W_\phi(t)W_\psi(t)^*), \phi(u)(t)) = 0$ for any l if well-defined, the same is true for $\lambda^l_\psi(W_\phi W_\psi^*)$ in place of $W_\phi W_\psi^*$. Then one finds a unitary W of B which nearly commutes with $\phi(u)$ such that

$$\phi(s_i) \approx W\psi(s_i)W^*$$

because by [Rørl], W is obtained by using such a path and a Rohlin tower for λ_ψ which is in the algebra generated by $\psi(s_{i_1}) \cdots \psi(s_{i_l})\psi(s_{j_l})^* \cdots \psi(s_{j_1})^*$ for a certain l.

Lemma 6.4 *If $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$ and $\theta \in G = \{g \in K_1(B) : (n-1)g = 0\} = \mathbb{Z}g_0$, there exists a homomorphism ψ of $\mathcal{O}_n \otimes C(\mathbb{T})$ into B such that*

(a) $[\psi(u)] = \theta$,

(b) $F(\lambda_\psi, \psi(u)) = 0$,

(c) $Sp(\psi(u)(t)) = \mathbb{T}$,

for any $t \in \mathbb{T}$.

Proof:

Since $k \in G$ there is a positive integer l such that $k = lg_0$.

There is a unitary $U \in M_{m^\infty}$ such that $Sp\, U = \mathbb{T}$. Using a unital homomorphism ψ_0 of $\mathcal{O}_n$ into $M_{g_0} \otimes \mathcal{O}_m$, define a homomorphism ψ of $\mathcal{O}_n \otimes C(\mathbb{T})$ into $M_{lm} \otimes M_{g_0} \otimes \mathcal{O}_m \otimes M_{m^\infty} \otimes C(\mathbb{T})$ by

$$\psi|\mathcal{O}_n \otimes 1 = \psi_0$$
$$\psi(u) = eUz + (1-e)U$$

where e is a projection of M_{lm} (embedded in the bigger algebra), and z is the canonical unitary of $C(\mathbb{T})$. Since $[\psi(u)] = (\dim e) \cdot g_0$ and $\theta \in \mathbb{Z}g_0$, one can choose e so that $[\psi(u)] = \theta$. Since $U \in M_{m^\infty}$ can be approximated by a unitary with finite spectrum in M_{m^∞}, which is invariant under λ_ψ, it follows that $F(\lambda_\psi, \psi(u)) = 0$. Since $\mathcal{O}_m \otimes M_{m^\infty} \cong \mathcal{O}_m$, ψ can be regarded as a homomorphism of $\mathcal{O}_n \otimes C(\mathbb{T})$ into B. This completes the proof.

Lemma 6.5 *If $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$, let $\theta_0 \in K_0(B)/(n-1)K_0(B)$, $\theta_1 \in K_1(B)$, and ψ be a homomorphism of $\mathcal{O}_n \otimes C(\mathbb{T})$ into B such that $F(\lambda_\psi, \psi(u)) = 0$, $Sp(\psi(u)(t)) = \mathbb{T}$ for any $t \in \mathbb{T}$. Then there is a homomorphism ϕ of $\mathcal{O}_n \otimes C(\mathbb{T})$ into B such that*

$$\begin{aligned} &(a) \quad [\phi(u)] = [\psi(u)], \\ &(b) \quad Sp(\phi(u)(t)) = \mathbb{T} \qquad \textit{for any } t \in \mathbb{T}, \\ &(c) \quad F(\lambda_\phi, \phi(u)) = \theta_0, \\ &(d) \quad [\sum_{i=1}^{n} \phi(s_i)\psi(s_i)^*] - \theta_1 \in (n-1)K_1(B). \end{aligned}$$

Proof:

We regard ψ as a homomorphism (of class δ with any $\delta > 0$) of $\mathcal{O}_n * C(\mathbb{T})$ into B and will find, for any $\delta > 0$, a homomorphism ϕ of class δ of $\mathcal{O}_n * C(\mathbb{T})$ into B such that

$$[\phi(u)] = [\psi(u)],\ F(\lambda_\phi, \phi(u)) = \theta_0,\ [\sum \phi(s_i)\psi(s_i)^*] = \theta_1,\ Sp\, \phi(u)(t) = \mathbb{T}.$$

First fix a set $\{T_1, \cdots, T_n\}$ of isometries of $M_{g_0} \otimes \mathcal{O}_m$ such that $\sum_{i=1}^{n} T_i T_i^* = 1$ and fix a unitary U of M_{m^∞} with $\mathrm{Sp}(U) = \mathbb{T}$. For a positive integer N with

$2\pi/N < \delta$, let u_N, v_N be unitaries of M_N such that $v_N u_N = e^{2\pi i/N} u_N v_N$ and embed u_N, v_N into M_{m^L} by

$$u_N \to u_N \oplus 1_{(m^L - N)}, \quad v_N \to v_N \oplus 1_{(m^L - N)},$$

where L is the smallest positive integer with $N \leq m^L$. Let e be a projection of $\mathbb{C}^{lm} \subset M_{lm}$ with $\dim(e) g_0 = [\psi(u)]$ and let p_i, $i = 1, 2$ be a projection in $\mathbb{C}^{lm} \otimes \mathbb{C}^{g_0} \subset M_{lm} \otimes M_{g_0}$. Define a homomorphism ϕ of $\mathcal{O}_n * C(\mathbb{T})$ into $M_{lm} \otimes M_{g_0} \otimes \mathcal{O}_m \otimes M_{m^L} \otimes M_{m^\infty} \otimes C(\mathbb{T}) \cong B$ by

$$\begin{aligned} \phi(s_1) &= (p_1 v_N + 1 - p_1) T_1 (p_2 z + 1 - p_2), \\ \phi(s_i) &= (p_1 v_N + 1 - p_1) T_i, \qquad 2 \leq i \leq n, \\ \phi(u) &= (p_1 u_N + 1 - p_1)(eUz + (1 - e)U). \end{aligned}$$

Then since $z \in 1 \otimes M_{g_0} \otimes \mathcal{O}_m \otimes M_{m^L} \otimes M_{m^\infty} \otimes C(\mathbb{T})$,

$$[\phi(u)] = \dim(e) \cdot [z] = [\psi(u)].$$

Since

$$\lambda_\phi(\phi(u)) = (e^{2\pi i/N} p_1 u_N + 1 - p_1)(eUz + (1 - e)U),$$

which is close to $\phi(u)$ in norm within $2\pi/N$, one obtains that

$$F(\lambda_\phi, \phi(u)) = \dim(p_1) + (n - 1) K_0(B).$$

Since

$$\sum \phi(s_i) \psi(s_i)^* = (\sum \phi(s_i) T_i^*)(\sum T_i \psi(s_i^*))$$

and

$$[\sum \phi(s_i) T_i^*] = [p_2 z + 1 - p_2] = \dim(p_2),$$

it follows that

$$[\sum \phi(s_i) \psi(s_i)^*] = \dim(p_2) + [\sum T_i \psi(s_i^*)].$$

Thus, by choosing p_1, and p_2 suitably, one obtains a homomorphism ϕ of class δ with the desired properties.

Now we use an intertwining argument as follows: For $\epsilon = \epsilon_j = 2^{-j}$, choose $\delta = \delta_j > 0$ as in Lemma 6.3. Choose a homomorphism ϕ_j of class δ_n such that

$$\begin{aligned} &[\phi_j(u)] = [\psi(u)], \\ &F(\lambda_{\phi_j}, \phi_j(u)) = \theta_0, \quad [\sum \phi_j(s_i) \psi(s_i^*)] = \theta_1 \\ &\mathrm{Sp}(\phi_j(u)(t)) = \mathbb{T}. \end{aligned}$$

Then applying Lemma 6.3 to the pair ϕ_1, ϕ_2, we obtain a unitary U_2 of B such that

$$\phi_1 \underset{\epsilon_1}{\approx} \operatorname{Ad} U_2 \circ \phi_2$$

i.e.,

$$\|\phi_1(s_i) - \operatorname{Ad} U_2 \circ \phi_2(s_i)\| < \epsilon_1\,, \qquad \|\phi_1(u) - \operatorname{Ad} U_2 \circ \phi_2(u)\| < \epsilon_1.$$

Applying Lemma 6.3 inductively, we obtain a unitary U_n of B for the pair $\operatorname{Ad} U_{n-1} \circ \phi_{n-1}$, ϕ_n;

$$\operatorname{Ad} U_{n-1} \circ \phi_{n-1} \underset{\epsilon_{n-1}}{\approx} \operatorname{Ad} U_n \circ \phi_n$$

Then the limit ϕ of $\operatorname{Ad} U_j \circ \phi_j$ exist on $s_1, \cdots, s_n, u$ and hence on $O_n * C(\mathbb{T})$. Since ϕ is a homomorphism of class δ with any $\delta > 0$, ϕ induces a homomorphism of $O_n \otimes C(\mathbb{T})$ into B.

Since

$$\begin{aligned}
&[\operatorname{Ad} U_j \circ \phi_j(u)] = [\phi_j(u)],\\
&F(\lambda_{\operatorname{Ad} U_j \circ \phi_j}, \operatorname{Ad} U_j \circ \phi_j(u)) = F(\operatorname{Ad} U_j \circ \lambda_{\phi_j} \circ \operatorname{Ad} U_j^*, \quad \operatorname{Ad} U_j \circ \phi_j(u))\\
&= F(\lambda_{\phi_j}, \phi_j(u))\\
&[\textstyle\sum \operatorname{Ad} U_j \circ \phi_j(s_i)\psi(s_i^*)] = [U_j \sum \phi_j(s_i)\psi(s_i)^* \lambda_\psi(U_j^*)]\\
&= [\textstyle\sum \phi_j(s_i)\psi(s_i^*)] - (n-1)[U_j],
\end{aligned}$$

it follows that ϕ has the desired properties.

Lemma 6.6 *There exists a $\delta > 0$ satisfying the following condition: If ϕ and ψ are unital homomorphism of $\mathcal{O}_n \otimes C(\mathbb{T})$ into $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$ and U is a unitary in B such that*

(i) $\|\phi(s_i) - \operatorname{Ad} U \circ \psi(s_i)\| < \delta$, $i = 1, \ldots, n$

(ii) $\|\phi(u) - \operatorname{Ad} U \circ \psi(u)\| < \delta$,

then it follows that

(a) $[\phi(u)] = [\psi(u)]$ $(\in K_1(B))$,

(b) $F(\lambda_\phi, \phi(u)) = F(\lambda_\psi, \psi(u))$ $(\in K_0(B)/(n-1)K_0(B))$,

(c) $[\sum \phi(s_i)\psi(s_i)^*] \in (n-1)K_1(B)$.

Proof:

Since
$$\left\|\sum_{i=1}^{n} \phi(s_i)U\psi(s_i)^*U^* - 1\right\| < n\delta,$$
λ_ϕ and $\lambda_{\mathrm{Ad}\, U\circ\psi}$ are homotopic in a small neighbourhood of λ_ϕ if δ is sufficiently small. Since the same is true for $\phi(u)$ and $\psi(u)$, one obtains that

$$F(\lambda_\phi, \phi(u)) = F(\lambda_{\mathrm{Ad}\, U\circ\psi}, \mathrm{Ad}\, U \circ \psi(u)).$$

Since $F(\lambda_{\mathrm{Ad}\, U\circ\psi}, \mathrm{Ad}\, U \circ \psi(u)) = F(\lambda_\psi, \psi(u))$, it follows that

$$F(\lambda_\phi, \phi(u)) = F(\lambda_\psi, \psi(u)).$$

The rest is clear.

Let ϕ and ψ be unital homomorphisms on $\mathcal{O}_n \otimes C(\mathbb{T})$ into $M_k \otimes \mathcal{O}_m$ and let E be the C^*-subalgebra of $(C([0,1]) \otimes M_k \otimes \mathcal{O}_m) \oplus (\mathcal{O}_n \otimes C(\mathbb{T}))$ defined by
$$E = \{(f,a) | f(0) = \phi(a), f(1) = \psi(a)\}.$$
Then one has a short exact sequence;

$$0 \to S(M_k \otimes \mathcal{O}_m) \to E \to \mathcal{O}_n \otimes C(\mathbb{T}) \to 0$$

where $S(M_k \otimes \mathcal{O}_m) = C((0,1)) \otimes M_k \otimes \mathcal{O}_m$. From this it follows that

$$0 \to K_0(M_k \otimes \mathcal{O}_m) \to K_1(E) \to K_1(\mathcal{O}_n \otimes C(\mathbb{T})) \to 0 \qquad (1)$$

is exact, where we have used that

$$K_1(S(M_k \otimes \mathcal{O}_m)) \cong K_0(M_k \otimes \mathcal{O}_n).$$

Since $K_1(\mathcal{O}_n \otimes C(\mathbb{T})) \cong \mathbb{Z}/(n-1)\mathbb{Z}$, and $\mathrm{Ext}(\mathbb{Z}/(n-1)\mathbb{Z}, G)$ is isomorphic to $G/(n-1)G$ for any abelian group G, we can represent (1) as an element of $K_0(M_k \otimes \mathcal{O}_m)/(n-1)K_0(M_k \otimes \mathcal{O}_m)$; let $g \in K_1(E)$ be a preimage of $1 \in \mathbb{Z}/(n-1)\mathbb{Z}$, and then $(n-1)g$ is the image of an element $h \in K_0(M_k \otimes \mathcal{O}_m)$. It follows that $h + (n-1)K_0(M_k \otimes \mathcal{O}_m)$ is independent of the choice of g. We denote this element by $\gamma(\phi, \psi)$.

Lemma 6.7 *Let ϕ and ψ be unital homomorphisms of $\mathcal{O}_n \otimes C(\mathbb{T})$ into $M_k \otimes \mathcal{O}_m$ and let $\gamma(\phi, \psi) \in K_0(M_k \otimes \mathcal{O}_m)/(n-1)K_0(M_k \otimes \mathcal{O}_m)$ be as above. It follows that $\gamma(\phi, \psi) = F(\lambda_\psi, \psi(u)) - F(\lambda_\phi, \phi(u))$.*

Proof:

Let $U = \{U_t\}$ be a continuous path of unitaries of $M_k \otimes \mathcal{O}_m$ such that $U_0 = \phi(u)$ and $U_1 = \psi(u)$, where u is the canonical unitary of $1 \otimes C(\mathbb{T}) \subset \mathcal{O}_n \otimes C(\mathbb{T})$. (Such exists since $K_1(M_k \otimes \mathcal{O}_m) = 0$ and $M_k \otimes \mathcal{O}_m$ is purely infinite.) We have to compute $[U^{n-1}]$ in $K_1(E)$.

Let $W = \sum \psi(s_i)\phi(s_i)^*$ and let $\{W_t\}$ be a path of unitaries such that $W_0 = 1$, $W_1 = W$. Let

$$V_t = \mathrm{Ad}\, W_t(\sum \phi(s_i)U_t\phi(s_i)^*)U_t^*.$$

Then one can show that $[V] = [U^{n-1}]$ (cf. [(Cun2]). There is a continuous path of unitaries from $\sum \phi(s_i)U_1\phi(s_i)^*U_1^*$ to U_1^{n-1} which depends continuously only on $\phi(s_i)$ and U_1, and thus one can use the same function for $W_t\phi(s_i)$ and U_t to obtain a path from V_t to U_t^{n-1} simultaneously in t.

Suppose that

$$\begin{array}{ll} U_t = 1, & 1/3 \le t \le 2/3 \\ W_t = 1 & 0 \le t \le 1/3 \\ W_t = W & 2/3 \le t \le 1. \end{array}$$

Then $[V] = [U_t^{(1)}] - [U_t^{(2)}]$ where $U^{(i)}$ is the unitary of E defined by

$$\begin{aligned} U_t^{(1)} &= \lambda_\phi(U_{t/3})U_{t/3}^* \\ U_t^{(2)} &= \lambda_\psi(U_{(1-t/3)})U_{(1-t/3)}^*. \end{aligned}$$

If $\mathrm{Sp}(\phi(u)) \neq \mathbb{T}$, one can conclude $[U^{(1)}] = 0$ for some choice of U_t. Otherwise we approximate $\phi(u)$ by a unitary U_ϕ with finite spectrum of the form

$$U_\phi = \sum_{j=1}^{N} e^{2\pi i j/N} P_j$$

for sufficiently large N, where $\{P_j\}$ is a partition of 1 into projections of class g_0. (Recall that g_0 is a generator of $G = \{g \in K(M_k \otimes \mathcal{O}_m);\ (n-1)g = 0\}$.) One chooses a set $\{T_i\}$ of isometries such that

$$P_jT_i = T_iP_j, \quad \sum_{i=1}^{n} T_iT_i^* = 1,$$

and let

$$V_\phi = \sum \phi(s_i) T_i^*.$$

Then for a suitable choice of U_t one can conclude that $U^{(1)}$ is equivalent to a path of unitaries:

$$1 \rightsquigarrow V_\phi U_\phi V_\phi^* U_\phi^* \rightsquigarrow 1$$

where the first part lies in a small neighbourhood of 1 and the second part is

$$V_\phi \sum_{j=1}^{N} e^{2\pi i j(1-t)/N} P_j V_\phi^* \sum_{j=1}^{N} e^{-2\pi i j(1-t)/N} P_j$$

from $t = 0$ to 1.

Let $\theta = F(\mathrm{Ad}\, V_\phi, U_\phi) \in F(\lambda_\phi, U_\phi)$. Let q_j be a subprojection of P_j with $[q_j] = \theta$ and let V' be a unitary such that $V'(1 - \sum q_j) = 1 - \sum q_j$ and

$$\mathrm{Ad}\, V'(q_j) = q_{j+1}$$

where $q_{N+1} = q_1$. Then $F(\mathrm{Ad}\, V', U_\phi) = \theta$ and hence V_ϕ is connected to V' by a path which nearly commutes with U_ϕ. Since $V' U_\phi V'^* U_\phi^* = e^{-2\pi i/N} \sum q_j + 1 - \sum q_j$, one can conclude that $[U^{(1)}] = -\theta$ (for this particular choice of U_t, $t \in [0, 1/3]$). By computing $[U^{(2)}]$ in a similar way one obtains that

$$[U^{n-1}] + (n-1)K_0(M_k \otimes \mathcal{O}_m) = F(\lambda_\psi, \psi(u)) - F(\lambda_\phi, \phi(u)),$$

where $[U^{n-1}]$ is regarded as an element of $K_0(M_k \otimes \mathcal{O}_m)$.

Theorem 4.1 easily follows from the following result which has more information on KK invariant:

Theorem 6.8 *(Uniqueness) For any sufficently small $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: For unital homomorphisms ϕ and ψ of $A = \mathcal{O}_n \otimes C(\mathbb{T})$ into $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$ such that $Sp(\phi(u)(t))$ and $Sp(\psi(u)(t))$ are δ-dense in $\mathbb{T}$ for any $t \in \mathbb{T}$, the following conditions are equivalent:*

(i) $KK(\phi) = KK(\psi)$

(ii) There exists a unitary U of B such that

$$\|\phi(s_i) - \mathrm{Ad}\, U \circ \psi(s_i)\| < \epsilon, \text{ and } \|\phi(u) - \mathrm{Ad}\, U \circ \psi(u)\| < \epsilon$$

(iii) $[\phi(u) = [\psi(u)]$
$F(\lambda_\phi, \phi(u)] = F(\lambda_\psi, \psi(u))$
$[\Sigma\phi(s_i)\psi(s_i)^*] \in (n-1)K_1(B)$.

Proof:

By choosing a sufficently small $\epsilon > 0$ one obtains (ii)⇒(iii) by Lemma 6.6. By choosing $\delta > 0$ as in Lemma 6.3 one obtains (iii)⇒(ii). Thus (ii) and (iii) are equivalent.

Suppose that $KK(\phi) = KK(\psi)$. Then since $\phi_* = \psi_*$ on $K_1(\mathcal{O}_n \otimes C(\mathbb{T}))$ one has that $[\phi(u)] = [\psi(u)]$. If we denote by ι the embedding of $\mathcal{O}_n$ into $\mathcal{O}_n \otimes C(\mathbb{T})$, one has that $KK(\phi \circ \iota) = KK(\psi \circ \iota)$. It follows from [Rør1] that

$$[\sum \phi(s_i)\psi(s_i)^*] \in (n-1)K_1(B).$$

Let π_t be the evaluation map of B onto $M_k \otimes \mathcal{O}_m$. Then since $KK(\pi_t \circ \phi) = KK(\pi_t \circ \psi)$, one has that $\gamma(\pi_t \circ \phi, \pi_t \circ \psi) = 0$, which says

$$F(\lambda_{\pi_t \circ \phi}, \pi_t \circ \phi(u)) = F(\lambda_{\pi_t \circ \psi}, \pi_t \circ \psi(u))$$

by Lemma 6.7. Since $F(\lambda_\phi, \phi(u))$ denotes $F(\lambda_{\pi_t \circ \phi}, \pi_t \circ \phi(u))$, we have obtained all the conditions in (iii).

Suppose (iii). Then by the reasons given above one obtains that

$$\begin{aligned} &\phi_* = \psi_* \quad \text{on} \quad K_1(\mathcal{O}_n \otimes C(\mathbb{T})) \\ &KK(\phi \circ \iota) = KK(\psi \circ \iota), \\ &KK(\pi_t \circ \psi) = KK(\pi_t \circ \psi). \end{aligned}$$

Since ϕ and ψ are unital, one has $\phi_* = \psi_*$ on $K_0(\mathcal{O}_n \otimes C(\mathbb{T}))$. Thus by the universal coefficient theorem, $KK(\phi) - KK(\psi)$ is regarded as an element of

$$\mathrm{Ext}(K_1(\mathcal{O}_n \otimes C(\mathbb{T})),\, K_0(B)) \oplus \mathrm{Ext}(K_0(\mathcal{O}_n \otimes C(\mathbb{T})),\, K_1(B))$$

Since the embedding ι singles out the second component and π_t does the first component, one can conclude that $KK(\phi) - KK(\psi) = 0$.

Theorem 4.2 can be obtained from the following:

Theorem 6.9 *(Existence) Let $A = \mathcal{O}_n \otimes C(\mathbb{T})$ and $B = M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$, where m and n are even positive integers and k is a positive integer such that $(m-1)|k(n-1)$. For any $g \in KK(A, B)$ which induces the map*

$$1 \in K_0(A) = \mathbb{Z}/(n-1)\mathbb{Z} \mapsto k \in K_0(B) = \mathbb{Z}/(m-1)\mathbb{Z}$$

there is a unital homomorphism ϕ of A into B such that $KK(\phi) = g$ and $\mathrm{Sp}(\phi(u)(t)) = \mathbb{T}$ *for any $t \in \mathbb{T}$.*

Proof:

This follows from Lemmas 6.4 and 6.5 and the uniqueness theorem.

7. $\mathcal{O}_k \otimes C(\mathbb{T}) \to \mathcal{O}_m \otimes \mathcal{O}_n$: Existence and uniqueness

Let m and n be even positive integers. Let k be an even positive integer such that $k-1$ is the greatest common divisor of $m-1$ and $n-1$. Note that by the Künneth formula as in Section 3 :

$$K_*(\mathcal{O}_m \otimes \mathcal{O}_n) \simeq K_*(\mathcal{O}_k \otimes C(\mathbb{T})) \simeq \mathbb{Z}/(k-1)\mathbb{Z} \oplus \mathbb{Z}/(k-1)\mathbb{Z}\,.$$

A unital homomorphism ψ of $\mathcal{O}_k * C(\mathbb{T})$ into $\mathcal{O}_m \otimes \mathcal{O}_n$ is said to be of class δ if

$$\|\lambda_\psi(\psi(u)) - \psi(u)\| < \delta$$

where u is the canonical unitary of $C(\mathbb{T})$ and λ_ψ is the endomorphism of $\mathcal{O}_m \otimes \mathcal{O}_n$ defined by

$$\lambda_\psi(x) = \sum_{i=1}^{k} \psi(s_i) x \psi(s_i)^*\,.$$

Lemma 7.1 *For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition : If $\{T_i : i = 1, \ldots, k\}$ is a set of isometries of $\mathcal{O}_m \otimes \mathcal{O}_n$ and U and V are unitaries of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that*

(i) $\sum_{i=1}^{k} T_i T_i^* = 1\,, \quad \|\lambda(U) - U\| < \delta\,, \quad \|\lambda(V) - V\| < \delta$

(ii) $[U] = [V], \quad F(\lambda, UV^*) = 0\,,$

(iii) $\mathrm{Sp}\, U$ and $\mathrm{Sp}\, V$ are δ−dense in $\mathbb{T}$, *where λ is the endomorphism of $\mathcal{O}_m \otimes \mathcal{O}_n$ defined by*

$$\lambda(x) = \sum T_i x T_i^* ,$$

then there exists a unitary W of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that

$$\|U - WVW^*\| < \epsilon, \qquad \|\lambda(W) - W\| < \epsilon .$$

Proof:

Suppose that $[U] = [V] = 0$. Since $\mathcal{O}_m \otimes \mathcal{O}_n$ is purely infinite, it follows by [Lin1] that there exists a unitary W of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $U \approx WVW^*$. Applying λ to this estimate and using $\lambda(U) \approx U$ and $\lambda(V) \approx V$ one obtains that $U \approx \lambda(W)V\lambda(W)^*$. Thus $W^*\lambda(W)$ nearly commutes with V and

$$\begin{aligned} F(\operatorname{Ad} W^*\lambda(W), V) &= F(\operatorname{Ad} W^* \circ \lambda \circ \operatorname{Ad} W, V) - F(\lambda, V) \\ &= F(\lambda, \operatorname{Ad} W(V)) - F(\lambda, V) \\ &= F(\lambda, U) - F(\lambda, V) = 0 . \end{aligned}$$

(since $(k-1)K_1(\mathcal{O}_m \otimes \mathcal{O}_n) = \{0\}$). Hence it also follows that

$$F(\operatorname{Ad} \lambda^l(W^*\lambda(W)), V) = 0$$

and that

$$F(\operatorname{Ad} W^*\lambda^l(W), V) = 0 ,$$

for $l = 1, 2, \ldots$, as far as this is well-defined. Since $[W^*\lambda^l(W)] = 0$, there is by [BEEK Theorem 8.1] a continuous path of unitaries from $W^*\lambda^l(W)$ to 1 in a set of elements almost commuting with V such that the length of the path is bounded by a universal constant. Hence by [Rørl] there is a unitary W_1 such that

$$W^*\lambda(W) \approx W_1\lambda(W_1^*) .$$

Since W_1 is constructed by using the above-mentioned paths and a Rohlin tower for λ which almost commutes with V, we can assume that W_1 almost commutes with V. Hence

$$WW_1V(WW_1)^* \approx WVW^* \approx U$$

and

$$\lambda(WW_1) \approx WW_1 .$$

This completes the proof when $[U] = [V] = 0$.

Since $\mathcal{O}_m$ (resp. $\mathcal{O}_n$) contains a central sequence of canonical sets of m (resp. n) isometries, $\mathcal{O}_m \otimes \mathcal{O}_n$ also contains a central sequence of canonical sets of k isometries. We choose k such isometries $S_1, \dots, S_k$ which nearly commute with T_i, U, V. By perturbing T_i, U, V slightly we assume that $E_j = S_j S_j^*$ exactly commutes with T_i, U, V. Then it follows that

$$\begin{aligned} F(\lambda, UV^*E_j + (1-E_j)) &= F(\lambda, S_jUV^*S_j^* + 1 - E_j) \\ &= F(\lambda, UV^*) \\ &= 0 \end{aligned}$$

since $S_jUV^*S_j^* + 1 - E_j$ is homotopic to UV^* by a path which is almost invariant under λ. Let

$$U_1 = UE_1 + V(1 - E_1).$$

Since $[V(1-E_1) + E_1] = (k-1)[V] = 0$, applying the first part to $\lambda|(1 - E_1)(\mathcal{O}_m \otimes \mathcal{O}_n)(1-E_1)$ and the unitaries $U(1-E_1)$, $V(1-E_1)$ of $(1-E_1)(\mathcal{O}_m \otimes \mathcal{O}_n)(1-E_1)$, we have that there is a unitary $W_1 \in (1-E_1)(\mathcal{O}_m \otimes \mathcal{O}_n)(1-E_1)$ such that $\lambda(W_1) \approx W_1$ and

$$U(1-E_1) \approx W_1 V(1-E_1) W_1^*.$$

Thus denoting $W_1 + E_1$ again by W_1 we obtain

$$U \approx W_1 U_1 W_1^*, \quad \lambda(W_1) \approx W_1.$$

Next apply the first part to $\lambda|(1-E_2)(\mathcal{O}_m \otimes \mathcal{O}_n)(1-E_2)$ and the unitaries $U_1(1-E_2)$, $V(1-E_2)$ of $(1-E_2)(\mathcal{O}_m \otimes \mathcal{O}_n)(1-E_2)$. Thus we obtain a unitary W_2 of $(1-E_2)(\mathcal{O}_m \otimes \mathcal{O}_n)(1-E_2)$ such that

$$U_1(1-E_2) \approx W_2 V(1-E_2) W_2^*, \quad \lambda(W_2) \approx W_2.$$

Denoting $W_2 + E_2$ again by W_2 we have

$$U \approx (W_1W_2)V(W_1W_2)^*, \quad \lambda(W_1W_2) \approx W_1W_2.$$

This completes the proof.

Lemma 7.2 *For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition : If unital homomorphisms ϕ and ψ of $\mathcal{O}_k * C(\mathbb{T})$ into $\mathcal{O}_m \otimes \mathcal{O}_n$ are of class δ and satisfy*

(i) $\mathrm{Sp}\,\phi(u)$ *and* $\mathrm{Sp}\,\psi(u)$ *are δ-dense in $\mathbb{T}$,*

(ii) $[\phi(u)] = [\psi(u)]$, *where u is the canonical unitary of $C(\mathbb{T})$,*

(iii) *There exists a central sequence $\{U_j\}$ of unitaries of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $[U_j] = [\phi(u)]$ and*

$$F(\lambda_\phi, \phi(u)U_j^*) = F(\lambda_\psi, \psi(u)U_j^*)$$

(iv) $\left[\sum_{i=1}^k \phi(s_i)\psi(s_i)^*\right] = 0$

then there exists a unitary U of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that

$$\begin{array}{ll} (a) & \|\phi(s_i) - \mathrm{Ad}\,U \circ \psi(s_i)\| < \epsilon\,, \quad i = 1, \dots, k \\ (b) & \|\phi(u) - \mathrm{Ad}\,U \circ \psi(u)\| < \epsilon\,. \end{array}$$

Proof:

¿From condition (iv) one finds that a unitary V of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that

$$\phi(s_i) \approx \mathrm{Ad}\,V \circ \psi(s_i)\,.$$

Since $\mathrm{Ad}\,V \circ \psi(u)U_j^* \approx \mathrm{Ad}\,V(\psi(u)U_j^*)$ for sufficiently large j, it follows that

$$\begin{aligned} F(\lambda_{\mathrm{Ad}\,V\circ\psi}, \mathrm{Ad}\,V \circ \psi(u)U_j^*) &= F(\mathrm{Ad}\,V \circ \lambda_\psi \circ \mathrm{Ad}\,V^*, \mathrm{Ad}\,V(\psi(u)U_j^*)) \\ &= F(\lambda_\psi, \psi(u)U_j^*)\,. \end{aligned}$$

Hence letting

$$\begin{aligned} \psi'(s_i) &= \phi(s_i) \\ \psi'(u) &= \mathrm{Ad}\,V \circ \psi(u)\,, \end{aligned}$$

it follows that $\psi' \approx \mathrm{Ad}\,V \circ \psi$ and

$$F(\lambda_\phi, \phi(u)\psi'(u)^*) = F(\lambda_\phi, \phi(u)U_j^*) - F(\lambda_\phi, \psi'(u)U_j^*) = 0\,.$$

By applying Lemma 7.1 we obtain a unitary U of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $\lambda_\phi(U) \approx U$ and $\mathrm{Ad}\,U \circ \psi'(u) \approx \phi(u)$. Hence we get the conclusion :

$$\phi \approx \mathrm{Ad}\,UV \circ \psi\,.$$

Lemma 7.3 *For any* $\delta > 0$ *and any* $\theta \in K_1(\mathcal{O}_m \otimes \mathcal{O}_n)$ *there exists a unital homomorphism* ψ *of* $\mathcal{O}_k * C(\mathbb{T})$ *into* $\mathcal{O}_m \otimes \mathcal{O}_n$ *of class* δ *such that*

$$[\psi(u)] = \theta, \quad \operatorname{Sp}\psi(u) = \mathbb{T}.$$

Proof: This follows from the following facts: There is a unital homomorphism of $\mathcal{O}_k$ into $\mathcal{O}_m \otimes \mathcal{O}_n$, and each equivalence class of unitaries of $\mathcal{O}_m \otimes \mathcal{O}_n$ contains a central sequence of unitaries.

Lemma 7.4 *Let* $\delta > 0$ *and* ψ *a unital homomorphism of* $\mathcal{O}_k * C(\mathbb{T})$ *into* $\mathcal{O}_m \otimes \mathcal{O}_n$ *of class* δ. *For any* $\theta_0 \in K_0(\mathcal{O}_m \otimes \mathcal{O}_n)$ *and* $\theta_1 \in K_1(\mathcal{O}_m \otimes \mathcal{O}_n)$ *there exists a unital homomorphism* ϕ *of* $\mathcal{O}_k * C(\mathbb{T})$ *into* $\mathcal{O}_m \otimes \mathcal{O}_n$ *such that*

(i) $\operatorname{Sp}\phi(u) = \mathbb{T}$, *where* u *is the canonical unitary of* $C(\mathbb{T})$,

(ii) $[\phi(u)] = [\psi(u)]$,

(iii) *There exists a central sequence* $\{U_j\}$ *of unitaries of* $\mathcal{O}_m \otimes \mathcal{O}_n$ *such that*

$$[U_j] = [\phi(u)]$$
$$F(\lambda_\phi, \phi(u)U_j^*) - F(\lambda_\psi, \psi(u)U_j^*) = \theta_0,$$

(iv) $[\sum \phi(s_i)\psi(s_i)^*] = \theta_1$.

Proof:

Let W be a unitary in $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $[W]_1 = \theta_1$ and W almost commutes with $\psi(s_i), \psi(u)$. Since $\mathcal{O}_m \otimes \mathcal{O}_n \simeq \mathcal{O}_m \otimes \mathcal{O}_n \otimes M_{(mn)^\infty}$, there are paths U_t, V_t of unitaries such that

$$U_0 = 1 = V_0, \quad [U_1, V_1] \approx 0$$

and U_t and V_t almost commute with $\psi(s_i)$, $\psi(u)$ and W. Define a unital homomorphism ϕ by

$$\phi(s_i) = V_1 W \psi(s_i)$$
$$\phi(u) = U_1 \psi(u).$$

Then $[\phi(u)] = [\psi(u)]$,

$$\begin{aligned} & F(\lambda_\phi, \phi(u)U_j^*) - F(\lambda_\psi, \psi(u)U_j^*) \\ = \; & F(\operatorname{Ad} V_1 W \circ \lambda_\psi, U_1\psi(u)U_j^*) - F(\lambda_\psi, \psi(u)U_j^*) \\ = \; & F(\operatorname{Ad} V_1 W, U_1) + F(\operatorname{Ad} V_1 W, \psi(u)U_j^*) \\ = \; & F(\operatorname{Ad} V_1, U_1) + F(\operatorname{Ad} W, \psi(u)U_j^*) \end{aligned}$$

where U_j is sufficiently central, and

$$\left[\sum \phi(s_i)\psi(s_i)^*\right] = [W].$$

Since ϕ can still be of class δ and condition (i) can be easily handled (in case $[\psi(u)] = 0$), this completes the proof by choosing appropriate U_1, V_1.

Let ϕ and ψ be unital homomorphisms of $\mathcal{O}_k \otimes C(\mathbb{T})$ into $\mathcal{O}_m \otimes \mathcal{O}_n$. When $[\phi(u)] = [\psi(u)]$, we define $\gamma(\phi, \psi)$ to be the equivalence class of the short exact sequence

$$0 \to K_0(\mathcal{O}_m \otimes \mathcal{O}_n) \to K_1(E) \to K_1(\mathcal{O}_k \otimes C(\mathbb{T})) \to 0$$

in $\operatorname{Ext}(K_1(\mathcal{O}_k \otimes C(\mathbb{T})), K_0(\mathcal{O}_m \otimes \mathcal{O}_n))$, where E is the C^*-subalgebra of $C([0,1]) \otimes (\mathcal{O}_m \otimes \mathcal{O}_n) \oplus \mathcal{O}_k \otimes C(\mathbb{T})$ defined by

$$\{(f, a) | f(0) = \phi(a), f(1) = \psi(a)\}.$$

We identify this extension group with $K_0(\mathcal{O}_m \otimes \mathcal{O}_n)$.

Lemma 7.5 *Let ϕ and ψ be unital homomorphisms of $\mathcal{O}_k \otimes C(\mathbb{T})$ into $\mathcal{O}_m \otimes \mathcal{O}_n$ such that*

$$[\phi(u)] = [\psi(u)], \quad [\sum \phi(s_i)\psi(s_i)^*] = 0.$$

Then there is a central sequence $\{U_j\}$ of unitaries of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $[U_j] = [\phi(u)]$, and

$$\gamma(\phi, \psi) = F(\lambda_\psi, \psi(u)U_j^*) - F(\lambda_\phi, \phi(u)U_j^*).$$

Proof:

The proof is about the same as that of Lemma 6.7 in the case $\mathcal{O}_n \otimes C(\mathbb{T}) \to M_k \otimes \mathcal{O}_m \otimes C(\mathbb{T})$. Adding the path V there the trivial path

$$\begin{aligned} 1 &\rightsquigarrow \lambda_\phi(U_j^*)U_j \\ &\rightsquigarrow \lambda_\psi(U_j^*)U_j \text{ (by the path } \operatorname{Ad} W_t \circ \lambda_\phi(U_j^*)U_j) \\ &\rightsquigarrow 1 \end{aligned}$$

all in a neighbourhood of 1, one sees (for a suitable choice of U_t) that $[V] = [U^{(1)}] - [U^{(2)}]$ where $U^{(1)}$ is the path of 1 to $\lambda_\phi(\phi(u)U_j^*)(\phi(u)U_j^*)^*$ in a neighbourhood of 1 and then to 1 as $\phi(u)$ tends to U_j, and $U^{(2)}$ is obtained similarly with ψ in place of ϕ. Thus we obtain the conclusion.

Theorem 7.6 *(Uniqueness) For any sufficiently small $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition : If unital homomorphisms ϕ and ψ of $\mathcal{O}_k \otimes C(\mathbb{T})$ into $\mathcal{O}_m \otimes \mathcal{O}_n$ satisfy that $\operatorname{Sp}\phi(u)$ and $\operatorname{Sp}\psi(u)$ are δ-dense in $\mathbb{T}$, then the following conditions are equivalent :*

(i) $KK(\phi) = KK(\psi)$

(ii) *There exists a unitary U of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that*

$$\begin{aligned} \|\phi(s_i) - \operatorname{Ad} U \circ \psi(s_i)\| &< \epsilon \\ \|\phi(u) - \operatorname{Ad} U \circ \psi(u)\| &< \epsilon \end{aligned}$$

(iii) $[\phi(u)] = [\psi(u)]$, *there exists a central sequence $\{U_j\}$ of unitaries of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $[U_j] = [\phi(u)]$ and*

$$F(\lambda_\phi, \phi(u)U_j^*) = F(\lambda_\psi, \psi(u)U_j^*)\,,$$

and $[\sum \phi(s_i)\psi(s_i)^*] = 0$.

Proof:

By Lemma 7.2 one has the implication (iii)⇒(ii).

Suppose (ii). Then for a sufficiently small $\epsilon > 0$ one has condition (iii) for ϕ and $\operatorname{Ad} U \circ \psi$ (in place of ψ), and then condition (iii) for ϕ and ψ.

Suppose (i). Then one has $[\phi(u)] = [\psi(u)]$. Denoting by ι the embedding of $\mathcal{O}_k$ into $\mathcal{O}_k \otimes C(\mathbb{T})$ one obtains $KK(\phi \circ \iota) = KK(\psi \circ \iota)$ which implies by [Rør1]

$$[\sum \phi(s_i)\psi(s_i^*)] = 0 \quad (= (k-1)K_1(\mathcal{O}_m \otimes \mathcal{O}_n))\,.$$

By Lemma 7.5 we obtain the other condition in (iii), concluding (i)⇒(iii).

Suppose (iii). Since $\phi_* = \psi_*$ on $K_*(\mathcal{O}_k \otimes C(\mathbb{T}))$, $KK(\phi) - KK(\psi)$ is represented as an element of

$$\mathrm{Ext}(K_1(\mathcal{O}_k \otimes C(\mathbb{T})), K_0(\mathcal{O}_m \otimes \mathcal{O}_n)) \oplus \mathrm{Ext}(K_0(\mathcal{O}_k \otimes C(\mathbb{T})), K_1(\mathcal{O}_m \otimes \mathcal{O}_n)).$$

By the reasoning given in the proof of (i)⇒(iii) we obtain that $KK(\phi) = KK(\psi)$.

Theorem 7.7 *(Existence) Let $A = \mathcal{O}_k \otimes C(\mathbb{T})$ and let $B = \mathcal{O}_m \otimes \mathcal{O}_n$, where m and n are even positive integers and $k-1$ is the greatest common divisor of $m-1$ and $n-1$. For any $g \in KK(A, B)$ which induces the map*

$$1 \in K_0(A) = \mathbb{Z}/(k-1)\mathbb{Z} \to 1 \in K_0(B) = \mathbb{Z}/(k-1)\mathbb{Z},$$

there is a unital homomorphism ϕ of A into B such that $KK(\phi) = g$ and $\mathrm{Sp}\,\phi(u) = \mathbb{T}$.

Proof: This follows from Lemmas 7.3 and 7.4 and the uniqueness theorem by using the intertwining arguments.

8. $\mathcal{O}_m \otimes \mathcal{O}_n \to \mathcal{O}_m \otimes \mathcal{O}_n$ or $\mathcal{O}_k \otimes C(\mathbb{T})$: Existence and uniqueness

Let m and n be even positive integers and let k be an (even) positive integer such that $k-1$ is the greatest common divisor of $m-1$ and $n-1$. Let $B = \mathcal{O}_k \otimes C(\mathbb{T})$ or $\mathcal{O}_m \otimes \mathcal{O}_n$; in either case $K_0(B) = \mathbb{Z}/(k-1)\mathbb{Z} = K_1(B)$ and the unit of B corresponds to $1 \in K_0(B)$. We say that a homomorphism ψ of $\mathcal{O}_m * \mathcal{O}_n$ into B is of class δ if

$$\max\{\|[\psi(s_i^1), \psi(s_j^2)]\|; i = 1, \ldots, m,\ j = 1, \ldots, n\} < \delta.$$

Lemma 8.1 *For any $\epsilon > 0$ there exists a $\delta > 0$ satisfying the following condition: If unital homomorphisms ϕ and ψ of $\mathcal{O}_m * \mathcal{O}_n$ into B is of class δ and satisfy*

(i) $\phi(s_i^1) = \psi(s_i^1)$, $i = 1, \dots, m$

(ii) $[W] = 0$ *in* $K_1(B)$

(iii) $F(\lambda_\phi^1, W) = 0$ *in* $K_0(B)$

where $W = \sum_{j=1}^n \phi(s_j^2)\psi(s_j^2)^*$ *and* λ_ϕ^1 *is the endomorphism of* B *defined by* $\lambda_\phi^1(x) = \sum_{i=1}^m \phi(s_i^1)x\phi(s_i^1)^*$, *then there exists a unitary* U *of* B *such that*

$$\|\phi(s_i^1) - \operatorname{Ad} U \circ \psi(s_i^1)\| < \epsilon, \quad \|\phi(s_j^2) - \operatorname{Ad} U \circ \psi(s_j^2)\| < \epsilon$$

for $i = 1, \dots, m$, $j = 1, \dots, n$.

Proof: ¿From (ii) and (iii), it follows by the following lemma 8.2 that there exists a continuous path $\{W_s; s \in [0,1]\}$ of unitaries of bounded length such that $W_0 = 1$, $W_1 = W$ and $\lambda_\phi^1(W_s) \approx W_s$. Then by a similar fact for $W\lambda_\phi^1(W)\dots\lambda_\phi^s(W)$ one obtains a unitary U of B such that $\lambda_\phi^1(U) \approx U$ and $\phi(s_j^2) \approx U\psi(s_j^2)U^*$, which implies that $\phi \approx \operatorname{Ad} U \circ \psi$.

Lemma 8.2 *For* $B = \mathcal{O}_k \otimes C(\mathbb{T})$ *or* $\mathcal{O}_m \otimes \mathcal{O}_n$, *there exists an* $L > 0$ *satisfying the following conditions: If* W *is a unitary of* B *and* $\{T_1, \dots, T_m\}$ *is a canonical set of isometries of* B *such that* $[W] = 0$, $\lambda(W) \approx W$ *and* $F(\lambda, W) = 0$ *where* $\lambda(x) = \sum T_i x T_i^*$, *then there is a continuous path* W_s *of unitaries of* B *such that* $W_0 = 1$, $W_1 = W$, $\lambda(W_s) \approx W_s$, *and length* $\{W_s\} \leq L$.

Proof:

Suppose that $B = \mathcal{O}_k \otimes C(\mathbb{T})$. Since $\mathcal{O}_k \cong \mathcal{O}_k \otimes M_{k^\infty}$, there is a sequence $\{{v_s}^l\}$ of paths of unitaries in $\mathcal{O}_k$ such that ${v_0}^l = 1$, $\operatorname{Sp} {v_1}^l = \mathbb{T}$, and $\{{v_s}^l\}$ are central uniformly in $s \in [0,1]$ as $l \to \infty$. By taking $V = {v_1}^l \otimes 1$ for sufficiently large l, one obtains an unitary V in $\mathcal{O}_k \otimes C(\mathbb{T})$ such that $[V] = 0$, $\lambda(V) \approx V$, $F(\lambda, V) = 0$, $\operatorname{Sp} V(t) = \mathbb{T}$, $\operatorname{Sp} V(t)W(t)$ is almost dense in $\mathbb{T}$ (for any $t \in \mathbb{T}$). Then there exists a unitary $v \in \mathcal{O}_k \otimes C(\mathbb{T})$ such that $vVv^* \approx VW$. Since $\lambda(V) \approx V$ and $\lambda(VW) \approx VW$, one obtains that $\lambda(v)V\lambda(v)^* \approx vVv^*$. Thus $\operatorname{Ad} v^*\lambda(v)(V) \approx V$ and

$$\begin{aligned}
F(\operatorname{Ad} v^*\lambda(v), V) &= F(\operatorname{Ad} v^*\lambda(v), V) + F(\lambda, V) \\
&= F(\operatorname{Ad} v^* \circ \lambda \circ \operatorname{Ad} v, V) \\
&= F(\lambda, \operatorname{Ad} v(V)) \\
&= F(\lambda, V) + F(\lambda, W) = 0\,.
\end{aligned}$$

Hence there is a continuous path of unitaries of bounded length from $v^*\lambda(v)$ to 1 in a set of unitaries which almost commute with V. Then one obtains a unitary w by [Rør1] such that $[w, V] \approx 0$ and $v^*\lambda(v) \approx w\lambda(w^*)$. Thus $\lambda(vw) \approx vw$ and $vwV(vw)^* \approx vVv^* \approx VW$. Taking vw instead of v one can assume that $\lambda(v) \approx v$ besides $vVv^* = VW$.

By using a set $\{e_{ij}\}$ of matrix units of order $k^l \geq 3$ which almost commute with v, V, W, T_i one finds a continuous path in a set of elements which are close to unitaries :

$$\begin{aligned} VWe_{11} + 1 - e_{11} &\rightsquigarrow VWe_{11} + ve_{22} + v^*e_{33} + (1 - e_{11} - e_{22} - e_{33}) \\ &\rightsquigarrow v^*VWe_{11} + ve_{22} + (1 - e_{11} - e_{22}) \\ &\rightsquigarrow v^*VWve_{11} + 1 - e_{11} \\ &\rightsquigarrow Ve_{11} + 1 - e_{11} \end{aligned}$$

where we use only appropriate rotations and a straight line. Thus there is a continuous path of length about $3\pi/2$ from $We_{11}+1-e_{11}$ to $1 = e_{11}+1-e_{11}$, and so there is a continuous path of unitaries of length about $(3\pi/2)\cdot k^l$ from W to 1. From the construction, the path almost commutes with T_i, or almost invariant under λ.

Suppose that $B = \mathcal{O}_m \otimes \mathcal{O}_n$. We only have to check the following :

(1) There exists a unitary V of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $[V] = 0$, $\lambda(V) \approx V$, $F(\lambda, V) = 0$, $\mathrm{Sp}\, V \approx \mathbb{T}$, $\mathrm{Sp}\, VW \approx \mathbb{T}$.

(2) With V above there exists a unitary $v \in \mathcal{O}_m \otimes \mathcal{O}_n$ such that $vVv^* \approx VW$.

(3) $\mathcal{O}_m \otimes \mathcal{O}_n$ has a central sequence of matrix algebras of order at least three.

One can show (1) and (3) as before since $\mathcal{O}_m \otimes \mathcal{O}_n \cong \mathcal{O}_m \otimes \mathcal{O}_n \otimes M_{(mn)^\infty}$. (2) follows from Lin's result [Lin1], since $\mathcal{O}_m \otimes \mathcal{O}_n$ is purely infinite.

Lemma 8.3 *For any $\epsilon > 0$ there exists $\delta > 0$ satisfying the following conditions : If unital homomorphisms ϕ and ψ of $\mathcal{O}_m * \mathcal{O}_n$ into B are of class δ and satisfy :*

(i) $[W_1] \in (k-1)K_1(B) = 0$ *where* $W_1 = \sum_{i=1}^m \phi(s_i^1)\psi(s_i^1)^*$.

(ii) $[W_2] = 0$ *where* $W_2 = \sum_{j=1}^{n} \phi(s_j^2)\psi(s_j^2)^*$.

(iii) *there is a central sequence* $\{T_j{}^l\}$ *of canonical sets of* n *isometries in* B *such that*

$$\left[\sum_{j=1}^{n} \phi(s_j^2) T_j{}^{l*}\right] = 0$$

$$F(\lambda_\phi^1, \sum \phi(s_j^2) T_j{}^{l*}) = F(\lambda_\psi^1, \sum \psi(s_j^2) T_j{}^{l*})$$

then there is a unitary U *of* B *such that*

$$\left\| \phi(s_i^1) - \operatorname{Ad} U \circ \psi(s_i^1) \right\| < \epsilon, \quad \left\| \phi(s_j^2) - \operatorname{Ad} U \circ \psi(s_j^2) \right\| < \epsilon$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Proof: By (i) there is a unitary V of B such that

$$\|\phi(s_i^1) - \operatorname{Ad} V \circ \psi(s_i^1)\| < \delta.$$

Define ψ_1 by

$$\psi_1(s_i^1) = \phi(s_i^1)$$

$$\psi_1(s_j^2) = V\psi(s_j^2)V^*.$$

Then ψ_1 is a homomorphism of class 2δ and satisfies $\phi(s_i^1) = \psi_1(s_i^1)$,

$$\begin{aligned}
& \left[\sum \phi(s_j^2)\psi_1(s_j^2)^*\right] \\
= & \left[\left(\sum \phi(s_j^2) T_j{}^{l*}\right)\left(\sum T_j{}^l V T_j{}^{l*}\right)\left(\sum T_j{}^l \psi(s_j^2)^*\right) \cdot V^*\right] \\
= & \left[\sum \phi(s_j^2) T_j{}^{l*}\right] + \left[\sum T_j{}^l \psi(s_j^2)^*\right] + (n-1)[V] \\
= & \left[\sum \phi(s_j^2)\psi(s_j^2)^*\right] = 0
\end{aligned}$$

and

$$\begin{aligned}
& F\left(\lambda_\phi^1, \sum \phi(s_j^2)\psi_1(s_j^2)^*\right) \\
= & F\left(\lambda_\phi^1, \sum \phi(s_j^2) T_j{}^{l*}\right) + F\left(\lambda_\phi^1, \left(\sum T_j{}^l V T_j{}^{l*}\right)\left(\sum T_j{}^l \psi(s_j^2)^*\right) V^*\right) \\
= & F\left(\lambda_\phi^1, \sum \phi(s_j^2) T_j{}^{l*}\right) \\
+ & F\left(\operatorname{Ad} V^* \circ \lambda_\phi^1 \circ \operatorname{Ad} V, V^*\left(\sum T_j{}^l V T_j{}^{l*}\right)\left(\sum T_j{}^l \psi(s_j^2)^*\right)\right) \\
= & F\left(\lambda_\phi^1, \sum \phi(s_j^2) T_j{}^{l*}\right) + F\left(\operatorname{Ad} V^* \circ \lambda_\phi^1 \circ \operatorname{Ad} V, \sum T_j{}^l \psi(s_j^2)^*\right) \\
= & F\left(\lambda_\phi^1, \sum \phi(s_j^2) T_j{}^{l*}\right) + F\left(\lambda_\psi^1, \sum T_j{}^l \psi(s_j^2)^*\right) = 0
\end{aligned}$$

for sufficiently large l, where we have used

$$V^*\left(\sum T_j{}^l V T_j{}^{l*}\right) \approx 1\,, \qquad \operatorname{Ad} V^* \circ \lambda^1_\phi \circ \operatorname{Ad} V \approx \lambda^1_\psi\,.$$

Applying Lemma 8.1 we obtain a unitary U of B such that $\phi \approx \operatorname{Ad} U \circ \psi_1$. Since $\psi_1 \approx \operatorname{Ad} V \circ \psi$, this concludes the proof.

Lemma 8.4 *There exists a unital homomorphism ψ of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B.*

Proof:

If $B = \mathcal{O}_m \otimes \mathcal{O}_n$, this is obvious. Suppose $B = \mathcal{O}_k \otimes C(\mathbb{T})$. For any $\delta > 0$ one can show that there is a unital homomorphism of $\mathcal{O}_m * \mathcal{O}_n$ into $\mathcal{O}_k \otimes C(\mathbb{T})$ of class δ because $\mathcal{O}_k$ has a central sequence of subalgebras which are isomorphic to itself. Given ϕ and ψ of $\mathcal{O}_m * \mathcal{O}_n$ into $\mathcal{O}_k \otimes C(\mathbb{T})$ of class δ, by replacing ϕ by ϕ_1 defined by

$$\begin{aligned}\phi_1(s^1_i) &= u^{\alpha_1}\phi(s^1_i)\\ \phi_1(s^2_j) &= u^{\alpha_2}\phi(s^2_j)\end{aligned}$$

with u the canonical unitary of $1 \otimes C(\mathbb{T}) \subset \mathcal{O}_k \otimes C(\mathbb{T})$, one can assume that the conditions (i) and (ii) in Lemma 8.3 are satisfied. There are unitaries $v, w \in 1 \otimes M_{k^\infty} \otimes 1 \subset \mathcal{O}_k \otimes M_{k^\infty} \otimes C(\mathbb{T}) \simeq \mathcal{O}_k \otimes C(\mathbb{T})$ such that $\|[v, w]\| \approx 0$ and v and w almost commute with $\phi_1(s^1_i)$ and $\phi_1(s^2_j)$. Replacing ϕ_1 by ϕ_2 defined by

$$\begin{aligned}\phi_2(s^1_i) &= v\phi_1(s^1_i)\\ \phi_2(s^2_j) &= w\phi_1(s^2_j)\end{aligned}$$

with suitable v, w, one can also assume the condition (iii) is satisfied. Since $[v] = [w] = 0$, conditions (i) and (ii) are still satisfied as well as (iii). By this change of ϕ, one can still assume that the new ϕ is of class δ. Thus for each $\epsilon_l = 2^{-l}$, choose $\delta_l > 0$ as in Lemma 8.3 and construct a unital homomorphism ψ_l of $\mathcal{O}_m * \mathcal{O}_n$ into $\mathcal{O}_k \otimes C(\mathbb{T})$ of class δ_l inductively such that the pair ψ_{l-1}, ψ_l satisfies the conditions in Lemma 8.3. Hence we assume that $\|\psi_{l-1}(s^1_i) - \psi_l(s^1_i)\| < \epsilon_{l-1}$ and $\left\|\psi_{l-1}(s^2_j) - \psi_l(s^2_j)\right\| < \epsilon_{l-1}$. Thus the limit of ψ_l exists and defines a unital homomorphism of $\mathcal{O}_m \otimes \mathcal{O}_n$ into $\mathcal{O} \otimes C(\mathbb{T})$.

Lemma 8.5 *Let ψ be a unital homomorphism of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B and let $\theta_1, \theta_2 \in K_1(B)$. Then there is a unital homomorphism ϕ of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B such that*

$$[W_1] = \theta_1\,, \quad [W_2] = \theta_2$$

where $W_1 = \sum \phi(s_i^1)\psi(s_i^1)^*$ and $W_2 = \sum \phi(s_j^2)\psi(s_j^2)^*$.

Proof:

Suppose that $B = \mathcal{O}_k \otimes C(\mathbb{T})$. Then one can define ϕ by

$$\phi(s_i^1) = u^{\theta_1}\psi(s_i^1)\,,$$
$$\phi(s_j^2) = u^{\theta_2}\psi(s_j^2)\,,$$

where u is the canonical unitary of $1 \otimes C(\mathbb{T}) \subset \mathcal{O}_k \otimes C(\mathbb{T})$.

Suppose that $B = \mathcal{O}_m \otimes \mathcal{O}_n$. Since $\mathcal{O}_m \otimes \mathcal{O}_n$ contains a central sequence of subalgebras which are isomorphic to itself, it contains a central sequence $\{u_l\}$ of unitaries of B which each generate $K_1(B)$. We define ϕ_l by

$$\phi_l(s_i^1) = u_l^{\theta_1}\psi(s_i^1)$$
$$\phi_l(s_j^2) = u_l^{\theta_2}\psi(s_j^2)$$

as a homomorphism of $\mathcal{O}_m * \mathcal{O}_n$ into $\mathcal{O}_m \otimes \mathcal{O}_n$. Then applying Lemma 8.3 to a subsequence of $\{\phi_l\}$ inductively, and taking the limit one obtains a homomorphism with the desired properties.

Lemma 8.6 *Let ψ be a unital homomorphism of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B and let $\theta_0 \in K_0(B)$. Then there is a unital homomorphism ϕ of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B such that*

$$[W_1] = 0\,, \quad [W_2] = 0\,.$$

where $W_1 = \sum \phi(s_i^1)\psi(s_i^1)^*$ and $W_2 = \sum \phi(s_j^2)\psi(s_j^2)^*$ and such that there is a central sequence $\{T_j{}^l\}$ of canonical sets of n isometries in B with

$$\left[\sum_j \phi(s_j^2)T_j{}^{l*}\right] = 0\,,$$
$$F\left(\lambda_\phi^1, \sum \phi(s_j^2)T_j{}^{l*}\right) = F\left(\lambda_\psi^1, \sum \psi(s_j^2)T_j{}^{l*}\right) + \theta_0\,.$$

Proof: Similar to the proof of Lemma 8.5. We use that $\mathcal{O}_k \cong \mathcal{O}_k \otimes M_{k^\infty}$ and $\mathcal{O}_m \otimes \mathcal{O}_n \cong \mathcal{O}_m \otimes \mathcal{O}_n \otimes M_{(mn)^\infty}$.

Lemma 8.7 *There exists a $\delta > 0$ such that if unital homomorphisms ϕ and ψ of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B satisfy*

$$\left\|\phi(s_i^1) - \psi(s_i^1)\right\| < \delta\,, \quad \left\|\phi(s_j^2) - \psi(s_j^2)\right\| < \delta$$

then $KK(\phi) = KK(\psi)$.

Proof:

Let U be a unitary of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $[U]$ generates $K_1(\mathcal{O}_m \otimes \mathcal{O}_n)$. We choose $\delta > 0$ so that $\|\phi(U) - \psi(U)\| < 1$ follows. Thus we assume $\phi_* = \psi_*$ as a map of $K_*(\mathcal{O}_m \otimes \mathcal{O}_n)$ into $K_*(B)$.

Define

$$E = \{(f, a) : f \in C[0,1] \otimes B, a \in \mathcal{O}_m \otimes \mathcal{O}_n, f(0) = \phi(a), f(1) = \psi(a)\}\ .$$

We have to show that the following two exact sequences split :

$$0 \to K_0(SB) \to K_0(E) \xrightarrow{q} K_0(\mathcal{O}_m \otimes \mathcal{O}_n) \to 0 \qquad (8.1)$$

$$0 \to K_1(SB) \to K_1(E) \xrightarrow{q} K_1(\mathcal{O}_m \otimes \mathcal{O}_n) \to 0\,. \qquad (8.2)$$

Sequence (8.1): We have to show that there is a $g \in K_0(E)$ such that $q(g) = 1$, and $(k-1)g = 0$ or $(m-1)g = 0 = (n-1)g$.

Define e_i^1 to be the projection of E obtained from

$$(1-t)\phi(s_i^1 s_i^{1*}) + t\psi(s_i^1 s_i^{1*})\,, \quad t \in [0,1]$$

by functional calculus, and let e_j^2 be the projection similarly defined from

$$(1-t)\phi(s_j^2 s_j^{2*}) + t\psi(s_j^2 s_j^{2*})\,.$$

This is possible if we assume, say, $\delta < 1/2(m+n)$. One sees that these projections are equivalent to 1 and we take these to be the g above. Then it easily follows that

$$mg = \sum_{i=1}^{m} [e_i^1] = g$$

$$ng = \sum_{j=1}^{n} [e_j^2] = g$$

if we take δ to be so small that $\{e_i^1\}$ (resp. $\{e_j^2\}$) are almost mutually orthogonal. This completes the proof.
Sequence (8.2): Let U be the unitary of $\mathcal{O}_m \otimes \mathcal{O}_n$ as above. Let $\hat{U}$ be the unitary of E obtained from

$$(1-t)\phi(U) + t\psi(U), \quad t \in [0,1]$$

by functional calculus. Then we have to show that

$$(m-1)[\hat{U}] = 0 = (n-1)[\hat{U}].$$

Let V_t be a continuous path of unitaries in B such that $V_0 = 1$, $V_1 = \lambda^1(U)U^*$. Then $\hat{U}^{n-1}$ is equivalent to the path in B defined by

$$\begin{aligned}
1 \overset{\phi(V_t)}{\rightsquigarrow}\ & \phi(\lambda^1(U)U^*) = \lambda_\phi^1(\phi(U))\phi(U)^* \\
\rightsquigarrow\ & \phi(U)^{n-1} \quad \text{(by the path } W_t \text{ depending on } \phi(s_i^1), \phi(U)) \\
\overset{\hat{U}(t)^{n-1}}{\rightsquigarrow}\ & \psi(U)^{n-1} \\
\rightsquigarrow\ & \lambda_\psi^1(\psi(U))\psi(U)^* \quad \text{(by } W_{1-t} \text{ depending on } \psi(s_i^1), \psi(U)) \\
\overset{\psi(V_{1-t})}{\rightsquigarrow}\ & 1.
\end{aligned}$$

Let $W = \sum \psi(s_i^1)\phi(s_i^1)^*$ and let $\hat{W}_t$ be the path of unitaries in B obtained from $(1-t)1 + tW$. (Note that $W \approx 1$.) Let

$$F(s,t) = W_t(\hat{W}_s\phi(s_i^1); \hat{U}(s))$$

where W_t is the function used in the above path. Since $F(s,t)$ is continuous, one sees that the above path is equivalent to

$$1 \rightsquigarrow \lambda_\phi^1(\phi(U))\phi(U)^* \overset{\lambda_t(\hat{U}(t))\hat{U}(t)^*}{\rightsquigarrow} \lambda_\psi^1(\psi(U))\psi(U)^* \rightsquigarrow 1$$

where $\lambda_t = \mathrm{Ad}\,\hat{W}_t \circ \lambda_\phi^1$. Since $\hat{W}_t \approx 1$, the middle part of the path is almost constant. Hence one can easily check that the path is trivial in the unitaries of E. This shows that $(n-1)[\hat{U}] = 0$. Similarly $(m-1)[\hat{U}] = 0$. This completes the proof.

Theorem 8.8 *(Uniqueness) For any unital homomorphisms ϕ and ψ of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B, the following conditions are equivalent :*

(i) $KK(\phi) = KK(\psi)$.

(ii) ϕ *and* ψ *are approximately unitarily equivalent.*

(iii) $[W_1] = [W_2] = 0$, *where* $W_\sigma = \sum \phi(s_i^\sigma)\psi(s_i^\sigma)^*$ *for* $\sigma = 1, 2$ *and there is a central sequence* $\{T_j{}^l\}$ *of canonical sets of* n *isometries such that* $[\sum \phi(s_j^2)T_j{}^{l*}] = 0$ *and*

$$F(\lambda_\phi^1, \sum \phi(s_j^2)T_j{}^{l*}) = F(\lambda_\psi^1, \sum \psi(s_j^2)T_j{}^{l*})\,.$$

Proof: (iii)⇒(ii). This follows from Lemma 8.3.
(ii)⇒(i). If ϕ is close to $\mathrm{Ad}\, U \circ \psi$ for some unitary U of B, we may suppose that $[U] = 0$ since $B = \mathcal{O}_k \otimes C(\mathbb{T})$ or $\mathcal{O}_m \otimes \mathcal{O}_n$. Thus $\mathrm{Ad}\, U \circ \psi$ is homotopic to ψ and so $KK(\mathrm{Ad}\, U \circ \psi) = KK(\psi)$. Thus we may suppose that ϕ and ψ are close to each other. We can then apply Lemma 8.7.
(ii)⇒(iii). Suppose that $\phi \approx \mathrm{Ad}\, U \circ \psi$ with some unitary U of B. Then

$$1 \approx \sum \phi(s_i^\sigma)U\psi(s_i^\sigma)^*U^* = W_\sigma \sum \psi(s_i^\sigma)U\psi(s_i^\sigma)^*U^* = W_\sigma \lambda_\psi^\sigma(U)U^*\,.$$

Hence

$$[W_1] = -[\lambda_\psi^1(U)U^*] = -(m-1)[U] = 0\,,$$
$$[W_2] = -[\lambda_\psi^2(U)U^*] = -(n-1)[U] = 0\,.$$

If $\{T_j{}^*\}$ is a central sequence of canonical sets of n isometries of B with $\left[\sum \phi(s_j^2)T_j{}^{l*}\right] = 0$, one has for sufficiently large l,

$$\begin{aligned} F\left(\lambda_\phi^1, \sum \phi(s_j^2)T_j{}^{l*}\right) &= F\left(\mathrm{Ad}\, U \circ \lambda_\psi^1 \circ \mathrm{Ad}\, U^*, \sum U\psi(s_j^2)U^*T_j{}^{l*}\right) \\ &= F(\left(\mathrm{Ad}\, U \circ \lambda_\psi^1 \circ \mathrm{Ad}\, U^*, \mathrm{Ad}\, U \left(\sum \psi(s_j^2)T_j{}^{l*}\right)\right) \\ &= F\left(\lambda_\psi^1, \sum \psi(s_j^2)T_j{}^{l*}\right)\,, \end{aligned}$$

which concludes the proof.
(i)⇒(ii) or (iii). Let ι_1 (resp. ι_2) be the natural embedding of $\mathcal{O}_m$ (resp. $\mathcal{O}_n$) into $\mathcal{O}_m \otimes \mathcal{O}_n$. Since $KK(\phi) = KK(\psi)$, one has

$$\begin{aligned} KK(\phi \circ \iota_1) &= KK(\psi \circ \iota_1)\,, \\ KK(\phi \circ \iota_2) &= KK(\psi \circ \iota_2) \end{aligned}$$

which imply $[W_1] = 0$ and $[W_2] = 0$ respectively (See [Rør1]).

Define a C^*-subalgebra E of $C[0,1] \otimes B \oplus \mathcal{O}_m \otimes \mathcal{O}_n$ by

$$\{(f,a) | \, f(0) = \phi(a), f(1) = \psi(a)\} \, .$$

Since $\phi_* = \psi_*$ on $K_*(\mathcal{O}_m \otimes \mathcal{O}_n)$, one obtains an exact sequence :

$$0 \to K_1(SB) \to K_1(E) \to K_1(\mathcal{O}_m \otimes \mathcal{O}_n) \to 0$$

from the exact sequence $0 \to SB \to E \to \mathcal{O}_m \otimes \mathcal{O}_n \to 0$. Since $KK(\phi) = KK(\psi)$, this sequence splits. Since $K_1(SB) = \mathbb{Z}/(k-1)\mathbb{Z} = K_1(\mathcal{O}_m \otimes \mathcal{O}_n)$, this implies that any $g \in K_1(E)$ satisfies $(k-1)g = 0$.

Let u be a unitary of $\mathcal{O}_m \otimes \mathcal{O}_n$ such that $\lambda^1(u) \approx u$. Since $[\phi(u)] = [\psi(u)]$, there is a unitary $\hat{u} \in E$ such that

$$\hat{u}(0) = \phi(u) \, , \quad \hat{u}(1) = \psi(u) \, .$$

We now want to compute $\hat{u}^{n-1}$ as in the proof of Lemma 8.7. It follows that $\hat{u}^{n-1}$ is equivalent to the path of unitaries in B defined by

$$\begin{array}{lll} 1 & \rightsquigarrow & \phi(\lambda^1(u)u^*) = \lambda^1_\phi(\phi(u))\phi(u)^* \text{ (in a neighbourhood of 1)} \\ & \overset{\lambda_t(\hat{u}(t))\hat{u}(t)^*}{\rightsquigarrow} & \lambda^1_\psi(\psi(u))\psi(u)^* \\ & \rightsquigarrow & 1 \quad \text{(in a neighbourhood of 1)} \end{array}$$

where $\lambda_t = \operatorname{Ad} \hat{W}_t \circ \lambda^1_\phi$ and $\hat{W}_t$ is a path of unitaries such that $\hat{W}_0 = 1$ and

$$\hat{W}_1 = \sum_{i=1}^{m} \psi(s_i^1)\phi(s_i^1)^* = W_1^* \, .$$

Let U be a unitary of B such that $[U] = [\phi(u)]$ and $\lambda_t(U) \approx U$. (If $B = \mathcal{O}_k \otimes C(\mathbb{T})$, U can be taken from the center and if $B = \mathcal{O}_m \otimes \mathcal{O}_n$, U can be taken from some central sequence of unitaries.) By adding the trivial path

$$1 \rightsquigarrow \lambda_0(U^*)U \overset{\lambda_t(U^*)U}{\rightsquigarrow} \lambda_1(U^*)U \rightsquigarrow 1 \, ,$$

one obtains that $\hat{u}^{n-1}$ is equivalent to

$$1 \rightsquigarrow \lambda_0(\phi(u)U^*)(\phi(u)U^*)^* \rightsquigarrow \lambda_1(\psi(u)U^*)(\psi(u)U^*)^* \rightsquigarrow 1 \, .$$

Suppose that

$$\hat{W}_t = \begin{cases} 1 & 0 \le t \le 1/3 \\ W_1^* & 2/3 \le t \le 1 \end{cases}$$
$$\hat{u}(t) = U \quad 1/3 \le t \le 2/3\,.$$

Then $[\hat{u}^{n-1}]$ is the difference of the equivalence classes of

$$1 \rightsquigarrow \lambda_\phi^1(\phi(u)U^*)(\phi(u)U^*)^* \rightsquigarrow 1 \tag{8.3}$$
$$1 \rightsquigarrow \lambda_\psi^1(\psi(u)U^*)(\psi(u)U^*)^* \rightsquigarrow 1\,. \tag{8.4}$$

By perturbing $\phi(u)U^*$ slightly, one assumes that $\phi(u)U^*$ is, for sufficiently large M,

$$U_0 = \sum_{j=0}^{M-1} e^{2\pi ij/M} p_j$$

where $\{p_j\}$ are mutually orthogonal projections with $[p_j] = 1$. Then we choose, for the path from U_0 to 1,

$$U_t = \sum_{j=0}^{M-1} e^{2\pi ij(1-t)/M} p_j\,.$$

Let $\{T_i\}$ be a canonical set of m isometries in the relative commutant of $\{p_j\}$ and let

$$V = \sum_{i=1}^{m} \phi(s_i^1) T_i{}^*\,.$$

Since $\lambda_\phi^1(U_t) = VU_tV^*$, the path (8.3) is equivalent to

$$1 \rightsquigarrow VU_0V^*U_0^* \overset{VU_tV^*U_t^*}{\rightsquigarrow} 1\,.$$

If $F(\lambda_\phi^1, U_0) = \theta \neq 0$, let q_j be a subprojection of p_j such that $[q_j] = \theta$, and define V_0 to be a unitary such that

$$V_0\,(1 - \sum q_j) = 1 - \sum q_j$$
$$V_0 q_j = q_{j+1} V_0$$

with $q_M = q_0$. Then $F(\mathrm{Ad}\, V_0, U_0) = \theta$, $F(\mathrm{Ad}\, V, U_0) = F(\lambda^1_\phi, U_0) = \theta$, and V_0 can be connected to V in a set of elements which nearly commute with U_0. Thus the above path is equivalent to

$$1 \rightsquigarrow V_0 U_0 V_0^* U_0^* = e^{-2\pi i/M} \left(\sum_{j=0}^{M} q_j \right) + 1 - \sum q_j \overset{V_0 U_t V_0^* U_t^*}{\rightsquigarrow} 1 .$$

Note that this is essentially a path in the matrix algebra M_M. By computing the winding number we conclude that the equivalence class is

$$-\theta \in K_0(B) \cong K_1(SB) \subset K_1(E) .$$

By computing the path (8.4) similarly, one obtains that

$$F(\lambda^1_\phi, \phi(u)U^*) = F(\lambda^1_\psi, \psi(u)U^*) \tag{8.5}$$

where U is a unitary of class $[\phi(u)]$ such that U is sufficiently central.

Suppose that $m = n = k$. Using the fact $[W_1] = 0$ we can assume that $\phi(s^1_i) \approx \psi(s^1_i)$. Take

$$u = \sum_{j=1}^{n} s^2_j s^{1*}_j w^*$$

where w is a unitary of $\mathcal{O}_m$ such that $\{w s^1_j\}$ nearly commute with s^1_j in $\mathcal{O}_m$. Since w can be chosen first, we can assume that $\phi(w) \approx \psi(w)$ too. From equation (8.5) it follows that

$$F(\lambda^1_\phi, \phi(u)\psi(u)^*) = 0$$

i.e.,

$$F\left(\lambda^1_\phi, \sum_{j=1}^{n} \phi(s^2_j)\psi(s^2_j)^* \right) = 0$$

since $\phi(w s^1_j) \approx \psi(w s^1_j)$. ¿From this one obtains the last condition of (iii).

In general let a and b be positive integers such that

$$m - 1 = a(k-1) , \quad n - 1 = b(k-1) .$$

Note that there are unital homomorphisms α of $\mathcal{O}_n$ into $M_a \otimes \mathcal{O}_m$ and β of $\mathcal{O}_m$ into $M_b \otimes \mathcal{O}_n$ since $(n-1)a = (m-1)b$.

Let us consider the unital homomorphisms

$$\phi' = i \otimes \phi : M_a \otimes \mathcal{O}_m \otimes \mathcal{O}_n \to M_a \otimes B$$
$$\psi' = i \otimes \psi : M_a \otimes \mathcal{O}_m \otimes \mathcal{O}_n \to M_a \otimes B$$

where i is the identity map of M_a. Since $KK(\phi) = KK(\psi)$, one obtains that $KK(\phi') = KK(\psi')$. From the previous arguments one obtains that for any unitary u of $M_a \otimes \mathcal{O}_m \otimes \mathcal{O}_n$ such that $\lambda^1(u) \approx u$,

$$F(\lambda^1_{\phi'}, \phi'(u)U^*) = F(\lambda^1_{\psi'}, \psi'(u)U^*)$$

where U is a sufficiently central unitary of $M_a \otimes B$ with $[U] = [\phi'(u)]$. By assuming $\{\alpha(s^2_j)\}$ nearly commutes with $\{s^1_i\}$ (which is possible because $\mathcal{O}_m$ contains a central sequence of subalgebras which is isomorphic to $\mathcal{O}_m$), we take

$$u = \sum_{j=1}^{n} \alpha(s^2_j)^* \otimes s^2_j \,.$$

Thus assuming $\phi(s^1_i) \approx \psi(s^1_i)$ one obtains that

$$F\left(\lambda^1_{\phi'}, \sum_{j=1}^{n} \phi'(s^2_j)\psi'(s^2_j)^*\right) = 0$$

which, as before, implies that the original ϕ, ψ as maps of $\mathcal{O}_m \otimes \mathcal{O}_n$ into $M_a \otimes B$ are approximately unitarily equivalent. Similarly one obtains that $\phi, \psi : \mathcal{O}_m \otimes \mathcal{O}_n \to M_b \otimes B$ are approximately unitarily equivalent.

Since a and b are relatively prime with each other, there are positive integers a', b' and l such that $aa' + bb' = k^l$. By regarding $M_{a'} \otimes M_a \oplus M_{b'} \otimes M_b$ as a subalgebra of M_{k^l} one concludes that

$$\phi, \psi : \mathcal{O}_m \otimes \mathcal{O}_n \to 1 \otimes B \subset M_{k^l} \otimes B$$

are approximately unitarily equivalent, That is, there exists a unitary U of $M_{k^l} \otimes B$ such that $\phi \approx \mathrm{Ad}\, U \circ \psi$. Whether $B = \mathcal{O}_k \otimes C(\mathbb{T})$ or $B = \mathcal{O}_m \otimes \mathcal{O}_n$, B contains a central sequence of subalgebras which are isomorphic to M_{k^l}. Then one takes a subalgebra C of B such that $C \cong M_{k^l}$ and $\phi(s^\sigma_i), \psi(s^\sigma_i)$, and U almost commute with any element of C. Then we take a unitary V of $M_{k^l} \otimes C$ such that $V(M_{k^l} \otimes 1)V^* = 1 \otimes C$. Then one obtains

$$\phi \approx \mathrm{Ad}\, V \circ \phi \approx \mathrm{Ad}\, VU \circ \psi \approx \mathrm{Ad}\, VUV^* \circ \psi \,.$$

Since VUV^* nearly commutes with $M_{k^l} \otimes 1$, one obtains a unitary U_0 of B such that $\phi \approx \mathrm{Ad}\, U_0 \circ \psi$, which concludes the proof.

Theorem 8.9 *(Existence) For any $g \in KK(\mathcal{O}_m \otimes \mathcal{O}_n, B)$ which maps $1 \in K_0(\mathcal{O}_m \otimes \mathcal{O}_n) = \mathbb{Z}/(k-1)\mathbb{Z}$ to $1 \in K_0(B) = \mathbb{Z}/(k-1)\mathbb{Z}$, there is a unital homomorphism ψ of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B such that $KK(\psi) = g$.*

Proof: ¿From the Universal Coefficient Theorem [RS] the cardinality of the set of such $g \in KK(\mathcal{O}_m \otimes \mathcal{O}_n, B)$ is $(k-1)^3$. By Lemmas 8.4–6 and the uniqueness theorem there exist exactly $(k-1)^3$ unital homomorphisms of $\mathcal{O}_m \otimes \mathcal{O}_n$ into B up to approximate unitary equivalence. This concludes the proof.

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