A Generalized Burnside theorem

by
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Introduction

In the paper [La] I have introduced a non-commutative deformation functor Def for a family $V = \{V_i\}_{i=1}^r$ of right A-modules, where A is a k-algebra, k a field. When $\dim_k \mathrm{Ext}_A^1(V_i, V_j) < \infty$ for all $i, j = 1, \ldots, r$, there is a hull – or a formal moduli H for V in the procategory of the category a_r of r-prointed finite dimensional k-algebras, on which this deformation functor is defined.

For $r = 1$ this is an obvious generalization, to the non commutative case, of the classical deformation theory of a module, see e.g. [La3]. It turns out, however, that for $r \geq 2$, the theory contains more sophisticated ingredients. First of all the category of extensions of extensions of the V_i's is "represented" by the category of finite representations of H.

Secondly there is a corresponding versal family of $H \otimes A^{op}$-modules $\mathcal{V}$. Let $\mathcal{A}(V)$ be $\mathrm{End}_H)\mathcal{V})$. Then there is a canonical homomorphism

$$\eta : A \to \mathcal{A}(V)\,.$$

If V is the family of all simple A-modules A an artinian k-algebra, and k algebraically closed, then:

$$\eta : A \underset{\simeq}{\to} \mathcal{A}(V)$$

generalizing the Burnside theorem for semisimple algebras.

§1. A generalized Burnside theorem and the closure operation $\mathcal{A}$

In [La], §(2.3) we proved the following result

Corollary (2.3) *Suppose the k-algebra A is of finite dimension and assume the family $V = \{V_i\}_{i=1}^r$ contains all simple A-modules, then the natural k-algebra homomorphism*

$$\eta : A \to H \hat{\otimes} \mathrm{End}_k(V) := (H_{ij} \otimes_k \mathrm{Hom}_k(V_i, V_j))$$

is injective.

Recall the classical Burnside-Wedderburn-Malcev theorems

Theorem (Burnside) *Let V be a finite dimensional k-vectorspace. Assume k is algebraically closed and let A be a subalgebra of $\mathrm{End}_k(V)$. If V is a simple A-module, then $A = \mathrm{End}_k(V)$.*

Theorem (Wedderburn) *Let A be a ring, and let V be a simple faithfull A-module. Put $D = \mathrm{End}_A(V)$ and assume V is a finite dimensional D-vector space. Then $A \simeq \mathrm{End}_D(V)$.*

Theorem (Wedderburn-Malcev) *Let A be a finite dimensional k-algebra, k-any field. Let $\underline{r}$ be the radical of A, and suppose the residue class algebra $A/\underline{r}$ is separable. Then there exists a semi-simple subalgebra S of A such that A is the semidirect sum of S and $\underline{r}$. If S_1 and S_2 are subalgebras such that $A = S_i \oplus \underline{r}$, $i = 1, 2$, then there exists an element $n \in \underline{r}$, such that $S_1 = (1-n) \cdot S_2 \cdot (1-n)^{-1}$.*

See e.g. [Lang] chap. XVII.

In this § we shall prove a generalization of the theorem of Burnside. In fact, assuming the field k is algebraically closed and that $V = \{V_i\}_{i=1}^r$ is the family of all simple A-modules we shall prove that the homomorphism η of the above Corollary (2.3), is an isomorphism.

When A is semi-simple we know that $\mathrm{Ext}_A^1(V_i, V_j) = 0$ for all $i, j = 1, \dots, r$, therefore the formal moduli H of V is isomorphic to k^r. This implies that

$$H \mathbin{\overline{\otimes}} \mathrm{End}_k(V) = \bigoplus_{i=1}^r \mathrm{End}_k(V_i),$$

which is the classical extension of Burnsides theorem.

We shall need the following elementary lemma

Lemma (1.1) *Let the k-algebra A be a direct sum of the right-A-modules V_i, $i = 1, \dots, d$ of the family $V = \{V\}_{i=1}^r$. Then left multiplication with an element $a \in A$ induces A-module homomorphisms*

$$a_{ij} \in \mathrm{Hom}_A(V_i, V_j), \qquad i, j = 1, \dots, d\,.$$

Moreover, any k-linear map $x : A \to A$ expressed as $x = (x_{ij}) \in \mathrm{End}_k(V) := (\mathrm{Hom}_k(V_i, V_j))$, commuting with all $\varphi = (\varphi_{ij}) \in \mathrm{End}_A(V) := (\mathrm{Hom}_A(V_i, V_j))$ is necessarily a right multiplication by some element $\tilde{x} \in A$.

Proof. Trivial, since x commuting with all $\varphi \in (\mathrm{Hom}_A(V_i, V_j))$ commutes with all left-multiplications by $a \in A$, and therefore $x(a) = a \cdot x(1)$, and we may put $\tilde{x} = x(1)$. QED

Corollary (1.2) *Assume that the family of right A-modules $V = \{V_i\}_{i=1}^r$ is such that*

(i) $A \simeq \bigoplus_{i=1}^{m} V_i^{n_i}$

(ii) $\mathrm{Hom}_A(V_i, V_j) = 0$ *for* $i \neq j$.

Then the canonical morphism of k-algebras

$$\eta : A \to \bigoplus_{i=1}^{n_i} \mathrm{End}_k(V_i)$$

is injective. Moreover, η induces an isomorphism

$$A \simeq \bigoplus_{i=1}^{n_i} \mathrm{End}_{D_i}(V_i)$$

where $D_i = \mathrm{End}_A(V_i)$.

This, in particular, proves the Wedderburn theorem for semisimple k-algebras A.

Theorem (1.3) (A generalized Burnside theorem) *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $V = \{V_i\}_{i=1}^r$ of simple A-modules, then*

$$A \simeq \mathrm{End}_H(\mathcal{V}) = H \mathbin{\overline{\otimes}} \mathrm{End}_k(V)$$

Proof. From [La], (2,3), we know that the canonical map

$$\eta : A \to H \mathbin{\overline{\otimes}} \mathrm{End}_k(V)$$

is injective. Since $\underline{r}(A)^n = 0$ for some n, we know that $\hat{A} = A$. The theorem therefore follows from the following lemmas.

Lemma (1.4) *Let A and B be finite type k-algebras and let $\varphi : A \to B$ be a homomorphism of k-algebras such that the induced morphism*

$$\varphi_2 : A \to B/r(B)^2$$

is surjective, then

$$\widehat{\varphi} : \hat{A} \to \hat{B}$$

is surjective.

Proof. Well-known.

Lemma (1.5) *Let A be a finite dimensional k-algebra, k an algebraically closed field. Let V_i, $i = 1, \dots, r$ be the different simple A-modules. Then*

$$\underline{r}(A)/\underline{r}(A)^2 \simeq (\mathrm{Ext}^1_A(V_i, V_j)^* \otimes_k \mathrm{Hom}_k(V_i, V_j)),$$

and, moreover, the homomorphism

$$\eta : A \to H \,\overline{\otimes}\, \mathrm{End}_k(V)$$

induces an isomorphism

$$t^*_\eta : \underline{r}(A)/\underline{r}(A)^2 \xrightarrow{\sim} (\mathrm{Ext}^1_A(V_i, V_j)^* \otimes_k \mathrm{Hom}_k(V_i, V_j)) = \underline{r}(H)/\underline{r}(H)^2 \,\overline{\otimes}_k\, \mathrm{End}_k(V)$$

Proof. We obviously have a homomorphism of k-algebras

$$A \twoheadrightarrow \bigoplus_{i=1}^{r} \mathrm{End}_k(V_i)$$

which by the classical Burnside theorem is surjective. According to the Wedderburn-Malcev theorem we may assume that

$$A \simeq (A_{ij})_{ij=1,\dots,r}$$

where, for each i, A_{ii} is a k-algebra such that

$$A_{ii}/r(A_{ii}) \simeq \mathrm{End}_k(V_i)$$

is a simple k-algebra.

Obviously $A/\underline{r}(A) \simeq \bigoplus_{i=1}^{r} \mathrm{End}_k(V_i)$ and

$$\underline{r}(A)/\underline{r}(A)^2 = (E_{ij})$$

each E_{ij} being an $\mathrm{End}_k(V_i)^{op} \otimes_k \mathrm{End}_k(V_j)$-module. This, however, means that

$$E_{ij} \simeq V_i^* \otimes V_j \otimes k^{r_{ij}}$$

as a right $\mathrm{End}_k(V_i)^{op} \otimes_k \mathrm{End}_k(V_j)$-module.

Now applying Hochschild cohomology as in [La] §2.1, we find: $\mathrm{Ext}^1_A(V_i, V_j) = HH^1(A, \mathrm{Hom}_k(V_i, V_j)) = \mathrm{Der}_k(A, \mathrm{Hom}_k(V_i, V_j))/im\, d^\circ$, where d° is the differential $\mathrm{Hom}_k(V_i, V_j) \to \mathrm{Der}_k(A, \mathrm{Hom}_k(V_i, V_j))$. Clearly any derivation $\xi \in \mathrm{Der}_k(A, \mathrm{Hom}_k(V_i, V_j))$ which is zero on $r(A)$ induces a derivation $\xi_0 \in \mathrm{Der}_k(A/\underline{r}(A), \mathrm{Hom}_k(V_i, V_j))$ which, since $A/r(A)$ is semisimple, obviously is a coboundary, i.e. an element of $im\, d^\circ$.

Moreover, any derivation $\xi \in \mathrm{Der}_k(A, \mathrm{Hom}_k(V_i, V_j))$ induces the zero map on $\underline{r}(A)^2$ since $\xi(r_1 \cdot r_2) = r_i\xi(r_2) + \xi(r_1)r_2 = 0$ for $r_1, r_2 \in \underline{r}(A)$, any coboundary $\nu \in im\, d^\circ$ must vanish on $\underline{r}(A)$ since $\nu(r) = \varphi r - r\varphi$ for some

$\varphi \in \mathrm{Hom}_k(V_i, V_j)$, and every $A^{op} \otimes_k A$-linear map $\underline{r}(A)/\underline{r}(A) \to \mathrm{End}_k(V_i, V_j)$ extends to a derivation of $\mathrm{Der}_k(A/\underline{r}(A)^2, \mathrm{Hom}_k(V_i, V_j))$. In fact, let φ be an $A^{op} \otimes_k A$-linear map

$$\underline{r}(A)/\underline{r}(A)^2 \to \mathrm{End}_k(V_i, V_j)$$

and define the map

$$\psi : A/\underline{r}(A)^2 = A/\underline{r}(A) \bigoplus \underline{k}(A)/\underline{r}(A)^2 \to \mathrm{End}_k(V_i, V_j)$$

by

$$\psi(s, r) = \varphi(r) + \varphi(\rho(r))$$

where ρ is the 1-Hochschild cochain on $A/\underline{r}(A)$ with values in $\underline{r}/\underline{r}^2$ that, according to the Wedderburn-Malcev theorem, exists. Then

$$\begin{aligned}\psi((s_1, r_1) \cdot (s_2, r_2)) &= \psi((s_1 \cdot s_2, s_1\rho(s_2) - \rho(s_1 \cdot s_2) + \rho(s_1)s_2 + s_1r_2 + r_1s_2)) \\ &= \varphi(s_1r_2 + r_1s_2 + s_1\rho(s_2) - \rho(s_1 \cdot s_2) + \rho(s_1) \cdot s_2) + \varphi(\rho(s_1 \cdot s_2)) \\ &= (s_1, r_1)\psi((s_2, r_2)) + \psi((s_1, r_1))(s_2, r_2)\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Ext}^1_A(V_i, V_j) &= \mathrm{Hom}_{A^{op}\otimes A}(\underline{r}(A)/\underline{r}(A)^2, \mathrm{Hom}_k(V_i, V_j)) \\ &= \{\varphi : \underline{r}(A)/\underline{r}(A)^2 \to \mathrm{Hom}_k(V_i, V_j) | \text{ for all } a \in A,\ r \in \underline{r}(A),\ s.t. \\ &\quad \varphi(a \cdot r) = a \cdot \varphi(r) \text{ and } \varphi(ra) = \varphi(r) \cdot a\}\end{aligned}$$

Since $\underline{r}(A)/\underline{r}(A)^2 \simeq (E_{ij})$ with

$$E_{ij} \simeq (V_i^* \otimes V_j)^{r_{ij}}$$

it is clear that

$$\begin{aligned}&\mathrm{Hom}_{A^{op}\otimes A}(\underline{r}(A)/\underline{r}(A)^2, \mathrm{Hom}_k(V_i, V_j)) \\ &\quad \simeq \mathrm{Hom}_{\mathrm{End}_k(V_i)^{op}\otimes_k \mathrm{End}_k(V_j)}((V_i^* \otimes V_j)^{r_{ij}}, (V_i^* \otimes V_j)) \simeq k^{r_{ij}}\end{aligned}$$

which means that

$$E_{ij} \simeq \mathrm{Ext}^1_A(V_i, V_j)^* \otimes_k \mathrm{Hom}_k(V_i, V_j), \qquad q.e.d.$$

Since η is an embedding it is clear that η induces an isomorphism on the tangent level,

$$t_\eta : \underline{r}(A)/\underline{r}(A)^2 \to \underline{r}(H)/\underline{r}(H)^2 \overline{\otimes} \mathrm{End}_k(V),$$

proving the theorem. QED

Now suppose, as above, that A is a finite dimensional k-algebra, and let $V = \{V_i\}_{i=1}^r$ be any family of finite dimensional A-modules. Obviously

$\dim_k \mathrm{Ext}^p_A(V_i, V_j) < \infty$ for all $p = 0, 1, 2, \dots$ and therefore the endomorphism ring

$$\mathcal{A}(V) := \mathrm{End}_H(\mathcal{V}_A)$$

is a k-algebra such that

$$\mathcal{A}(V)/\underline{r} = \bigoplus_{i=1}^{r} \mathrm{End}_k(V_i)$$

$\underline{r}$ being the radical.

This implies that $V = \{V_i\}_{i=1}^r$ is the family of all simple $\mathcal{A}(V)$-modules, provided the k-algebra $\mathcal{A}(V)$ is known to be of finite k-dimension. In this case the generalized Burnside theorem implies that the operation

$$(A, V) \mapsto (\mathcal{A}_A(V_A), V) =: (\mathcal{A}(V), V)$$

is a closure operation, i.e.

$$(\mathcal{A}_{\mathcal{A}(V)}(V_{\mathcal{A}(V)}), V) = (\mathcal{A}(V), V).$$

Moreover, we have the following,

Proposition (1.6) *Let $\tau : A \to B$ be any homomorphism of finite dimensional k-algebras. Consider a family $V_B = \{V_i\}_{i=1}^r$ of finite dimensional B-modules and let V_A be the corresponding family of A-modules.*

Suppose moreover that V_B is the family of all simple B-modules. Then there exists an, up to isomorphisms, unique homomorphism of k-algebras

$$\mathcal{A}(\tau) : \mathcal{A}(V_A) \to \mathcal{A}(V_B) \simeq B$$

extending τ.

Proof. There is an obvious forgetful functor defining a morphism of functors on $\underline{a}_r$,

$$\tau^* : \mathrm{Def}_{B,V} \to \mathrm{Def}_{A,V}$$

which in its turn induces a k-algebra homomorphism

$$\eta : H_{A,V} \to H_{B,V}$$

unique up to isomorphisms, and therefore a k-algebra homomorphism

$$\mathcal{A}(V_A) := H_{A,V} \,\overline{\otimes}\, \mathrm{End}_k(V_A) \to H_{B,V} \,\overline{\otimes}\, \mathrm{End}_k(V_B) =: \mathcal{A}(V_B)$$

obviously extending τ. By the generalized Burnside theorem, $\mathcal{A}(V_B) \simeq B$, and the Proposition follows. QED

Remark (1.7) Up to now we have only considered finite families of A-modules such that

$$\dim_k \operatorname{Ext}^p_A(V_i, V_j) < \infty, \qquad p = 1, 2.$$

Neither of these conditions are essential. Introducing natural topologies we may, as in [La2], treat general families of finite type A-modules.

Notice also that if $r_1 \leq r_2$, there is an obvious canonical faithfull morphism

$$\underline{a}_{r_1} \to \underline{a}_{r_2}$$

inducing a morphism of functors

$$\operatorname{Def}_{A,V(1)} \to \operatorname{Def}_{A,V(2)}$$

where $V(1) = \{V_i\}_{i=1}^{r_1}, V(2) = \{V_i\}_{i=1}^{r_2}$. Therefore we obtain an up to isomorphisms unique k-algebra homomorphism

$$r_{2,1} : H_{A,V(2)} \to H_{A,V(1)}.$$

However, this "restriction" morphism is not in general unique. The resulting problems will be dealt with in a forthcoming paper.

Proposition (1.8) *Let A be any k-algebra, $V = \{V_i\}_{i=1}^r$ any family of A-modules such that*

$$\dim_k \operatorname{Ext}^1_A(V_i, V_j) < \infty \quad \textit{for all } i, j = 1, \ldots, r\,.$$

Then the category (of isomorphism classes) of extensions of extensions of the V_i's is isomorphic to the category (of isomorphism classes) of finite dimensional representations of H_V.

Proof. Clearly any morphism $\varphi : H \to R$ in $\underline{a}_r$ correspond to an extension of extensions of the V_i's. Conversely, we prove by induction on the length of the extension that any extension corresponds to an object R of $\underline{a}_r$ and a morphism $\varphi : H \to R$. QED

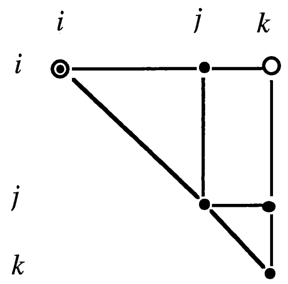
Example (1.9) Consider the extension E_{ijk} of length 3 given as the composite extension of $\xi_{ij} : 0 \leftarrow V_i \leftarrow E_{ij} \leftarrow V_j \leftarrow 0$ and
$\xi : 0 \leftarrow E_{ij} \leftarrow E_{ijk} \leftarrow V_k \leftarrow 0$.

Take the pullback ξ_{ik} of ξ via $V_j \to E_{ij}$ and consider the diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 & V_i \arrow[l] \arrow[u] & E_{ij} \arrow[l] \arrow[u] & V_j \arrow[l] \arrow[u] \arrow[d, dashed, bend left] & 0 \arrow[l] \\
 & E_{ik} \arrow[u, dashed] & E_{ijk} \arrow[l, dashed] \arrow[u] & E_{jk} \arrow[l] \arrow[u] & \\
 & V_k \arrow[u, dashed] \arrow[r, equal] & V_k \arrow[u] \arrow[r, equal] & V_k \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Obviously the cup-product $\xi_{ij} \cup \xi_{jk}$ is 0 in $\mathrm{Ext}^2_A(V_i, V_k)$. This is also the criterion for the existence of ξ. Moreover if ξ and ξ' are two extension $0 \leftarrow E_{ij} \leftarrow E_{ijk} \leftarrow V_k \leftarrow 0$ with the same pullback ξ_{jk}, then there is an extension $\xi_{ik} : 0 \leftarrow V_i \leftarrow E_{ik} \leftarrow V_k \leftarrow 0$, the pullback via $E_{ij} \to V_i$ of which is the difference $\xi - \xi'$.

Consider for the extension of extensions E_{ijk}, the diagram,



corresponding to the k^r-algebra R

$$R = \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ k \\ \\ \\ \end{array}
\begin{array}{c} \begin{array}{ccccc} & i & j & k & \end{array} \\
\begin{pmatrix}
k & 0 & 0 & 0 & 0 \\
\ddots & \cdots & \cdots & \cdots & \vdots \\
 & k & k & k & 0 \\
 & \ddots & \cdots & \cdots & \vdots \\
 & & k & k & 0 \\
 & & \ddots & \cdots & \vdots \\
 & & & k & 0 \\
 & & & \ddots & \cdots \\
 & & & & k
\end{pmatrix} \end{array}$$

Notice that E_{ijk} then corresponds to a morphism

$$\varphi : H \to R\,,$$

unique modulo the radical $\underline{r}(R)$ squared, where

$$\underline{r}(R)^2 = \begin{array}{cc} & \begin{array}{ccccc} & i & j & k & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \ddots & \cdots & \cdots & \cdots & \vdots \\ & 0 & 0 & k & 0 \\ & \ddots & \cdots & \cdots & \vdots \\ & & 0 & 0 & 0 \\ & & \ddots & \cdots & \vdots \\ & & & 0 & 0 \\ & & & \ddots & \cdots \\ & & & & 0 \end{pmatrix} \end{array}$$

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