States and shifts on infinite free products of C^*-algebras

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Dedicated to Richard V. Kadison on occasion of his 70th birthday

Abstract. We study quasi equivalence of states of free products of C^*-algebras together with free shifts, compute their entropy and show a strong form of unique ergodicity.

1 Introduction

Let for each element ι in index set I, A_ι be a unital C^*-algebra and ϕ_ι a state on A_ι. Let $(A, \phi) = (*A_\iota, *\phi_\iota)_{\iota \in I}$ be the corresponding free product C^*-algebra as defined in [6, 1.5]. In the present paper we shall study states on A, and if $I = \mathbf{Z}$ and all the pairs (A_ι, ϕ_ι) are equal, the shift automorphism on A arising from the shift $\iota \to \iota + 1$. Our results will, except for those in the last section, extend those in [5] for the II_1-factor $L(\mathbf{F}_\infty)$ defined by the left regular representation of the free group $\mathbf{F}_\infty$ in infinite number of generators. Our main result is for general infinite products and shows the existence of a universal function $r : (0, 1] \to \mathbf{N}$ such that whenever (A, ϕ) is as above and ω is a state whose GNS-representation is quasi contained in that of ϕ, then there is for each $\varepsilon > 0$ a subset $J \subset I$ of cardinality card $J \leq r(\varepsilon)$, such that $\|(\phi - \omega)|_{A_\iota}\| < \varepsilon$ for all $\iota \notin J$.

In the two last sections we assume $I = \mathbf{Z}$ and all the (A_ι, ϕ_ι) are equal and let α denote the free shift of A which arises as mentioned above from the shift on $\mathbf{Z}$. Analogously to the free shift on $L(\mathbf{F}_\infty)$ we use the above result to show that the entropy in the sense of Connes, Narnhofer and Thirring [1], called CNT-entropy in the sequel, of α with respect to the invariant state ϕ is zero. Then in the last section we show that α satisfies a very strong unique

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ergodicity property. Namely, if (B, β, μ) is a unital C^*-dynamical system and λ is an $\alpha \otimes \beta$-invariant state on $A \otimes B$ such that $\lambda(1 \otimes b) = \mu(b)$ for $b \in B$, then $\lambda = \phi \otimes \mu$. An immediate corollary of this is that the entropy of Sauvageot and Thouvenot of α with respect to ϕ is also zero.

We remark that it is not necessary for the above to restrict attention to the free shift. Our arguments work for an arbitrary infinite index set I and an automorphism arising from a bijection σ of I such that for all finite subsets $J \subset I$ there exists $p \in \mathbf{N}$ such that the sets $\sigma^{pn}(J), n \in \mathbf{N}$, are all disjoint.

We refer the reader to the book [6] of Voiculescu, Dykema and Nica for the theory of free products of C^*-algebras.

2 States on free products

Let I be an index set, and for each $\iota \in I$ let A_ι be a unital C^*-algebra and ϕ_ι a state on A_ι. Following [6, 1.5.1] we shall define the free product $(A, \phi) = (*A_\iota, *\phi_\iota)_{\iota \in I}$ with its canonical cyclic representation π.

Let $(\pi_\iota, \mathcal{H}_\iota, \xi_\iota)$ be the GNS-representation of $\phi_\iota, \iota \in I$. Let $\mathcal{H}_\iota^\circ = \mathcal{H}_\iota \ominus \mathbf{C}\xi_\iota$, and $(\mathcal{H}, \xi) = *_{\iota \in I}(\mathcal{H}_\iota, \xi_\iota)$. Put

$$\mathcal{H}(\iota) = \mathbf{C}\xi \oplus \bigoplus_{n \geq 1} \Big(\bigoplus_{\substack{\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n \\ \iota_1 \neq \iota}} \mathcal{H}_{\iota_1}^\circ \otimes \cdots \otimes \mathcal{H}_{\iota_n}^\circ \Big).$$

We have unitary operators $V_\iota : \mathcal{H}_\iota \otimes \mathcal{H}(\iota) \to \mathcal{H}$ defined by

$$\xi_\iota \otimes \xi \to \xi$$
$$\mathcal{H}_\iota^\circ \otimes \xi \to \mathcal{H}_\iota^\circ \text{ by } \eta \otimes \xi \to \eta$$
$$\xi_\iota \otimes (\mathcal{H}_{\iota_1}^\circ \otimes \cdots \otimes \mathcal{H}_{\iota_n}^\circ) \to \mathcal{H}_{\iota_1}^\circ \otimes \cdots \otimes \mathcal{H}_{\iota_n}^\circ \text{ by } \xi_\iota \otimes \eta \to \eta, \iota_1 \neq \iota$$
$$\mathcal{H}_\iota^\circ \otimes (\mathcal{H}_{\iota_1}^\circ \otimes \cdots \otimes \mathcal{H}_{\iota_n}^\circ) \to \mathcal{H}_\iota^\circ \otimes \mathcal{H}_{\iota_1}^\circ \otimes \cdots \otimes \mathcal{H}_{\iota_n} \text{ by } \psi \otimes \eta \to \psi \otimes \eta, \iota_1 \neq \iota$$

The representation $\lambda_\iota : A_\iota \to B(\mathcal{H})$ is defined by

$$\lambda_\iota(a) = V_\iota(\pi_\iota(a) \otimes 1_{\mathcal{H}(\iota)})V_\iota^*, \ a \in A_\iota.$$

The free product representation $\pi = *\pi_\iota : *A_\iota \to B(\mathcal{H})$ is the $*$-homomorphism of the free product C^*-algebra $(*A_\iota, *\lambda_\iota) \to B(\mathcal{H})$, using the universal property of the free product. When we write $(A, \phi) = (*A_\iota, *\phi_\iota)_{\iota \in I}$ we shall mean $*A_\iota$ in the representation π i.e. we shall mean $\pi(*A_\iota) \subset B(\mathcal{H})$.

We can now state the main result of this section, which is a direct generalization of [5, Lem. 2.4]. Since C^*-algebras isomorphic to the scalars are redundant in the definition of free products, we shall in order to avoid complications assume the C^*-algebras in the product to have linear dimension at least 2. Hence we shall exclude homomorphisms in the theorem. We denote by $|J|$ the cardinality of a set J.

Theorem 1. *For each $\varepsilon \in (0,1]$ let $r(\varepsilon) = [100\varepsilon^{-2}]+1$. Then the following holds. Let $(A,\phi) = (*A_\iota, *\phi_\iota)_{\iota \in I}$ be a free product of unital C^*-algebras A_ι with states ϕ_ι which are not homomorphisms. Suppose ω is a state of A of the form $\omega = \omega' \circ \pi$ with ω' a normal state on $\pi(A)''$. Then for each $\varepsilon \in (0,1]$ there exists a subset $J = J(\omega, \varepsilon) \subset I$ with $|J| \leq r(\varepsilon)$ such that*

$$\|(\phi - \omega)|_{A_\iota}\| < \varepsilon \quad \forall \iota \notin J, \iota \in I.$$

Proof. We first assume the state ω' is a vector-state ω_η. We use the convention that whenever we write $J \subset I$ we mean a finite ordered subset of I of the form

$$J = \{\iota_1, \iota_2, \cdots, \iota_n\}, \ \iota_1 \neq \iota_2 \neq \cdots \neq \iota_n.$$

Here $n = |J|$. For each $\iota \in I$ we let

$$I(\iota) = \{J \subset I : \iota_1 = \iota\}.$$

Since each vector ξ_ι is cyclic for $\pi_\iota(A_\iota)$ we may (by approximation) assume

$$\eta = \lambda\xi + \sum_{\iota \in I} \sum_{J \in I(\iota)} \eta_J,$$

where

$$\eta_J = \eta_{\iota_1} \otimes \eta_{\iota_2} \otimes \cdots \otimes \eta_{\iota_n}, \ J = \{\iota_1, \cdots, \iota_n\},$$

and

$$\eta_{\iota_k} = \pi_{\iota_k}(a^J_{\iota_k})\xi_{\iota_k}, \qquad k = 1, \cdots, n, \ a^J_{\iota_k} \in A_{\iota_k}.$$

Furthermore, $\eta_{\iota_k} \in \mathcal{H}^\circ_{\iota_k}$, so that $\phi_{\iota_k}(a^J_{\iota_k}) = 0$, and we have that all vectors η_J are mutually orthogonal since they are the orthogonal projections of η on the Hilbert spaces $\mathcal{H}^\circ_{\iota_1} \otimes \cdots \otimes \mathcal{H}^\circ_{\iota_n}$, hence in particular, if $J = K$ then the corresponding vectors η_J and η_K coincide.

From now on we fix an index $\iota_0 \in I$, and let $a \in A_{\iota_0}$ be self-adjoint with $\phi(a) = \phi_{\iota_0}(a) = 0$. Denote by $\lambda_0 = \lambda_{\iota_0}$, $\pi_0 = \pi_{\iota_0}$, $\xi_0 = \xi_{\iota_0}$, $V_0 = V_{\iota_0}$. Computing we find

$$\begin{aligned}
\omega(a) &= |\lambda|^2\phi(a) + 2Re\lambda \sum_{\iota\in I}\sum_{J\in I(\iota)} (\lambda_0(a)\xi, \eta_J) \\
&+ \sum_{\iota}\sum_{J\in I(\iota)} (\lambda_0(a)\eta_J, \eta_J) + \sum_{\iota,\rho\in I}\sum_{J\in I(\iota)}\sum_{\substack{K\in I(\rho)\\ K\neq J}} (\lambda_0(a)\eta_J, \eta_K) \\
&= \sum_{\iota}\sum_{J\in I(\iota)} (\lambda_0(a)\eta_J, \eta_J) + 2Re\lambda \sum_{\iota\in I}\sum_{J\in I(\iota)} (\pi_0(a)\otimes 1(\xi_0\otimes\xi), V_0^*\eta_J) \\
&+ \sum_{\iota,\rho\in I}\sum_{J\in I(\iota)}\sum_{\substack{K\in I(\rho)\\ K\neq J}} ((\pi_0(a)\otimes 1)V_0^*\eta_J, V_0^*\eta_K).
\end{aligned}$$

We shall compute the scalar products case by case. We use the notation when $|J| > 1$, $J = \{\iota_1, \cdots, \iota_n\}$.

$$\eta_J^1 = \eta_{\iota_2}\otimes\eta_{\iota_3}\otimes\cdots\otimes\eta_{\iota_n}.$$

(1) $J \in I(\iota_0)$. Then

$$\begin{aligned}
(\pi_0(a)\otimes 1(\xi_0\otimes\xi), V_0^*\eta_J) &= \begin{cases} (\pi_0(a)\xi_0\otimes\xi, \pi_0(a_{\iota_0}^J)\xi_0\otimes\eta_J^1) & \text{if } |J| > 1 \\ \phi(a_{\iota_0}^{J*}a) & \text{if } |J| = 1 \end{cases} \\
&= \begin{cases} 0 & \text{if } |J| > 1 \\ \phi(a_0^*a) & \text{if } |J| = 1, \end{cases}
\end{aligned}$$

where we denote by a_0 the element $a_{\iota_0}^J \in A_{\iota_0}$ when $J = \{\iota_0\}$. We next consider

$$X = ((\pi_0(a)\otimes 1)V_0^*\eta_J, V_0^*\eta_K) = (\lambda_0(a)\eta_J, \eta_K).$$

(2) $J = K = \{\iota_0\}$. Then as in (1)

$$X = \phi(a_0^*aa_0).$$

(3) $J = \{\iota_0\}, K \in I(\iota_0), |K| > 1$. Then

$$X = (\pi_0(a)\otimes 1(\eta_{\{\iota_0\}}\otimes\xi), \eta_K) = (\pi_0(a)\pi_0(a_0)\xi_0, \pi_0(a_{\iota_0}^K)\xi_0)(\xi, \eta_K^1) = 0$$

(4) $J, K \in I(\iota_0), |J| > 1, K = \{\iota_0\}$. Then as in (3) $X = 0$.

(5) $J, K \in I(\iota_0)$, $|J| > 1$, $|K| > 1$. Then $V_0^*\eta_J = \eta_J, V_0^*\eta_K = \eta_K$. Thus

$$\begin{aligned}
X &= (\pi_0(a)\pi_0(a_{\iota_0}^J)\xi_0, \pi_0(a_{\iota_0}^K)\xi_0)(\eta_J^1, \eta_K^1) \\
&= \phi(a_{\iota_0}^{K*}aa_{\iota_0}^J)(\eta_J^1, \eta_K^1) \\
&= \begin{cases} 0 & \text{if } J \neq K \\ \phi(a_{\iota_0}^{J*}aa_{\iota_0}^J)\|\eta_J^1\|^2 & \text{if } J = K. \end{cases}
\end{aligned}$$

(6) $J = \{\iota_0\}$, $K \notin I(\iota_0)$. Then

$$X = (\pi_0(a) \otimes 1\eta_{\{\iota_0\}} \otimes \xi, \xi_0 \otimes \eta_K) = 0.$$

(7) $J \notin I(\iota_0)$, $K = \{\iota_0\}$. Then similarly $X = 0$.

(8) $J \in I(\iota_0)$, $|J| > 1$, $K \notin I(\iota_0)$. Then

$$\begin{aligned} X &= ((\pi_0(a) \otimes 1)\eta_J, \xi_0 \otimes \eta_K) \\ &= \phi(aa^J_{\iota_0})(\eta^1_J, \eta_K). \end{aligned}$$

(9) $J \notin I(\iota_0), K \in I(\iota_0), |K| > 1$. Then as in (8)

$$X = \phi(a^{K^*}_{\iota_0} a)(\eta_J, \eta^1_K).$$

(10) $J, K \notin I(\iota_0)$. Then since $\phi(a) = 0$,

$$\begin{aligned} X &= (\pi_0(a) \otimes 1(\xi_0 \otimes \eta_J), \xi_0 \otimes \eta_K) \\ &= \phi(a)(\eta_J, \eta_K) \\ &= 0. \end{aligned}$$

Summing up (1)–(10) we obtain

$$\begin{aligned} \omega(a) = \phi(a_0^* a a_0) &+ \sum_{\substack{J \in I(\iota_0) \\ |J|>1}} \phi(a^{J^*}_{\iota_0} a a^J_{\iota_0}) \|\eta^1_J\|^2 \\ &+ 2Re\lambda\phi(a_0^* a) + \\ &+ \sum_{\substack{J \in I(\iota_0) \\ |J|>1}} \sum_{\iota \neq \iota_0} \sum_{K \in I(\iota)} \phi(aa^J_{\iota_0})(\eta^1_J, \eta_K) \\ &+ \sum_{\substack{K \in I(\iota_0) \\ |K|>1}} \sum_{\iota \neq \iota_0} \sum_{J \in I(\iota)} \phi(a^{K^*}_{\iota_0} a)(\eta_J, \eta^1_K) . \end{aligned}$$

If $J = \{\iota_1, \ldots, \iota_n\}$ put $J_1 = \{\iota_2, \ldots, \iota_n\}$. Then $(\eta^1_J, \eta_K) = 0$ unless $K = J_1$.

We therefore have for $\|a\| \le 1$,

$$\begin{aligned} |\omega(a)| \le 2|\lambda| \|\eta_{\{\iota_0\}}\| &+ \|\eta_{\{\iota_0\}}\|^2 + \sum_{\substack{J \in I(\iota_0) \\ |J|>1}} \|\eta_J\|^2 \\ &+ \sum_{\substack{J \in I(\iota_0) \\ |J|>1}} |\phi(a^J_{\iota_0})| \|\eta^1_J\| \|\eta_{J_1}\| + \sum_{\substack{J \in I(\iota_0) \\ |J|>1}} |\phi(a^{J^*}_{\iota_0})| \|\eta_{J_1}\| \|\eta^1_J\| \\ &\le 2|\lambda| \|\eta_{\{\iota_0\}}\| + \sum_{J \in I(\iota_0)} \|\eta_J\|^2 + 2 \sum_{\substack{J \in I(\iota_0) \\ |J|>1}} \|\eta_J\| \|\eta_{J_1}\|, \end{aligned}$$

where the last term arises from the fact that

$$|\phi(a^J_{\iota_0})(\eta^1_J, \eta_K)| \le \|\eta^J_{\iota_0}\| \|\eta^1_J\| \|\eta_K\| = \|\eta_J\| \|\eta_K\|.$$

Since $\sum_{J \notin I(\iota_0)} \|\eta_J\|^2 \leq 1$ we thus have from the Cauchy-Schwarz inequality and the fact that the sum $\|\eta_{J_1}\|^2$ is the same as over K's with $K \notin I(\iota_0)$,

$$
\begin{aligned}
|\omega(a)| &\leq 2|\lambda|\|\eta_{\{\iota_0\}}\| + \sum_{J \in I(\iota_0)} \|\eta_J\|^2 + 2\Big(\sum_{\substack{J \in I(\iota_0) \\ |J|>1}} \|\eta_J\|^2\Big)^{1/2} \Big(\sum_{\substack{J \in I(\iota_0) \\ |J|>1}} \|\eta_{J_1}\|^2\Big)^{1/2} \\
&\leq 2|\lambda|(\|\eta_{\{\iota_0\}}\|^2)^{1/2} + 3\Big(\sum_{J \in I(\iota_0)} \|\eta_J\|^2\Big)^{1/2} \qquad (11) \\
&\leq 5\Big(\sum_{J \in I(\iota_0)} \|\eta_J\|^2\Big)^{1/2}.
\end{aligned}
$$

Note that we have

$$1 = \|\eta\|^2 = |\lambda|^2 + \sum_{\iota \in I} \sum_{J \in I(\iota)} \|\eta_J\|^2. \qquad (12)$$

If $C_\iota \geq 0 \ \forall \iota \in I$ and $\sum_{\iota \in I} C_\iota \leq 1$, then given $\delta > 0$ and $r(\delta) = [100\delta^{-2}] + 1$ then there exists a subset J of I with $|J| \leq r(\delta)$ such that

$$C_\iota < \frac{\delta^2}{100}, \qquad \iota \notin J.$$

Thus by (12) there is a subset $J(\omega, \varepsilon)$ of I with $|J(\omega, \varepsilon)| \leq r(\varepsilon)$ such that

$$\sum_{J \in I(\iota)} \|\eta_J\|^2 < \frac{\varepsilon^2}{100} \qquad \text{for} \quad \iota \notin J(\omega, \varepsilon)$$

In particular by (11)

$$|\omega(a)| < \varepsilon/2 \quad \text{for} \quad \iota_0 \notin J(\omega, \varepsilon).$$

We have thus shown the essence of the theorem in the case $\omega = \omega' \circ \pi$ with ω' a vector state. For the general case let $\xi_i \in \mathcal{H}$, $\|\xi_i\| = 1$, $i \in \mathbf{N}$. Let $\alpha_i \geq 0$, $\sum_{i=1}^{\infty} \alpha_i = 1, i \in \mathbf{N}$. Put $\omega = \sum_{i=1}^{\infty} \alpha_i \omega_{\xi_i} \circ \pi$, i.e. $\omega' = \sum_{i=1}^{\infty} \alpha_i \omega_{\xi_i}$. Then

$$1 = \|\omega(1)\| = \sum_{i=1}^{\infty} \alpha_i \|\xi_i\|^2.$$

Write as before

$$\xi_i = \lambda_i \xi + \sum_{\iota \in I} \sum_{J_i \in I(\iota)} \eta_{J_i}.$$

Thus

$$1 = \|\xi_i\|^2 = |\lambda_i|^2 + \sum_{\iota \in I} \sum_{J_i \in I(\iota)} \|\eta_{J_i}\|^2,$$

hence

$$1 = \sum_i \alpha_i |\lambda_i|^2 + \sum_{\iota \in I} \sum_i \alpha_i \sum_{J_i \in I(\iota)} \|\eta_{J_i}\|^2.$$

As above given $\varepsilon > 0$ there is $J = J(\omega, \varepsilon) \subset I$ with $|J| \leq r(\varepsilon)$ such that

$$\sum_i \sum_{J_i \in I(\iota)} \alpha_i \|\eta_{J_i}\|^2 < \frac{\varepsilon^2}{100}, \qquad \iota \notin J\,.$$

Therefore, if $a \in A_{\iota_0}$, $\iota_0 \notin J$, $\phi(a) = 0$, we have by (11)

$$\begin{aligned}
|\omega(a)| &\leq \sum_i \alpha_i |\omega_{\xi_i}(\pi(a))| \\
&\leq 5\|a\| \sum_i \alpha_i \Big(\sum_{J_i \in I(\iota_0)} \|\eta_{J_i}\|^2 \Big)^{\frac{1}{2}} \\
&= 5\|a\| \sum_i \alpha_i^{\frac{1}{2}} \Big(\sum_{J_i \in I(\iota_0)} \alpha_i \|\eta_{J_i}\|^2 \Big)^{\frac{1}{2}} \\
&\leq 5\|a\| \Big(\sum_i \alpha_i \Big)^{\frac{1}{2}} \Big(\sum_i \sum_{J_i \in I(\iota_0)} \alpha_i \|\eta_{J_i}\|^2 \Big)^{\frac{1}{2}} \\
&= 5\|a\| \Big(\sum_i \sum_{J_i \in I(\iota_0)} \alpha_i \|\eta_{J_i}\|^2 \Big)^{\frac{1}{2}} \\
&< 5\|a\| \frac{\varepsilon}{10} \\
&= \varepsilon/2\|a\|.
\end{aligned}$$

Finally, if $a \in A_{\iota_0}$, $a = \phi(a)1 + a_0$ with $\phi(a_0) = 0$, $a_0 \in A_{\iota_0}$. Then $\|a_0\| \leq \|a\| + |\phi(a)| \leq 2\|a\|$. Thus if $\iota_0 \notin J$

$$|(\phi - \omega)(a)| = |\omega(a_0)| < \frac{\varepsilon}{2}\|a_0\| \leq \varepsilon\|a\|.$$

Thus $\|(\phi - \omega)|_{A_{\iota_0}}\| < \varepsilon$ if $\iota_0 \notin J(\omega, \varepsilon) = J$. QED.

Following [1] if B is a C^*-algebra and μ a state of B we denote by $\|x\|_\mu = \mu(x^*x)^{1/2}$. Then $\|x\|_\mu \leq \|x\|$. If A is another C^*-algebra and $\rho : A \to B$ a linear map we put

$$\|\rho\|_\mu = \sup_{\|x\| \leq 1} \|\rho(x)\|_\mu.$$

Corollary 1. *Let $(A, \phi), r$ be as in Theorem 1. Let B be a finite dimensional Abelian C^*-algebra generated by its minimal projections $p_1, \cdots, p_n$. Suppose $P : A \to B$ is a positive unital linear map, so that $P(x) = \sum_{i=1}^{n} \psi_i(x) p_i$*

with ψ_i a state of A. Suppose μ is a state of B such that $\mu \circ P = \phi$. Then given $\varepsilon > 0$ there exists $J = J(P, \varepsilon) \subset I$ with $|J| \leq r(\varepsilon)$ such that

$$\|P(x) - \phi(x)1\|_\mu < \varepsilon \|x\|, \qquad x \in A_\iota, \ \iota \notin J.$$

Proof. We have

$$\phi(x) = \mu \circ P(x) = \sum_{i=1}^n \psi_i(x) \mu(p_i).$$

Thus if $\mu(p_i) \neq 0$, $\psi_i \leq \mu(p_i)^{-1}\phi$, so that $\psi_i = \omega_{\xi_i} \circ \pi$ with ξ_i a unit vector in $\mathcal{H}$. Therefore, if $a \in A_{\iota_0}$, is self-adjoint and $\phi(a) = 0$, it follows by (11) that

$$|\psi_i(a)|^2 \leq 25 \sum_{J_i \in I(\iota_0)} \|\eta_{J_i}\|^2,$$

where the notation is as in the proof of Theorem 1 and $\xi_i = \lambda_i \xi + \sum_{\iota \in I} \sum_{J_i \in I(\iota)} \eta_{J_i}$. Given $\varepsilon > 0$ choose $J = J(P, \varepsilon) \subset I$ with $|J| \leq r(\varepsilon)$ such that

$$\sum_i \Big(\sum_{J_i \in I(\iota_0)} \|\eta_{J_i}\|^2 \Big) \mu(p_i) < \frac{\varepsilon^2}{100} \qquad \text{for} \quad \iota_0 \notin J.$$

Thus we have for $\iota_0 \notin J$, since the cases $\mu(p_i) = 0$ don't matter,

$$\begin{aligned} \sum_i \psi_i(a)^2 \mu(p_i) &\leq 25\|a\|^2 \sum_i \Big(\sum_{J_i \in I(\iota_0)} \|\eta_{J_i}\|^2 \Big) \mu(p_i) \\ &< 25\|a\|^2 \cdot \frac{\varepsilon^2}{100} \\ &= \|a\|^2 (\varepsilon/2)^2. \end{aligned}$$

Since $\phi(a) = 0$ and a in self-adjoint,

$$\|P(a) - \phi(a)1\|_\mu^2 = \sum_{i=1}^n \psi_i(a)^2 \mu(p_i) < \|a\|^2 \left(\frac{\varepsilon}{2}\right)^2.$$

For general self-adjoint $a \in A_{\iota_0}$, $a = \phi(a) + a_0$, $\phi(a_0) = 0$. Hence if $\iota_0 \notin J$,

$$\begin{aligned} \|P(a) - \phi(a)1\|_\mu &= \mu((\phi(a) + P(a_0) - \phi(a))^2)^{\frac{1}{2}} \\ &= \mu(P(a_0)^2)^{\frac{1}{2}} \\ &< \|a_0\| \varepsilon/2 \\ &\leq 2\|a\|\varepsilon/2 \\ &= \varepsilon\|a\|. \end{aligned}$$

The conclusion follows. QED

3 Entropy of free shifts

In [5] it was shown that the entropy of the free shift on $L(\mathbf{F}_\infty)$ is zero. We shall now generalize this result to arbitrary free shifts.

Theorem 2. *Let A_0 be a unital C^*-algebra and ϕ_0 a state of A_0. Let $A_i = A_0$, $\phi_i = \phi_0$, $i \in \mathbf{Z}$, and let $(A, \phi) = (*A_i, *\phi_i)_{i\in\mathbf{Z}}$. Let α be the free shift on A, i.e. α is the automorphism of A arising from the shift $n \to n+1$ on $\mathbf{Z}$. Then the CNT-entropy of α with respect to $\phi, h_\phi(\alpha) = 0$.*

Proof. If ϕ_0 is a homomorphism each ϕ_i can be identified with its GNS-representation, hence the GNS-representation of ϕ is one dimensional, so ϕ is a homomorphism, thus $h_\phi(\alpha) = 0$. We therefore assume each ϕ_i is not a homomorphism.

Let C be a finite dimensional C^*-algebra and $\gamma : C \to A$ a unital completely positive map. Let $\varepsilon > 0$ and $r = r(\varepsilon/2)$ as in Theorem 1. Choose $k \in \mathbf{N}$ so large that

$$k^{-1} r S(\phi \circ \gamma) < \varepsilon .$$

Let B be a finite dimensional Abelian C^*-algebra generated by its minimal projections $p_1, \cdots, p_n$. Suppose $P : A \to B$ is a positive unital map and μ a state of B such that $\mu \circ P = \phi$. For each $i \in \{1, \cdots, n\}$ there is a state ψ_i of A such that

$$P(x) = \sum_{i=1}^{n} \psi_i(x) p_i.$$

Then

$$\phi = \mu \circ P = \sum_{1}^{n} \mu(p_i)\psi_i,$$

is ϕ written as a convex combination of states. In the notation of [1] we have

$$\varepsilon_\mu(P) = \sum_{1}^{n} \mu(p_i) S(\phi|\phi_i)$$
$$s_\mu(P) = S(\mu) - \varepsilon_\mu(P).$$

Since $A = *A_i$ and C is finite dimensional there is $n_0 \in \mathbf{N}$ such that if $x \in C$ then there is $a \in \tilde{A}_0 = \underset{i=-n_0}{\overset{n_0}{*}} A_i$ such that $\|\gamma(x) - a\| < \varepsilon/4\|x\|$, and $\|a\| \leq \|x\|$. Let $p = 2n_0 + 1$ and $\tilde{A}_m = \alpha^{mp}(\tilde{A}_0)$, $\tilde{\phi}_0 = \underset{i=-n_0}{\overset{n_0}{*}} \phi_i$, $\tilde{\phi}_m = \tilde{\phi}_0 \circ \alpha^{mp}$. By distributivity [6, 2.5.5] and the uniqueness of the GNS-representation of ϕ, we can write (A, ϕ) as the free product

$$(A, \phi) = (*\tilde{A}_m, *\tilde{\phi}_m)_{m\in\mathbf{Z}}.$$

Furthermore, α^p acts as a free shift on A.

For each $j \in \{0, \cdots, k-1\}$ let B_j be a C^*-subalgebra of B, and let $E_j : B \to B_j$ be the unique μ-invariant conditional expectation of B onto

B_j. Then E_j satisfies the Cauchy-Schwarz inequality $E_j(x^*x) \geq E_j(x)^* E_j(x)$. If Φ is a self-adjoint linear map of a C^*-algebra D into B, we have for $y \in D$,

$$\begin{aligned} \|E_j \circ \Phi(y)\|_\mu^2 &= \mu(E_j(\Phi(y^*))E_j(\Phi(y))) \\ &\leq \mu(E_j(\Phi(y^*)\Phi(y))) \\ &= \mu(\Phi(y)^*\Phi(y)) \\ &= \|\Phi(y)\|_\mu^2. \end{aligned}$$

Now put

$$P_j = E_j \circ P \circ \alpha^{jp} \circ \gamma : C \to B_j.$$

Choose the set $J \subset \mathbf{N}$ with $|J| \leq r(\varepsilon/2)$ corresponding to P as in Corollary 1. Let $J_0 = J \cap \{0, 1, \cdots, k\}$. Then

$$\|P(x) - \phi(x)1\|_\mu < \varepsilon/2\|x\|, \; x \in (\tilde{A}_j), j \notin J_0.$$

For $x \in C$, $\|x\| \leq 1$ choose $y \in \tilde{A}_j$ such that $\|y\| \leq 1$, and

$$\|y - \alpha^{jp} \circ \gamma(x)\| < \varepsilon/4.$$

Then we have by the above estimates applied to $\Phi_j = (P - \phi) \circ \alpha^{jp} \circ \gamma$,

$$\begin{aligned} \|P_j(x) - \phi \circ \gamma(x)\|_\mu &= \|E_j \circ P \circ \alpha^{jp} \circ \gamma(x) - E_j \circ \phi \circ \alpha^{jp} \circ \gamma(x)\|_\mu \\ &\leq \|(P - \phi) \circ \alpha^{jp} \circ \gamma(x)\|_\mu \\ &\leq \|(P - \phi)(y)\|_\mu + \|(P - \phi)(y - \alpha^{jp}\gamma(x))\|_\mu \\ &< \varepsilon/2\|y\| + \varepsilon/2\|x\| \\ &< \varepsilon, \end{aligned}$$

for $j \notin J_0$, since $\|P - \phi\|_\mu \leq 2$.

If we in the notataion of [1, VI.2] let $\rho = P_j$, $\rho' = \phi \circ \gamma$ then we have for $j \notin J_0$

(13) $$|s_\mu(P_j) - s_\mu(\phi \circ \gamma)| \leq \delta(n, d, \varepsilon),$$

where d is the linear dimension of C and $\lim_{\varepsilon \to 0} \delta(n, d, \varepsilon) = 0$. By definition the entropy of the Abelian model (B, E_j, P, μ) for $(A, \phi, \gamma, \gamma \circ \alpha^p, \cdots, \gamma \circ \alpha^{p(k-1)})$ is

$$S(\mu | \bigvee_0^{k-1} B_j) - \sum_{j=0}^{k-1} s_\mu(P_j).$$

By subadditivity of S and [1, III.3] we therefore have that the entropy of the Abelian model is smaller than

$$\begin{aligned}
& S(\mu| \bigvee_{j\in J_0} B_j) + S(\mu| \bigvee_{j\notin J_0} B_j) - \sum_{j\in J_0} s_\mu(P_j) - \sum_{j\notin J_0} s_\mu(P_j) \\
& = (S(\mu| \bigvee_{j\in J_0} B_j) - \sum_{j\in J_0} s_\mu(P_j)) + (S(\mu| \bigvee_{j\notin J_0} B_j) - \sum_{j\notin J_0} s_\mu(P_j)) \\
& \le \sum_{j\in J_0} S(\phi\circ\gamma^{jp}) + \text{ Entropy of Abelian model } (B, E_j, P, \mu;\ j\notin J_0).
\end{aligned}$$

As in the proof [1, VI. 3] it follows by (13) that the entropy of the Abelian model $(B, E_j, P, \mu; j \notin J_0)$ differs from that defined by $\phi\circ\gamma$ by at most $(k - |J_0|)\varepsilon' \le k\varepsilon'$, where $\varepsilon' > 0$ is a number which converges to zero with ε. It follows that the entropy of the Abelian model (B, E_j, P, μ) differs from that of (B, ϕ) by less than

$$r(\varepsilon/2)S(\phi\circ\gamma) + k\varepsilon'.$$

We therefore have

$$|H_\phi(\gamma, \alpha^p\circ\gamma, \cdots, \alpha^{p(k-1)}\circ\gamma) - H_\phi(\phi\circ\gamma)| < r(\varepsilon/2)S(\phi\circ\gamma) + k\varepsilon'.$$

By [1, III.3] $H_\phi(\phi\circ\gamma) \le S(\phi\circ\gamma)$. Therefore, with our original choice of k as satisfying $k^{-1}r(\varepsilon/2)S(\phi\circ\gamma) < \varepsilon$, we find

$$\begin{aligned}
\frac{1}{k}H_\phi(\gamma, \alpha^p\circ\gamma, \cdots, \alpha^{p(k-1)}\circ\gamma) & \le \frac{1}{k}S(\phi\circ\gamma) + \frac{1}{k}r(\varepsilon/2)S(\phi\circ\gamma) + \varepsilon' \\
& < \varepsilon + \varepsilon + \varepsilon',
\end{aligned}$$

which can be made arbitrarily small. As in [5, Lem. 3.4] this means that $\frac{1}{m}H_\phi(\gamma, \alpha\circ\gamma, \cdots, \alpha^{m-1}\circ\gamma)$ can be made arbitrarily small, hence $H_{\phi,\gamma}(\alpha) = 0$ for all γ, i.e. $h_\phi(\alpha) = 0$. QED

4 Unique ergodicity of free shifts

In ergodic theory an automorphism of $C(X)$, X compact Hausdorff, is said to be uniquely ergodic if there exists a unique invariant probability measure on X, or equivalently a unique invariant state. For free shifts we shall prove a much stronger property. Our proof is a modification of an argument of Powers [3].

Theorem 3. *Let A_0 be a unital C^*-algebra with a state ϕ_0. Let $A_i = A_0$, $\phi_i = \phi_0$, $i \in \mathbf{Z}$, and put $(A, \phi) = (*A_i, *\phi_i)_{i \in \mathbf{Z}}$. Let α be the free shift and suppose (B, β, μ) is a C^*-dynamical system (i.e. B is a C^*-algebra, β an automorphism and μ a β-invariant state). Suppose λ is an $\alpha \otimes \beta$-invariant state on $A \otimes B$ such that $\lambda(1 \otimes b) = \mu(b)$ for $b \in B$. Then $\lambda = \phi \otimes \mu$.*

Proof. For each $i \in \mathbf{Z}$ let

$$\begin{aligned} \mathring{A}_i &= \{a \in A_i : \phi_i(a) = 0\} \\ \mathring{A} &= \operatorname{span}\{\prod_{k=1}^{n} a_{i_k} : a_{i_k} \in \mathring{A}_{i_k}, i_k \neq i_{k+1}\}. \end{aligned}$$

Then $\mathbf{C}1 + \mathring{A}$ is dense in A, and $\phi(a) = 0$ for all $a \in \mathring{A}$. For $k \in \{1, \cdots, s\}$ fix $a_k \in \mathring{A}$, $b_k \in B$, such that $a = \sum_{k=1}^{s} a_k \otimes b_k$ is self-adjoint in $\mathring{A} \otimes B \subset A \otimes B$. We shall show $\lambda(a) = 0$. We have

$$a_k = \sum_l \prod_{k_i} a_{k_i,l} \qquad \text{with} \quad a_{k_i,l} \in \mathring{A}_{k_i}$$

is a finite sum of finite products of operators in different $\mathring{A}_i$'s. Let

$$J = \{j \in \mathbf{Z} : a_{k_i,l} \in \mathring{A}_j \quad \text{for some} \quad a_k\},$$

i.e. J is the set of indices i such that some $a'_i \in \mathring{A}_i$ appears in the decomposition of a into a finite sum of finite products of elements in the $\mathring{A}_i$.

If we represent (A, ϕ) in its GNS-representation we may assume (A, ϕ) acts on the Hilbert space $\mathcal{H}$, where

$$\mathcal{H} = \mathbf{C}\xi \bigoplus \bigoplus_{i_1 \neq i_2 \neq \cdots \neq i_r} \mathcal{H}^\circ_{i_1} \otimes \cdots \otimes \mathcal{H}^\circ_{i_n}$$

For each $n \in \mathbf{Z}$ let

$$\mathcal{H}(n) = \bigoplus_{i_1 = n} \mathcal{H}^\circ_{i_1} \otimes \cdots \otimes \mathcal{H}^\circ_{i_r}$$

(Thus $\mathcal{H} = \mathbf{C}\xi \oplus \bigoplus_{n=-\infty}^{\infty} \mathcal{H}(n)$). Let

$$\mathcal{H}_J = \bigoplus_{n \in J} \mathcal{H}(n) \subset \mathcal{H}.$$

Then

$$\mathcal{H}_J^{\perp} = \mathbf{C}\xi \bigoplus \bigoplus_{m \in J^C} \mathcal{H}(m),$$

where J^C is the complement of J in $\mathbf{Z}$.

If $\eta = \eta_{i_1} \otimes \cdots \otimes \eta_{i_r} \in \mathcal{H}(m)$, $m \in J^C$, and $i \in J$ then if $a_i \in \mathring{A}_i$, we have (see section 2),

$$\begin{aligned} \pi(a_i)\eta &= \lambda_i(a_i)\eta = V_i(\pi_i(a_i) \otimes 1)V_i^*\eta \\ &= V_i(\pi_i(a_i) \otimes 1)\xi_i \otimes \eta \\ &= V_i(\pi_i(a_i)\xi_i \otimes \eta) \\ &= \pi_i(a_i)\xi_i \otimes \eta \in \mathcal{H}_J. \end{aligned}$$

If $j \neq i, j \in J$, then similarly for $a_j \in A_j$,

$$\begin{aligned} \pi(a_j a_i)\eta &= \pi(a_j)\pi(a_i)\eta \\ &= \pi(a_j)(\pi_i(a_i)\xi_i \otimes \eta) \\ &= \pi_j(a_j)\xi_j \otimes \pi_i(a_i)\xi_i \otimes \eta \in \mathcal{H}_J. \end{aligned}$$

An easy induction argument shows that with $a_k \in \mathring{A}$ as in the beginning of the proof, then $\pi(a_k)\eta \in \mathcal{H}_J$, hence we have

$$\pi(a_k)\mathcal{H}_J^\perp \subset \mathcal{H}_J.$$

Suppose B acts on the Hilbert space K. Let

$$M_J = \mathcal{H}_J \otimes K.$$

Then $M_J^\perp = \mathcal{H}_J^\perp \otimes K$, so $a = \sum a_k \otimes b_k$ satisfies

$$\pi \otimes id(a) M_J^\perp \subset M_J.$$

Since α is the free shift there exist integers $0 = n_1 < n_2 < \cdots < n_{20}$ such that if α_0 denotes the shift on $\mathbf{Z}$ then the sets $\alpha_0^{n_i}(J)$, $i = 1, \cdots, 20$, are all disjoint. Put

$$b = \frac{1}{20}\sum_{i=1}^{20}(\alpha \otimes \beta)^{n_i}(a).$$

Then $b \in \mathring{A} \otimes B$. For simplicity of notation identify a and $\pi \otimes id(a)$ and similarly for $(\alpha \otimes \beta)^{n_r}(a)$. Put

$$\begin{aligned} \mathcal{H}_{J_r} &= \bigoplus_{n \in \alpha_0^{n_r}(J)} \mathcal{H}(n), \\ M_{J_r} &= \mathcal{H}_{J_r} \bigotimes K. \end{aligned}$$

Let e_r denote the orthogonal projection of $H = \mathcal{H} \otimes K$ onto M_{J_r}. Since the sets $\alpha_0^{n_i}(J)$ are mutually disjoint, the projections e_r, $r = 1, \ldots, 20$, are mutually orthogonal. Furthermore

$$(\alpha \otimes \beta)^{n_r}(a) : e_r^\perp(H) \to e_r(H).$$

Corollary 2. *The Sauvageot-Thouvenot entropy of a free shift is zero.*

Proof. Take B to be abelian in the above theorem. Then it follows as in [2] easily from [4] that the entropy vanishes.

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